

RELATIVELY CONGRUENCE MODULAR QUASIVARIETIES OF MODULES

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ABSTRACT. We show that the quasiequational theory of a relatively congruence modular quasivariety of left R -modules is determined by a two-sided ideal in R together with a filter of left ideals. The two-sided ideal encodes the identities that hold in the quasivariety, while the filter of left ideals encodes the quasiidentities. The filter of left ideals defines a generalized notion of torsion.

It follows from our result that if R is left Artinian, then any relatively congruence modular quasivariety of left R -modules is axiomatizable by a set of identities together with at most one proper quasiidentity, and if R is a commutative Artinian ring then any relatively congruence modular quasivariety of left R -modules is a variety.

Dedicated to Don Pigozzi

1. INTRODUCTION

This paper is inspired by Problem 9.13 of Don Pigozzi's paper, *Finite basis theorems for relatively congruence-distributive quasivarieties*, (Pigozzi 1988). The problem asks:

Is it true that every finitely generated and relatively congruence modular quasivariety is finitely based?

This question is still open. Pigozzi's paper shows the answer to be affirmative if "modular" is strengthened to "distributive". Analogous problems for varieties were shown to have positive solutions in (Baker 1977; McKenzie 1987; R. Willard 2000; Kearnes, Szendrei, and R. D. Willard to appear), and a related problem for quasivarieties was shown to have an affirmative solution in (Maróti and McKenzie 2004). The best partial answer to Problem 9.13 that now exists is Theorem 8 of (Dziobiak et al. 2009), which says that if a quasivariety \mathcal{K} and the variety it generates are finitely generated and relatively congruence modular, then \mathcal{K} is finitely based.

A relatively congruence distributive quasivariety is nothing other than a relatively congruence modular quasivariety in which no member has a nontrivial abelian congruence, so Pigozzi's paper solves the part of Problem 9.13 that does not involve abelian

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congruences. Tools for dealing with abelian congruences in relatively congruence modular quasivarieties were developed in (Kearnes and McKenzie 1992; Kearnes and Szendrei 1998), but they have not yet yielded a full solution to the problem. What these tools show is that abelian congruences in such quasivarieties are quasiaffine, which means the blocks of an abelian congruence support a module-like structure. In this paper we study the purest relatively congruence modular quasivarieties which are not distributive, namely quasivarieties of modules. Our main result is a description of all relatively congruence modular quasivarieties of modules.

2. THE CLASSIFICATION THEOREM

For a unital ring R let $R\text{-Mod}$ be the variety of left R -modules. If \mathcal{K} is a subquasivariety of $R\text{-Mod}$ and $M \in \mathcal{K}$, then a \mathcal{K} -submodule (or *relative submodule*) of M is an R -submodule $S \leq M$ such that $M/S \in \mathcal{K}$. \mathcal{K} is *relatively congruence modular (RCM)* if every $M \in \mathcal{K}$ has a modular lattice of \mathcal{K} -submodules.

For the simplest example of these definitions take $R = \mathbb{Z}$, so that $R\text{-Mod}$ is the variety of abelian groups. Let \mathcal{K} be the subquasivariety of $\mathbb{Z}\text{-Mod}$ consisting of torsion-free abelian groups. The only relative submodules on, say, $\mathbb{Z} \in \mathcal{K}$ are (0) and \mathbb{Z} . That is, \mathbb{Z} is *relatively simple* (although \mathbb{Z} is far from being a simple module). It can be shown that this quasivariety, the quasivariety of torsion-free abelian groups, is a minimal quasivariety which happens to be RCM.

Let's generalize the example above in an artificial way. Let $R = \mathbb{Z}[t]$ and let \mathcal{K} be the subquasivariety of $R\text{-Mod}$ consisting of all torsion-free abelian groups considered as R -modules by defining t to act as zero on any module in \mathcal{K} . Then \mathcal{K} is axiomatized by the identity $tx = 0$ together with a family of quasiidentities of the form

$$nx = 0 \rightarrow x = 0.$$

This is essentially the same as the preceding example, so in particular it is RCM.

The point of this paper is to show that every RCM quasivariety of modules looks like the one from the previous paragraph. For any RCM quasivariety \mathcal{K} of R -modules there is a set Σ of 1-variable identities along with a specific torsion notion which realizes \mathcal{K} as the subquasivariety of $R\text{-Mod}$ consisting of the torsion-free R -modules that satisfy Σ . Σ corresponds to a two-sided ideal in R while the torsion notion corresponds to a filter in the lattice of left ideals of R .

Let's begin by identifying the role played by Σ .

Lemma 2.1. *Let \mathcal{V} be a subvariety of $R\text{-Mod}$ and let $I = \{r \in R \mid \mathcal{V} \models rx = 0\}$ be its annihilator. Then*

- (1) I is a two-sided ideal in R ,
- (2) $\Sigma := \{rx = 0 \mid r \in I\}$ axiomatizes \mathcal{V} relative to $R\text{-Mod}$, and
- (3) \mathcal{V} is definitionally equivalent to $R/I\text{-Mod}$.

We imagine applying this in the situation where \mathcal{K} is a subquasivariety of $R\text{-Mod}$ and \mathcal{V} is the variety generated by \mathcal{K} .

We do not prove Lemma 2.1, but do point out that a key idea in the proof is that a single module identity $r_1x_1 + \cdots + r_kx_k = 0$ has the same strength as the set $\{r_1x_1 = 0, \dots, r_kx_k = 0\}$ of 1-variable module identities.

Lemma 2.1 allows us to pass from R to R/I and henceforth consider only the situation where \mathcal{K} generates $R\text{-Mod}$. We shall make this assumption as we work out the main result of the paper.

Next we describe the torsion concept that plays a role in this paper.

Definition 2.2. Let \mathcal{L} be the poset of finitely generated left ideals of R ordered by inclusion. A *torsion notion* for R is a subset $\mathcal{F} \subseteq \mathcal{L}$ satisfying the following conditions:

- (1) \mathcal{F} is a nonempty order filter in \mathcal{L} . ($A \in \mathcal{F}$, $B \in \mathcal{L}$, and $A \subseteq B$ implies $B \in \mathcal{F}$.)
- (2) \mathcal{F} is downward directed. ($A, B \in \mathcal{F}$ implies there is a $C \in \mathcal{F}$ such that $C \subseteq A$ and $C \subseteq B$.)
- (3) If $X, Y \subseteq R$ are finite subsets such that the left ideals (X) and (Y) belong to \mathcal{F} , then the left ideal (XY) belongs to \mathcal{F} .
- (4) For all $A \in \mathcal{F}$ and $r \in R$ there is $B \in \mathcal{F}$ such that $Br \subseteq A$.
- (5) (Regularity of elements of \mathcal{F}) If $A \in \mathcal{F}$, $r \in R$, and $Ar = 0$, then $r = 0$.

Given a torsion notion \mathcal{F} we say that an element m of an R -module M is an \mathcal{F} -torsion element if $Am = 0$ for some $A \in \mathcal{F}$. If M has no nonzero \mathcal{F} -torsion elements, then it is \mathcal{F} -torsion-free. This may also be expressed by saying that $M \models Ax = 0 \rightarrow x = 0$ for each $A \in \mathcal{F}$. It is easy to see that a statement of the form $Ax = 0 \rightarrow x = 0$ for a finitely generated left ideal A is equivalent to a quasiidentity. Namely if $A = (a_1, \dots, a_m)$, then $Ax = 0 \rightarrow x = 0$ is satisfied if and only if

$$(q_A) \quad (a_1x = 0) \wedge \cdots \wedge (a_mx = 0) \rightarrow (x = 0)$$

is satisfied, so the class of \mathcal{F} -torsion-free R -modules is a quasivariety.

Any torsion notion \mathcal{F} contains R , and the set $\{R\}$ is always a torsion notion. Every R -module is torsion-free with respect to this trivial torsion notion.

Items (1)–(4) of Definition 2.2 simplify quite a bit when R is commutative. Namely, (4) automatically holds when R is commutative, since we can choose $B = A$. Item (3) now asserts that \mathcal{F} is closed under multiplication. When this holds, (2) will also hold, since for commutative rings the product of two ideals is contained in each of them. Thus (1)–(4) merely say that \mathcal{F} is a multiplicatively closed order filter in the poset of finitely generated ideals. (For any ring R , item (5) of the definition asserts that the free R -module, R , is \mathcal{F} -torsion-free.)

Here is the statement of the main theorem of the paper.

Theorem 2.3. *Let \mathcal{K} be a quasivariety of R -modules such that the variety generated by \mathcal{K} is all of $R\text{-Mod}$. Then \mathcal{K} is RCM iff there is a torsion notion \mathcal{F} such that \mathcal{K} is the quasivariety of \mathcal{F} -torsion-free R -modules.*

We prove Theorem 2.3 in the next two sections, but here we derive a corollary.

Corollary 2.4. *Let R be a left Artinian ring. If \mathcal{K} is an RCM quasivariety of R -modules, then \mathcal{K} may be axiomatized relative to $R\text{-Mod}$ by a finite set of identities together with at most one proper quasiidentity. In particular, if R is a finitely presentable ring, then \mathcal{K} is finitely axiomatizable.*

Proof. By the Hopkins-Levitski Theorem, a left Artinian ring is left Noetherian, so every left ideal of R is finitely generated. In particular, we can effect the passage from R -modules to R/I -modules (as indicated in Lemma 2.1) by imposing finitely many identities on the variety $R\text{-Mod}$. Thus we may assume henceforth that \mathcal{K} generates $R\text{-Mod}$ as a variety and our goal is now to prove that \mathcal{K} can be axiomatized relative to $R\text{-Mod}$ by at most one quasiidentity.

Let \mathcal{F} be the torsion notion guaranteed by Theorem 2.3. Since R is Artinian, \mathcal{F} is generated as an order filter by its minimal elements. By item (2) of Definition 2.2, \mathcal{F} is a principal order filter in \mathcal{L} , say \mathcal{F} is the order filter generated by $A \in \mathcal{L}$. Now the notion of ‘ \mathcal{F} -torsion-free’ is expressible by $Ax = 0 \rightarrow x = 0$, or equivalently by the single quasiidentity q_A . (What has been left unsaid so far is that if $A \subseteq B$, then $Ax = 0 \rightarrow x = 0$ is stronger than $Bx = 0 \rightarrow x = 0$.)

The last assertion of the corollary follows from the fact that $R\text{-Mod}$ is finitely axiomatizable when R is finitely presentable. \square

The one proper quasiidentity mentioned in Corollary 2.4 can be eliminated when R is commutative.

Corollary 2.5. *Let R be a commutative Artinian ring. Any RCM quasivariety of R -modules is a variety.*

Proof. Let \mathcal{K} be an RCM quasivariety of R -modules. We shall argue the proof for arbitrary R until it is necessary to appeal to commutativity.

Using Lemma 2.1 we may reduce to the case where the variety generated by \mathcal{K} is all of $R\text{-Mod}$. In the proof of Corollary 2.4 we showed that the torsion notion \mathcal{F} associated to \mathcal{K} is a principal filter in the poset of finitely generated left ideals of R . Let A be the generator of this principal filter.

Item (4) of the definition of ‘torsion notion’ implies that A is a two-sided ideal. For if $r \in R$, then there must be a $B \in \mathcal{F}$ such that $Br \subseteq A$. Since $A \subseteq B$ this yields $Ar \subseteq Br \subseteq A$.

Another special property that A must satisfy is that $A^2 = A$. To see this, choose a finite set X that generates A as a left ideal. Then item (3) implies that $(X^2) \in \mathcal{F}$. But clearly $(X^2) \subseteq A^2$, so $A \subseteq (X^2) \subseteq A^2 \subseteq A$.

Now we invoke the commutativity hypothesis. A finitely generated idempotent ideal is generated by an idempotent element, so $A = (e)$ for some element e satisfying $e^2 = e$. For $r = 1 - e$ we have $Ar = 0$, so item (5) of the definition of ‘torsion notion’ yields that $1 - e = 0$, i.e. $e = 1$, or equivalently $A = R$. This forces $\mathcal{F} = \{R\}$. As noted after Definition 2.2, this implies that every R -module is \mathcal{F} -torsion-free, so $\mathcal{K} = R\text{-Mod}$. \square

Corollaries 2.4 and 2.5 are not true if you weaken ‘Artinian’ to ‘Noetherian’, since the quasivariety of torsion-free abelian groups is not finitely axiomatizable and is not a variety. Also, Corollary 2.5 is not true without the commutativity hypothesis. To see this, let R be the ring of upper triangular 2×2 matrices over some field. If the matrix units in R are e_{11}, e_{12}, e_{22} , then the quasivariety of R -modules axiomatized relative to $R\text{-Mod}$ by $(e_{11}x = 0) \wedge (e_{12}x = 0) \rightarrow (x = 0)$ is RCM and is not a variety.

3. RCM \implies TORSION NOTION

In this section we prove that if \mathcal{K} is an RCM quasivariety of R -modules and the variety generated by \mathcal{K} is all of $R\text{-Mod}$, then there is a torsion notion \mathcal{F} such that \mathcal{K} is the quasivariety of \mathcal{F} -torsion-free R -modules. This is one direction of the proof of Theorem 2.3.

To prove what is needed we make use of the fact, proved in (Kearnes and McKenzie 1992), that an RCM quasivariety has an ‘almost equational axiomatization’, and that the \mathcal{K} -extension of a submodule can be computed easily with the aid of that axiomatization. Here the \mathcal{K} -extension of a submodule $S \leq M$ is the least \mathcal{K} -submodule $\bar{S} \leq M$ that contains S .

We recall the necessary concept from (Kearnes and McKenzie 1992). A Δ -axiom is a first-order sentence, involving pairs of terms $(p_j(x, y, \bar{u}, \bar{v}, \bar{z}), q_j(x, y, \bar{u}, \bar{v}, \bar{z}))$, $j < n$, expressing that

(1) the identities

$$p_j(x, x, \bar{u}, \bar{u}, \bar{z}) = q_j(x, x, \bar{u}, \bar{u}, \bar{z})$$

hold for $j < n$, and

(2) the quasiidentity

$$\bigwedge (p_j(x, y, \bar{u}, \bar{u}, \bar{z}) = q_j(x, y, \bar{u}, \bar{u}, \bar{z})) \rightarrow (x = y)$$

holds.

We label this Δ -axiom $\Delta(p, q)$.

The two theorems from (Kearnes and McKenzie 1992) that we will use are:

Theorem 3.1. *(Theorem 5.1 of (Kearnes and McKenzie 1992)) Let \mathcal{K} be an RCM quasivariety. \mathcal{K} is axiomatized by a set of Δ -axioms combined with a set of identities.*

For the following theorem, the \mathcal{K} -extension of a congruence θ is the least \mathcal{K} -congruence containing θ .

Theorem 3.2. (Theorem 5.2 of (Kearnes and McKenzie 1992)) Let \mathcal{K} be an RCM quasivariety. Let $\mathbf{A} \in \mathcal{K}$, $\theta \in \text{Con}(\mathbf{A})$, and $u, v \in A$. Then (u, v) belongs to the \mathcal{K} -extension of θ iff there is some Δ -axiom $\Delta(p, q)$ valid in \mathcal{K} , some pairs $(a_i, b_i) \in \theta$, and some elements \bar{c} such that $p_i(u, v, \bar{a}, \bar{b}, \bar{c}) = q_i(u, v, \bar{a}, \bar{b}, \bar{c})$ for all i .

When dealing with quasivarieties of modules it is possible to code a Δ -axiom $\Delta(p, q)$ as a left ideal in such a way that the following are true.

Theorem 3.3. Let $\Delta(p, q)$ be a Δ -axiom and let A be its encoding as a left ideal.

- (1) An R -module M satisfies $\Delta(p, q)$ iff it satisfies $Ax = 0 \rightarrow x = 0$.
- (2) If \mathcal{K} is an RCM quasivariety of R -modules, $M \in \mathcal{K}$, $S \leq M$ is a submodule, and \bar{S} is its \mathcal{K} -extension, then $m \in M$ can be shown to lie in \bar{S} using $\Delta(p, q)$ (in the way described in Theorem 3.2) iff $Am \subseteq S$.

In order to prove the theorem we must first describe how to encode a Δ -axiom as a left ideal.

The first step of the construction uses the fact that equations of the form $p = q$ can be rewritten as equations of the form $(p - q) = 0$. So take a Δ -axiom for R -modules, $\Delta(p, q)$, and rewrite its pairs as differences

$$\begin{aligned} D_j(x, y, \bar{u}, \bar{v}, \bar{z}) &:= p_j(x, y, \bar{u}, \bar{v}, \bar{z}) - q_j(x, y, \bar{u}, \bar{v}, \bar{z}) \\ &= a_j x + b_j y + \sum_i c_{ij} u_i + \sum_i d_{ij} v_i + \sum_i e_{ij} z_i, \end{aligned}$$

where $a_j, b_j, c_{ij}, d_{ij}, e_{ij} \in R$. Item (1) from the definition of a Δ -axiom now reads

$$(1)' \quad D_j(x, x, \bar{u}, \bar{u}, \bar{z}) = 0 = (a_j + b_j)x + \sum_i (c_{ij} + d_{ij})u_i + \sum_i e_{ij}z_i.$$

We will be working in the situation where \mathcal{K} is a quasivariety of modules and the variety it generates is all of $R\text{-Mod}$. For (1)' to hold in such a quasivariety the coefficients in the righthand expression must all be zero, i.e. $a_j + b_j = c_{ij} + d_{ij} = e_{ij} = 0$. Thus

$$D_j(x, y, \bar{u}, \bar{v}, \bar{z}) = a_j(x - y) + \sum_i c_{ij}(u_i - v_i)$$

(no dependence on the last block of variables). Introducing new variables X, U_i to represent $x - y, u_i - v_i$ we shall find that the module term operation

$$(3.1) \quad E_j(X, \bar{U}) = a_j X + \sum_i c_{ij} U_i$$

can be used to replace the pair (p_j, q_j) in the definition of ' Δ -axiom'. That is, $\Delta(p, q)$ can be rewritten in a reduced form in an obvious way using the terms $E_j(X, \bar{U})$.

The left ideal associated to $\Delta(p, q)$ is defined to be $A = (a_0, \dots, a_{n-1})$, the left ideal generated by the coefficients of X in the module terms $E_j(X, \bar{U})$, $j < n$.

Proof of Theorem 3.3. For part (1) of Theorem 3.3 consider a Δ -axiom $\Delta(p, q)$ and write it using the terms from (3.1). Condition (1)' of the definition of a Δ -axiom now reads

$$(1)'' \quad E_j(0, \bar{0}) = 0,$$

which must hold simply because E_j is a module term. Condition (2) of the definition of a Δ -axiom reads

$$\bigwedge_j (E_j(X, \bar{0}) = 0) \rightarrow (X = 0).$$

This is equivalent to $Ax = 0 \rightarrow x = 0$ for $x = X$. This establishes Theorem 3.3 (1).

Now we turn to Theorem 3.3 (2). Suppose that $M \in \mathcal{K}$, $S \leq M$ and $m \in \bar{S}$. Choose $\Delta(p, q)$ witnessing that $m \in \bar{S}$. With $\Delta(p, q)$ written in terms of the E_j 's we have that there exist a tuple \bar{s} with entries in S such that $E_j(m, \bar{s}) = a_j m + \sum_i c_{ij} s_{ij}$ belongs to S for all j . This means that $a_j m \in -\sum_i c_{ij} s_{ij} + S = S$ for all j , or $Am \subseteq S$ for A equal to the associated left ideal. Conversely, assume that $Am \subseteq S$. Then for $\bar{s} = \bar{0}$ we have that $E_j(m, \bar{0}) \in S$ for all j , so the form of $\Delta(p, q)$ that uses the terms $E_j(X, \bar{U})$ shows that $m \in \bar{S}$. This establishes Theorem 3.3 (2). \square

Now we state and prove the main theorem of this section.

Theorem 3.4. *Let \mathcal{K} be an RCM quasivariety such that the variety generated by \mathcal{K} is all of $R\text{-Mod}$. If \mathcal{F} is the set of left ideals of R that code Δ -axioms true in \mathcal{K} , then \mathcal{F} is a torsion notion for R -modules and \mathcal{K} is the quasivariety of \mathcal{F} -torsion-free R -modules.*

Proof. According to Theorem 3.1, \mathcal{K} is axiomatized relative to $R\text{-Mod}$ by a set of Δ -axioms together with a set of identities. Since the variety generated by \mathcal{K} is all of $R\text{-Mod}$, we will not use any identities other than those that hold in $R\text{-Mod}$. According to Theorem 3.3, the Δ -axioms true in \mathcal{K} are equivalent to a family of statements of the form $Ax = 0 \rightarrow x = 0$ where A is a finitely generated left ideal. Let \mathcal{F} be the set of finitely generated left ideals of R such that $Ax = 0 \rightarrow x = 0$ holds in \mathcal{K} . Since the subset of these left ideals that arise from Δ -axioms already serves to axiomatize \mathcal{K} relative to $R\text{-Mod}$, the full set also serves to axiomatize \mathcal{K} relative to $R\text{-Mod}$. It follows from this that, if we show that \mathcal{F} is a torsion notion, then \mathcal{K} must be the quasivariety of \mathcal{F} -torsion-free R -modules.

Item (1) from definition of a 'torsion notion' is the claim that \mathcal{F} is an order filter in the poset of finitely generated left ideals of R . That is, if A and B are finitely generated left ideals of R , $A \subseteq B$, and $Ax = 0 \rightarrow x = 0$ holds in \mathcal{K} , then $Bx = 0 \rightarrow x = 0$ also holds in \mathcal{K} . This is true because $A \subseteq B$ implies that $Bx = 0 \rightarrow Ax = 0$.

Item (5) is the next easiest to verify. Since the variety generated by \mathcal{K} is $R\text{-Mod}$, both \mathcal{K} and $R\text{-Mod}$ have the same free modules. Hence the 1-generated free module R belongs to \mathcal{K} . Hence R satisfies $Ax = 0 \rightarrow x = 0$ for each $A \in \mathcal{F}$, which is exactly what (5) claims.

Item (2) asserts \mathcal{F} is down directed. Choose $A, B \in \mathcal{F}$. We shall apply Theorem 3.3 to the situation $M := R \oplus R (\in \mathcal{K})$ and $S := A \oplus B \leq M$. Note that the pair $(1, 0) \in M$ belongs to the \mathcal{K} -extension of S , since $A \in \mathcal{F}$ and $A(1, 0) \subseteq A \oplus B$. Similarly $(0, 1)$ belongs to the \mathcal{K} -extension of S , since $B \in \mathcal{F}$ and $B(0, 1) \subseteq A \oplus B$. Since \overline{S} is a submodule, the element $(1, 0) + (0, 1) = (1, 1)$ must belong to the \mathcal{K} -extension of $A \oplus B$. Hence there must exist $C \in \mathcal{F}$ such that $C(1, 1) \subseteq A \oplus B$. Necessarily $C \subseteq A \cap B$.

Item (4) asserts that for all $A \in \mathcal{F}$ and $r \in R$ there is a $B \in \mathcal{F}$ such that $Br \subseteq A$. To prove this we again apply the second part of Theorem 3.3. Let $M = R \in \mathcal{K}$ and let $S = A$. The element $1 \in R (= M)$ belongs to the \mathcal{K} -extension of $A (= S)$, since $A \cdot 1 \subseteq S$. The \mathcal{K} -extension of S is a submodule, so for any $r \in R$ we have that $r \cdot 1 = r \in M$ also belongs to the \mathcal{K} extension of $S = A$. Theorem 3.3 guarantees the existence of $B \in \mathcal{F}$ such that $B \cdot r \subseteq A$, which is what item (4) requires.

Item (3) asserts that if $X, Y \subseteq R$ are finite subsets such that the left ideals (X) and (Y) belong to \mathcal{F} , then the left ideal (XY) belongs to \mathcal{F} . To prove this, assume that $X = \{a_0, \dots, a_{m-1}\}$ and $Y = \{b_0, \dots, b_{n-1}\}$. The fact that $(X), (Y) \in \mathcal{F}$ implies that \mathcal{K} satisfies the quasiidentities

$$\bigwedge_i (a_i x = 0) \rightarrow (x = 0) \quad \text{and} \quad \bigwedge_j (b_j x = 0) \rightarrow (x = 0).$$

But this means that \mathcal{K} satisfies

$$(3.2) \quad \bigwedge_j \left(\bigwedge_i (a_i (b_j x) = 0) \right) \rightarrow (x = 0).$$

For, if $\bigwedge_i (a_i (b_j x) = 0)$ holds for a fixed j , then the quasiidentity associated to X guarantees that $b_j x = 0$. But if this holds for all j , then the quasiidentity associated to Y guarantees that $x = 0$. Now (3.2) is just the quasiidentity associated to $XY = \{a_i b_j \mid i < m, j < n\}$. Since we have shown that it holds in \mathcal{K} we conclude that $(XY) \in \mathcal{F}$. \square

4. TORSION NOTION \implies RCM

In this section we prove that if \mathcal{F} is a torsion notion for R -modules, then the quasivariety of \mathcal{F} -torsion-free R -modules is RCM and the variety it generates is all of $R\text{-Mod}$. This is other direction of the proof of Theorem 2.3.

Lemma 4.1. *Assume that \mathcal{F} is a torsion notion for R -modules, and that \mathcal{K} is the quasivariety of \mathcal{F} -torsion-free R -modules. If $M \in \mathcal{K}$ and $S \leq M$ is a submodule of M , then the \mathcal{K} -extension of S is the set*

$$\overline{S} := \{m \in M \mid \exists A \in \mathcal{F} (Am \subseteq S)\}.$$

(In this lemma we are not assuming that \mathcal{K} is RCM, so we cannot refer to Theorem 3.3.)

Proof. The set \overline{S} defined in the statement contains S because S is a submodule. (One can take $A = R \in \mathcal{F}$ to prove any $m \in S$ belongs to \overline{S} .)

Let's prove that \overline{S} is closed under addition. If $x, y \in \overline{S}$, then there exist $A, B \in \mathcal{F}$ such that $Ax, By \subseteq S$. By the down directedness of \mathcal{F} there is a $C \subseteq A \cap B$ such that $C \in \mathcal{F}$. For this C we have

$$C(x + y) \subseteq Cx + Cy \subseteq Ax + By \subseteq S,$$

yielding $x + y \in \overline{S}$.

Now we argue that \overline{S} is closed under scalar multiplication. Assume that $x \in \overline{S}$ and $r \in R$. Since $x \in \overline{S}$ there is some $A \in \mathcal{F}$ such that $Ax \subseteq S$. By item (4) of Definition 2.2 there exists $B \in \mathcal{F}$ such that $Br \subseteq A$. Thus

$$B(rx) \subseteq Ax \subseteq S,$$

yielding $rx \in \overline{S}$.

Next we argue that \overline{S} is a \mathcal{K} -submodule of M . For this we must show that $M/\overline{S} \in \mathcal{K}$, or that M/\overline{S} is \mathcal{F} -torsion-free. This can be established by showing that if $A \in \mathcal{F}$, $m \in M$, and $Am \subseteq \overline{S}$, then $m \in \overline{S}$. Suppose that $A = (a_0, \dots, a_{m-1})$ as a left ideal. The statement $Am \subseteq \overline{S}$ now means $\{a_0m, \dots, a_{m-1}m\} \subseteq \overline{S}$. For each k there must exist $A_k \in \mathcal{F}$ such that $A_k(a_k m) \subseteq S$. By the down directedness of \mathcal{F} there is a $B \subseteq \cap A_k$, and this B has the property that $Ba_k m \subseteq S$ for all k . Suppose that $B = (b_0, \dots, b_{n-1})$ as a left ideal. By item (3) of Definition 2.2 the left ideal C generated by the set $\{b_j a_i \mid i < m, j < n\}$ belongs to \mathcal{F} . Cm is the submodule of M generated by all elements $b_j a_i m$, all of which belong to S . Thus $Cm \subseteq S$. This forces $m \in \overline{S}$, concluding the proof that M/\overline{S} is \mathcal{F} -torsion-free.

We have shown that \overline{S} is a \mathcal{K} -submodule extending S , but still must show that it is the least such. For this it suffices to observe that, from the definition of \overline{S} , if $m \in \overline{S}$, then for any submodule $T \leq M$ satisfying $S \leq T \leq \overline{S}$ we have that m/T is an \mathcal{F} -torsion element of M/T . \square

For the next theorem, which is the main result of the section, we need another fact from (Kearnes and McKenzie 1992). In Theorem 4.1 of that paper it is shown that a quasivariety is RCM if it satisfies the 'extension principle' and the 'relative shifting lemma'. The second of these properties will hold for any subquasivariety of an RCM quasivariety. Thus, since $R\text{-Mod}$ is RCM, any subquasivariety of $R\text{-Mod}$ satisfies the 'relative shifting lemma'. The 'extension principle' is not typically inherited by subquasivarieties.

The *extension principle* for a quasivariety \mathcal{K} of modules is the property that, for $M \in \mathcal{K}$, the function mapping a submodule $S \leq M$ to its \mathcal{K} -extension \overline{S} is a lattice homomorphism from the lattice of submodules of M to the lattice of \mathcal{K} -submodules of

M . In the presence of the ‘relative shifting lemma’, the extension principle is equivalent to the *weak extension principle*, which asserts that if $S \cap T = 0$ for submodules $S, T \leq M$, $M \in \mathcal{K}$, then $\overline{S} \cap \overline{T} = 0$. (The equivalence of the weak and full extension principles for quasivarieties satisfying the ‘relative shifting lemma’ is explained at the foot of page 482 of (Kearnes and McKenzie 1992).)

Altogether, this means that a subquasivariety of $R\text{-Mod}$ is RCM iff it satisfies the weak extension principle. We need this fact to prove the following theorem.

Theorem 4.2. *If \mathcal{F} is a torsion notion for R -modules, then the quasivariety \mathcal{K} of \mathcal{F} -torsion-free R -modules is RCM and the variety it generates is $R\text{-Mod}$.*

Proof. As discussed before the statement of the theorem, to prove that the quasivariety of \mathcal{F} -torsion-free modules is RCM it suffices to establish the weak extension principle. So choose an \mathcal{F} -torsion-free module $M \in \mathcal{K}$ and two submodules $S, T \leq M$ satisfying $S \cap T = 0$. Let’s prove that their \mathcal{K} -extensions \overline{S} and \overline{T} are disjoint.

Choose $m \in \overline{S} \cap \overline{T}$. By Lemma 4.1 there exist $A, B \in \mathcal{F}$ such that $Am \subseteq S$ and $Bm \subseteq T$. By the down directedness of \mathcal{F} there is a $C \subseteq A \cap B$ that belongs to \mathcal{F} , and for this C we have $Cm \subseteq Am \cap Bm \subseteq S \cap T = 0$, so m is an \mathcal{F} -torsion element. This forces $m = 0$, as desired. We conclude that the quasivariety of \mathcal{F} -torsion-free R -modules is RCM.

To show that the variety generated by the \mathcal{F} -torsion-free R -modules is all of $R\text{-Mod}$, it suffices to note that the 1-generated free R -module is \mathcal{F} -torsion-free. This is the content of item (5) of Definition 2.2. Thus $R \in \mathcal{K}$, so the variety generated by \mathcal{K} is $R\text{-Mod}$. \square

5. FINAL STATEMENT

Given a fixed ring R , we now know that a typical RCM quasivariety \mathcal{K} of R -modules can be described by a pair (I, \mathcal{F}) where I is an ideal – the annihilator of \mathcal{K} – and \mathcal{F} is a torsion notion for R/I . This information can be expressed entirely in terms of the left ideal structure of R by replacing \mathcal{F} with the set \mathcal{G} defined to consist of all $\nu^{-1}(A)$ for $A \in \mathcal{F}$ and $\nu: R \rightarrow R/I$ the natural map. This yields the following statement.

Theorem 5.1. *Let R be a ring. A quasivariety \mathcal{K} of R -modules is RCM iff there is a pair (I, \mathcal{G}) such that \mathcal{K} is the collection of R -modules satisfying $Ix = 0$ and $Ax = 0 \rightarrow x = 0$ for all $A \in \mathcal{G}$. Here we require that I be a two-sided ideal of R and \mathcal{G} be a family of left ideals of R , each containing I and finitely generated over I , such that items (1)–(4) of Definition 2.2 hold, along with*

(5)' (Regularity modulo I of elements of \mathcal{G}) *If $A \in \mathcal{G}$, $r \in R$ and $Ar \subseteq I$, then $r \in I$.*

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