

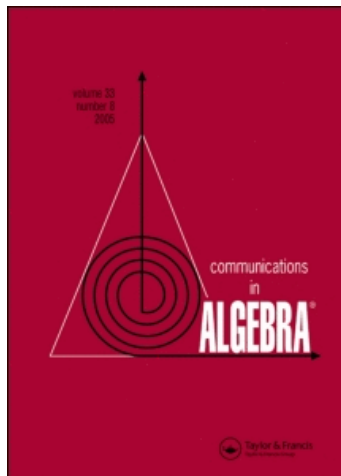
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CARDINALITIES OF RESIDUE FIELDS OF NOETHERIAN INTEGRAL DOMAINS

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We determine the relationship between the cardinality of a Noetherian integral domain and the cardinality of a residue field. One consequence of the main result is that it is provable in Zermelo–Fraenkel Set Theory with Choice (ZFC) that there is a Noetherian domain of cardinality \aleph_1 with a finite residue field, but the statement “There is a Noetherian domain of cardinality \aleph_2 with a finite residue field” is equivalent to the negation of the Continuum Hypothesis.

Key Words: Generalized continuum hypothesis; Integral domain; Noetherian ring; Prime spectrum; Residue field.

2000 Mathematics Subject Classification: Primary 13E05; Secondary 13B25, 13H99.

1. INTRODUCTION

Let R be a commutative Noetherian ring, and suppose that M_1, \dots, M_n are finitely many of the maximal ideals of R . The only general relationships that hold among the cardinals $|R|, |R/M_1|, \dots, |R/M_n|$ are that:

- (i) Each $|R/M_i|$ must be a prime power or infinite (since it is the size of a field);
- (ii) $|R|$ must be at least as large as $\prod |R/M_i|$ (since $|R| \geq |R/\cap M_i| = \prod |R/M_i|$);
and
- (iii) If R is finite, then $|R|$ is divisible by $\prod |R/M_i|$ (same reason as in (ii)).

One sees that no other relationships hold by considering the case where R is a finite product of fields of appropriate cardinalities.

Now drop the condition that R is Noetherian and add the condition that R is an integral domain. Again there are no interesting relationships between the cardinals $|R|, |R/M_1|, \dots, |R/M_n|$. If R is finite, then R is a field, $n = 1$, and $|R| = |R/M_1|$ is a prime power. If $|R|$ is to have infinite cardinality ρ , then we can take $R = \mathbb{Z}[X]$, where X is a set of ρ indeterminates. Such a ring has many homomorphisms onto fields of cardinality κ for any $\kappa \leq \rho$ that is a prime power or is infinite. Thus R

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has maximal ideals M_1, \dots, M_n such that $|R/M_i| = \kappa_i$ for any sequence of cardinals $(\kappa_1, \dots, \kappa_n)$, where each κ_i is a prime power or is infinite and dominated by ρ .

The situation when R is both Noetherian and an integral domain is different. Either R is a finite field, or else the following is true: if $|R| = \rho$ and $|R/M_i| = \kappa_i$, then as before each κ_i must be a prime power or infinite, but now it must be that

$$\kappa_i + \aleph_0 \leq \rho \leq \kappa_i^{\aleph_0} \tag{1.1}$$

for each i . We explain why this must hold in Lemma 2.1. We shall show conversely that if $(\rho, \kappa_1, \dots, \kappa_n)$ is any finite sequence of cardinals satisfying (1.1) for each i , and satisfying the condition that each κ_i is a prime power or is infinite, then there is a Noetherian integral domain R of cardinality ρ that has exactly n maximal ideals whose residue fields satisfy $|R/M_i| = \kappa_i$, $1 \leq i \leq n$.

All of the arguments needed to support the statements made in the previous paragraph appear in the articles of Shah [6, 7]. However, Shah draws the wrong conclusions from these arguments, because she assumes that $\kappa^{\aleph_0} = \kappa$ whenever κ exceeds the size of the continuum. This error in cardinal arithmetic leads Shah to conclude that a Noetherian domain R of size larger than the continuum has the property that any residue field R/M has the same size as R (not true), hence all residue fields have the same size as each other (also not true). This erroneous conclusion invalidates her description of $\text{Spec}(R[x])$ when R is a semilocal domain of dimension 1.

We tried to locate Shah to inform her of the error and found that, sadly, she passed away in 2005. Since no correction to her articles exists in the literature and her work is still being referenced (cf. [2]), and since this error concerns a very basic property of rings, we felt it necessary to record the corrections.

2. CORRECTIONS CONCERNING CARDINALITY

In this article, all rings are commutative with 1.

Lemma 2.1. *Let R be a Noetherian integral domain that is not a finite field, and let I be a proper ideal of R . If $|R| = \rho$ and $|R/I| = \kappa$, then $\kappa + \aleph_0 \leq \rho \leq \kappa^{\aleph_0}$.*

Proof. If R is an integral domain that is not a finite field, then R must be infinite, so $\rho \geq \aleph_0$. Combining this with the inequality $\rho = |R| \geq |R/I| = \kappa$, we obtain that $\rho \geq \max(\aleph_0, \kappa) = \kappa + \aleph_0$. This is the lefthand inequality of the conclusion of the lemma.

For each finite n , the R/I -module I^n/I^{n+1} is Noetherian, hence is generated by some finite number of elements, say $f(n)$ elements. The module I^n/I^{n+1} is a quotient of a free R/I -module on $f(n)$ generators, so $|I^n/I^{n+1}| \leq |R/I|^{f(n)}$. This yields $|R/I^n| = |R/I| \cdot |I/I^2| \cdots |I^{n-1}/I^n| \leq |R/I|^{F(n)} = \kappa^{F(n)}$, where $F(n) = f(0) + \cdots + f(n-1)$ is some finite number.

The product map $\pi : R \rightarrow \prod_{m \in \omega} R/I^m$ induced by the natural maps in each coordinate has kernel $\bigcap_{m \in \omega} I^m$. Since R is a Noetherian integral domain, the Krull Intersection Theorem guarantees that this kernel is zero. Hence π is injective, and we have $\rho = |R| \leq \prod_{m \in \omega} |R/I^m| \leq \prod_{m \in \omega} \kappa^{F(m)} = \kappa^{\sum_{m \in \omega} F(m)} \leq \kappa^{\aleph_0}$, which is the righthand inequality of the conclusion of the lemma. \square

We are interested in the case of Lemma 2.1 where R/I is a residue field, hence $|R/I| = \kappa$ is a prime power or is infinite. In this case, there is an easy converse to Lemma 2.1.

Lemma 2.2. *Suppose that κ and ρ are cardinals satisfying $\kappa + \aleph_0 \leq \rho \leq \kappa^{\aleph_0}$, and that κ is a prime power or is infinite. There is a Noetherian integral domain R with a maximal ideal M such that $|R| = \rho$ and $|R/M| = \kappa$.*

Proof. Let F be a field of cardinality κ , and let $F[[t]]$ be the ring of formal power series over F in the variable t . The underlying set of $F[[t]]$ is the set of all functions from ω into F , whence $|F[[t]]| = \kappa^{\aleph_0}$. The quotient field of $F[[t]]$ is the field $F((t))$ of formal Laurent series in the variable t . There is a field K of cardinality ρ such that $F(t) \subseteq K \subseteq F((t))$ for any ρ satisfying $|F(t)| = \kappa + \aleph_0 \leq \rho \leq \kappa^{\aleph_0} = |F((t))|$. Note that $F[[t]]$ is a discrete valuation ring (DVR) on $F((t))$, $K \subseteq F((t))$, and $F[[t]] \cap K$ is not a field (since t is not invertible). It follows that $F[[t]] \cap K$ is a DVR on K (whence also has cardinality ρ) with maximal ideal $M = (t) \cap K$. Clearly, F maps injectively into $(F[[t]] \cap K)/M$ and $(F[[t]] \cap K)/M$ maps injectively into $F[[t]]/(t) \cong F$. It follows that $|(F[[t]] \cap K)/M| = \kappa$ and the proof is complete. \square

Remark 2.3. Here is an alternative proof of Lemma 2.2: Suppose that R is any elementary subring of $F[[t]]$ (i.e., a subring where every first-order sentence with parameters from R that is true in R is also true in $F[[t]]$). If $F[t] \subseteq R \subseteq F[[t]]$, then it is not hard to see that such an R must be a DVR with maximal ideal $M = R \cap (t)$ and with residue field isomorphic to F . Hence R is a Noetherian integral domain with a maximal ideal M such that $|R/M| = |F| = \kappa$. The Downward Löwenheim–Skolem Theorem guarantees the existence of rings R satisfying $F[t] \subseteq R \subseteq F[[t]]$, R an elementary subring of $F[[t]]$, and $|R| = \rho$ for any ρ satisfying $|F[t]| = \kappa \leq \rho \leq \kappa^{\aleph_0} = |F[[t]]|$.

In more detail, for countable first-order languages L , the Downward Löwenheim–Skolem Theorem guarantees that if A is an infinite subset of an L -structure B , then A can be enlarged to an elementary substructure A' of B without altering its cardinality ($|A| = |A'|$). Apply this to any set A satisfying $F[t] \subseteq A \subseteq F[[t]] =: B$ and $|A| = \rho$ to produce the desired ring R .

Corrections 2.4. Lemma 2.5 of [6] and Lemma 2.1 of [7] are versions of Lemma 2.1 from above, although they are stated only in the case where R is a Noetherian local domain and $I = M$ is a maximal ideal of R . The proofs of all three are essentially the same, although the important inequality $\kappa + \aleph_0 \leq \rho \leq \kappa^{\aleph_0}$ does not appear in [6] or [7]. Instead, the lemmas from [6, 7] assert that $|R| \leq \sup(|R/M|, c)$ and that, if $|R| > c$, then $|R| = |R/M|$. (Here $c = 2^{\aleph_0}$ is the cardinality of the continuum.) These assertions are incorrect, as Lemma 2.2 witnesses.

Corollary 2.2 to Lemma 2.1 of [7] asserts that if R is a Noetherian ring of size larger than the continuum, then comparable prime ideals have the same index (i.e., if $P \subseteq M$ are both prime, then $|R/P| = |R/M|$). This corollary is derived from Lemma 2.1 of [7] by factoring by P , localizing at the prime M/P , and then applying the incorrect lemma to the resulting Noetherian local domain. This corollary is also incorrect, as our Lemma 2.2 witnesses. (Take R and M to be as in that lemma and take $P = (0)$.)

Remark 2.5. If κ is finite, then the inequality $\kappa + \aleph_0 \leq \rho \leq \kappa^{\aleph_0}$ from Lemma 2.1 reduces to

$$\aleph_0 \leq \rho \leq 2^{\aleph_0}, \tag{2.1}$$

while if κ is infinite it reduces to

$$\kappa \leq \rho \leq \kappa^{\aleph_0}. \tag{2.2}$$

It follows from (2.1) that the set of possible cardinalities of an infinite Noetherian integral domain with a finite residue field depends on the cardinality of the continuum, i.e., the value of α for which $\aleph_\alpha = 2^{\aleph_0}$. There exist models of ZFC showing that 2^{\aleph_0} can equal any \aleph_α of uncountable cofinality, hence the interval $[\aleph_0, 2^{\aleph_0}]$ can be as large as desired.

It follows from (2.2) that the set of possible cardinalities of a Noetherian integral domain with an infinite residue field of size κ depends on the size of the interval $[\kappa, \kappa^{\aleph_0}]$. Here are some remarks about the possible sizes of this interval. For these remarks, the notation κ^+ denotes the successor of cardinal κ . (That is, if $\kappa = \aleph_\alpha$, then $\kappa^+ = \aleph_{\alpha+1}$.) The *Continuum Hypothesis (CH)* is the assumption $2^{\aleph_0} = \aleph_1^+$. The *Generalized Continuum Hypothesis (GCH)* is the assumption $2^\kappa = \kappa^+$ for all infinite κ . Both are known to be independent of ZFC.

- (1) (It is provable in ZFC that the interval $[\kappa, \kappa^{\aleph_0}]$ is trivial arbitrarily often.) If $\kappa = \lambda^{\aleph_0}$ for some λ , then $\kappa^{\aleph_0} = \lambda^{\aleph_0 \cdot \aleph_0} = \lambda^{\aleph_0} = \kappa$, so setting $\kappa = \lambda^{\aleph_0}$, and letting λ run through all cardinals, we get arbitrarily large cardinals κ satisfying the equation $\kappa^{\aleph_0} = \kappa$.
- (2) (It is provable in ZFC that the interval $[\kappa, \kappa^{\aleph_0}]$ is nontrivial arbitrarily often.) Let κ be an infinite cardinal, and let $\text{cf } \kappa$ be the cofinality of κ . It is well known that $\kappa^{\text{cf } \kappa} > \kappa$ (see Corollary 5.14 of [4], for example). Thus if $\text{cf } \kappa = \aleph_0$, then $\kappa^{\aleph_0} > \kappa$. Since $\aleph_{\alpha+\omega}$ has countable cofinality for every α , setting $\kappa = \aleph_{\alpha+\omega}$, and letting α run through all ordinals yields arbitrarily large cardinals κ where the equation $\kappa^{\aleph_0} = \kappa$ fails to hold.
- (3) (It is consistent with ZFC that all of the intervals $[\kappa, \kappa^{\aleph_0}]$ are small.) Assume that GCH holds, and let κ be an infinite cardinal. Then $\kappa^{\aleph_0} \leq \kappa^\kappa = 2^\kappa = \kappa^+$, so $\kappa^{\aleph_0} = \kappa$ or $\kappa^{\aleph_0} = \kappa^+$. Hence $[\kappa, \kappa^{\aleph_0}]$ has size 1 or 2 for all infinite κ .
- (4) (It is consistent with ZFC that some interval $[\kappa, \kappa^{\aleph_0}]$ is large.) Choose any $\aleph_\alpha < \aleph_\beta$ and let $\kappa = \aleph_\alpha$. There is a model of ZFC where $2^{\aleph_0} > \aleph_\beta$. In this model $\kappa^{\aleph_0} \geq 2^{\aleph_0} > \aleph_\beta$, so the interval $[\kappa, \kappa^{\aleph_0}]$ contains $[\aleph_\alpha, \aleph_\beta]$, which may be arbitrarily large.
- (5) (It is not consistent with ZFC that some interval $[\kappa, \kappa^{\aleph_0}]$ is (truly) large.) By “truly large” we mean that the interval contains a lot of alephs, all of which are larger than the continuum. (Remark (4) showed that it is consistent for $[\kappa, \kappa^{\aleph_0}]$ to contain an arbitrary number of alephs, but some were below the size of the continuum.) Suppose that $\kappa = \aleph_\omega$ and that κ^{\aleph_0} is not the size of the continuum. (For example, suppose κ itself is larger than the continuum.) Then it follows from a celebrated theorem of Shelah that $\kappa^{\aleph_0} < \aleph_{\omega_4}$. (Theorem 5.13 in Volume II, Chapter 14 of [1].) So, if $\kappa = \aleph_\omega$ exceeds the size of the continuum,

then there is an upper bound on which aleph κ^{\aleph_0} can be. (Similar results hold for $\kappa = \aleph_\delta$ where δ is any limit ordinal of countable cofinality.)

- (6) (The statement: "There is a Noetherian integral domain of size \aleph_2 with a finite residue field" is equivalent to the negation of CH.) Let κ be a prime power. By Lemmas 2.1 and 2.2, there exists a Noetherian integral domain D of size \aleph_2 with a residue field of size κ if and only if $\aleph_2 \leq \kappa^{\aleph_0} = 2^{\aleph_0}$ if and only if CH fails. Note that $\aleph_1 \leq 2^{\aleph_0}$ in any model of ZFC, and so the existence of a Noetherian integral domain of size \aleph_1 with a finite residue field is provable in ZFC.

Until this point, we have compared the size of a Noetherian integral domain R to the size of a single residue field R/M . Next we turn to the comparison of $|R|$ to the cardinalities of finitely many residue fields, $|R/M_i|$, $i = 1, \dots, n$, which is one of the problems focused on in [6, 7]. The result is simply that these cardinals can be arbitrary subject to the restriction in Lemma 2.1.

Theorem 2.6. *Let $(\rho, \kappa_1, \dots, \kappa_n)$ be a finite sequence of cardinals where each κ_i is a prime power or is infinite and*

$$\kappa_i + \aleph_0 \leq \rho \leq \kappa_i^{\aleph_0}$$

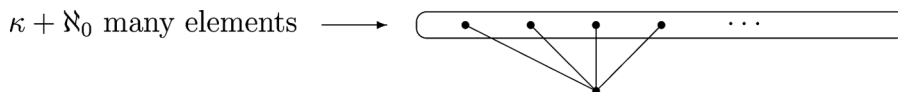
holds for all i . Then there is a Noetherian integral domain R of dimension 1 with exactly n maximal ideals M_1, \dots, M_n such that $|R| = \rho$ and $|R/M_i| = \kappa_i$ for all i .

Proof. When $\rho \leq 2^{\aleph_0}$ this result is stated and proved correctly in Theorem 2.3 of [7]. When $\rho > 2^{\aleph_0}$ the result is stated and proved incorrectly because of the influence of the erroneous Lemma 2.1 of [7]. The correct thing to do when $\rho > 2^{\aleph_0}$ is to simply notice that this cardinality restriction has no bearing on the construction used earlier. That is, the construction that worked when $\rho \leq 2^{\aleph_0}$ also works when $\rho > 2^{\aleph_0}$. In fact, when $\rho > 2^{\aleph_0}$ one can omit the use of special number fields in the construction and just use \mathbb{Q} , as in the argument from Example 2.10 of [6]. □

3. CORRECTIONS CONCERNING SPECTRA

If R is a commutative ring, then the inclusion order on prime ideals can be recovered from the topology on $\text{Spec}(R)$, since $\mathfrak{p} \subseteq \mathfrak{q}$ holds exactly when \mathfrak{q} lies in the closure of $\{\mathfrak{p}\}$. Conversely, if R is Noetherian, then the topology of $\text{Spec}(R)$ can be determined from the order on prime ideals, since a subset of $\text{Spec}(R)$ is topologically closed iff it is a finitely generated order filter in the poset of primes under inclusion. Thus, there is no harm in considering $\text{Spec}(R)$ exclusively as a poset when R is Noetherian, which we do in this section.

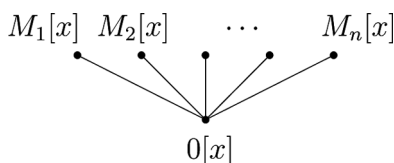
If F is a field of cardinality κ , then $\text{Spec}(F[x])$ considered as an ordered set is easily seen to be a *fan of size $\kappa + \aleph_0$* :



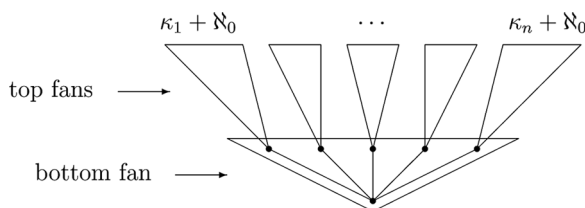
by which we mean a subset of pairwise incomparable elements (an antichain) of size $|F[x]| = \kappa + \aleph_0$ with a bottom element adjoined. Here we are using the facts that the prime ideals of $F[x]$ are (0) and $(f(x))$ where $f(x) \in F[x]$ is irreducible, and that there are $|F[x]|$ of the latter type.

The primary goal of the articles [6, 7] is to determine the structure of the ordered set $\text{Spec}(R[x])$ when R is a Noetherian integral domain of dimension 1 that has finitely many maximal ideals. This problem was solved by Heinzer and Wiegand in [3] in the case where R is countable. Shah's efforts in [6, 7] are directed towards extending that result to uncountable rings.

The statement that R is a Noetherian integral domain of dimension 1 that has finitely many maximal ideals means that it is a Noetherian ring whose spectrum is a finite fan. Suppose that this is so and that the nonzero primes of R are the maximal ideals M_1, \dots, M_n . The ideals $0[x], M_1[x], \dots, M_n[x]$ are prime in $R[x]$ and form a fan contained in $\text{Spec}(R[x])$, which we call the *bottom fan* of $\text{Spec}(R[x])$:

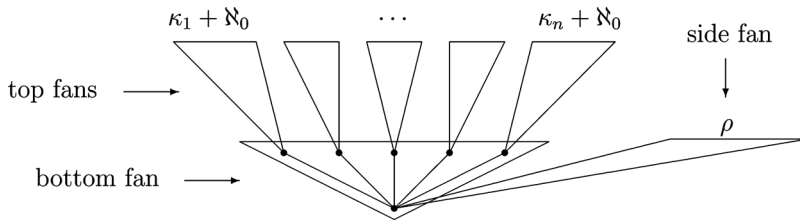


The primes of $R[x]$ containing some given $M_i[x]$ are in 1-1 order-preserving correspondence with the primes of $R[x]/M_i[x] = (R/M_i)[x]$, and hence these primes form a fan of size $\kappa_i + \aleph_0$ for $\kappa_i = |R/M_i|$ according to our observation about spectra of polynomial rings over fields. We call these the *top fans* of $\text{Spec}(R[x])$:



We depict the top fans as if they are disjoint, and indeed they must be, since $M_i + M_j = R$ implies $M_i[x] + M_j[x] = R[x]$, so no prime of $R[x]$ contains distinct $M_i[x]$ and $M_j[x]$. We also depict $\text{Spec}(R[x])$ as if it had height 2, which of course it must be by dimension theory for Noetherian rings. This fully describes that part of $\text{Spec}(R[x])$ consisting of the primes comparable to the $M_i[x]$'s.

The remaining primes are exactly the nonzero primes of $R[x]$ whose restriction to R is zero. But a prime of $R[x]$ restricts to zero in R iff it survives the localization of $R[x]$ at the multiplicatively closed set $S = R - \{0\}$, hence these primes are in 1-1 correspondence with the primes of $S^{-1}R[x] \cong K[x]$ where K is the quotient field of R . This shows that the remaining primes form one last fan (of size $|K| + \aleph_0 = |R| + \aleph_0 = |R| =: \rho$), that we call the *side fan* of $\text{Spec}(R[x])$:



It is now clear why it is important to know the relationship between the cardinality ρ of R and the cardinalities $\kappa_i = |R/M_i|$ of its residue fields: these cardinals determine the sizes of the fans that appear in $\text{Spec}(R[x])$ when $\text{Spec}(R)$ is itself a finite fan.

The only undetermined aspect of $\text{Spec}(R[x])$ is the order relationship between the top elements of the side fan and the top elements of the top fans. So let S denote the top elements of the side fan, let T be the set of top elements of the top fans (the height two elements), let $\mathcal{P}(T)$ be the power set of T , and let $\chi : S \rightarrow \mathcal{P}(T)$ be the function that assigns to each top element p of the side fan the set of primes in T that lie strictly above p . What is shown in Theorem 2.4 of [6] is that this comparability function χ must behave in exactly one of two different ways, depending on whether R is a Henselian local ring. If R is not such a ring, then

(α) $\chi : S \rightarrow \mathcal{P}(T)$ maps onto the set of finite (possibly empty) subsets of T , and for any finite set $T_0 \subseteq T$ the fiber $\chi^{-1}(T_0)$ has size ρ .

But if R is a Henselian local ring, then

(β) $\chi : S \rightarrow \mathcal{P}(T)$ maps onto the subsets of T that have size at most 1, and for any set $T_0 \subseteq T$ of size at most 1 the fiber $\chi^{-1}(T_0)$ has size ρ .

This information completes the description of the possible isomorphism types of $\text{Spec}(R[x])$ when R is a Noetherian integral domain of dimension 1 with finitely many maximal ideals. The final theorem is the following one.

Theorem 3.1. *Let R be a Noetherian integral domain of dimension 1 with n maximal ideals M_1, \dots, M_n . If $\rho = |R|$ and $\kappa_i = |R/M_i|$, then $\text{Spec}(R[x])$ has the following properties, which determine it up to isomorphism.*

Case 1. *If R is not a Henselian local ring, then:*

- (a) *the bottom fan has size n ;*
- (b) *the top fans have sizes $\kappa_i + \aleph_0$ for $1 \leq i \leq n$;*
- (c) *the side fan has size ρ ; and*
- (d) *χ behaves as in (α) above.*

Case 2. *If R is a Henselian local ring, then $n = 1$, (a)–(c) hold along with*

- (d) *χ behaves as in (β) above.*

Unfortunately, both [6, 7] go one step further and claim that if ρ exceeds the size of the continuum, then $\rho = \kappa_1 = \dots = \kappa_n$, which, as we have seen, is false. The only restrictions on the sequence $(\rho, \kappa_1, \dots, \kappa_n)$ are those listed in Theorem 2.6.

As a final comment, Theorem 2.6 provides examples of Noetherian integral domains with n maximal ideals realizing any cardinal sequence $(\rho, \kappa_1, \dots, \kappa_n)$ satisfying the restrictions of that theorem. These cardinal sequences determine the possibilities for $\text{Spec}(R[x])$ in Case (1) of Theorem 3.1. For Case (2) of Theorem 3.1, Theorem 2.6 determines which cardinal sequences (ρ, κ) arise for local domains. We did not take care in Theorem 2.6 to distinguish whether or not our examples are Henselian, but in fact there exist both Henselian and non-Henselian local domains (R, M) such that $|R| = \rho$ and $|R/M| = \kappa$ for any ρ and κ satisfying the cardinality restrictions of Theorem 2.6, as the next example shows.

Example 3.2. Let F be a field of cardinality κ , and let $F[[t]]$ be the ring of formal power series over F in the variable t . As in the proof of Lemma 2.2, we will consider rings of the form $R = F[[t]] \cap K$, where K is a field satisfying $F(t) \subseteq K \subseteq F((t))$. As shown in that lemma, for any such choice of K the resulting ring R is a DVR with maximal ideal $M = (t) \cap K$ for which $|R| = |K|$ and $|R/M| = |F| = \kappa$.

The polynomial $x^2 - x - t \in (F[[t]])[x]$ has two roots that sum to 1 in $F[[t]]$; call them r and $1 - r$. But $x^2 - x - t$ has no roots in $F[t]$, hence $r \notin F(t)$. Let L be a field satisfying $F(t) \subseteq L \subseteq F((t))$ that is maximal for the condition that $r \notin L$. The existence of L follows from Zorn's Lemma, and it is not hard to see that the maximality of L implies that $F((t))$ is algebraic over L . This implies that $|L| = |F((t))| = \kappa^{\aleph_0}$.

Let us further restrict the choice of field K from the first paragraph of this example so that $F(t) \subseteq K \subseteq L$. For any ρ satisfying $|F(t)| = \kappa + \aleph_0 \leq \rho \leq \kappa^{\aleph_0} = |L|$ there is such a field K satisfying $|K| = \rho$. If $R = F[[t]] \cap K$, then R is a DVR of cardinality ρ with maximal ideal $M = (t) \cap K$, and $|R/M| = \kappa$.

We argue that R is not Henselian. If it were so, then for any polynomial $f(x) \in R[x]$ that factors modulo M into coprime factors, $\bar{f}(x) = \bar{g}(x)\bar{h}(x)$, there would exist coprime polynomials $g(x), h(x) \in R[x]$ such that $f(x) = g(x)h(x)$ in $R[x]$, $\bar{g}(x) = g(x) + M[x]$ and $\bar{h}(x) = h(x) + M[x]$. But there do not exist such factors for the polynomial $f(x) = x^2 - x - t$, which factors modulo M as $\bar{x}(\bar{x} - 1)$. For if this factorization could be pulled back to $R[x]$, then $x^2 - x - t$ would have roots in R . Since $R \subseteq F((t))$, these roots could only be r and $1 - r$, and in the previous paragraph we arranged that these elements were not in R . This shows that R is not Henselian, completing the argument that there exist non-Henselian local rings (R, M) with cardinal parameters $\rho = |R|$ and $\kappa = |R/M|$ for any (ρ, κ) satisfying the cardinality restrictions of Theorem 2.6.

The Henselian example satisfying the same cardinality restrictions may be obtained from the non-Henselian example by Henselization. This does not affect the isomorphism type of the residue field (Theorem 43.3 of [5]) or the cardinality of the ring (since the Henselization of R is algebraic over R).

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