THE DOLD-KAN CORRESPONDENCE

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ABSTRACT

The Dold-Kan correspondence refers to an equivalence between the category of simplicial objects and the category of connective chain complexes, both associated to some fixed abelian category $\mathcal{A}$. We begin this note by introducing these categories and then moving on to establish this correspondence. The significance of this is that it allows us to use the toolset of homological algebra in the study of simplicial homotopy theory. We conclude with an example of this.

CONTENTS

I. Introduction: The categories .................................................. 1
II. Main Theorem: Dold-Kan Correspondence .............................. 4
III. A quick application: Eilenberg MacLane spaces .................. 9

I. INTRODUCTION: THE CATEGORIES

In effort to be as self-contained as efficiently possible, we recall here the relevant definitions and categories. We begin by introducing the general setting for homological algebra, an abelian category.

Definition 1. An abelian category is a category $\mathcal{C}$ satisfying:
AB0) $\mathcal{C}$ has finite direct sums, that is, it has finite products and coproducts, and these coincide via a canonical isomorphism. We denote the direct sum of $A, B \in \mathcal{C}$ by $A \oplus B$.

AB1) $\mathcal{C}$ has kernels and cokernels. Explicitly, for any morphism $f : A \to B$ in $\mathcal{C}$ there are objects $\ker(f)$ and $\coker(f)$ together with morphisms $\iota : \ker(f) \to A$ and $\pi : B \to \coker(f)$ that are the universal morphisms such that $f \circ \iota = 0$ and $\pi \circ f = 0$.

AB2) Images and coimages coincide via a canonical isomorphism, where for a morphism $f : A \to B$ in $\mathcal{C}$, we define the image of $f$ to be $\ker(\coker(f))$ and the coimage of $f$ to be $\coker(\ker(f))$.

Some explanation is in order. We note that a consequence of axiom AB0 is the existence of a zero object, denoted 0, which is simply the product/coproduct of an empty collection of objects of $\mathcal{C}$. As such, its defining property is that for any object $A \in \mathcal{C}$, there is a unique morphism $A \to 0$ and a unique morphism $0 \to A$. A further consequence is that for any two objects $A, B \in \mathcal{C}$ there is a zero map $A \xrightarrow{0} B$ which is simply the composition $A \to 0 \to B$ of the unique maps. This is the map that is referred to in axiom AB1 above. In fact, it turns out that these conditions not only give us a zero object in every hom-set $\text{Hom}(A, B)$, but a full-fledged abelian group structure. This is important, but a bit of a diversion, so we simply state is as an theorem and refer the reader to [Wei95] for a reference.

Theorem 1. If $\mathcal{C}$ is an abelian category, then for any two objects $A, B \in \mathcal{C}$, the set $\text{Hom}(A, B)$ of morphisms $A \to B$ forms an abelian group with addition satisfying:

$$h \circ (g + g') \circ f = h \circ g \circ f + h \circ g' \circ f.$$ 

Here are some examples of abelian categories:
1. The category of abelian groups.
2. The category of vector spaces.
3. The category of $R$-modules for a ring $R$.
4. The category of presheaves valued in an abelian category.

The reader is welcome to replace the term abelian category with their preferred example throughout this note. Now we move on to chain complexes.

**Definition 2.** A chain complex, $(A, \partial)$ in an abelian category $\mathcal{A}$ is a sequence of morphisms

$$\cdots \to A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \to \cdots$$

such that $\partial_n \circ \partial_{n+1} = 0$, or more succinctly $\partial^2 = 0$.

We will mainly be interested in connective chain complexes which simply means that $A_k = 0$ for $k < 0$ (I should mention that some people understand 'connective' as the lighter restriction that the homology of the chain complex vanish in negative degrees). Now we want to form a category out of these objects, so we must introduce morphisms between them. Given two chain complexes $(A, \partial^A)$ and $(B, \partial^B)$, a chain map between them is a sequence of maps $f_n : A_n \to B_n$ such that for all $n$, the square below commutes.

$$
\begin{array}{ccc}
A_n & \xrightarrow{\partial_n} & A_{n-1} \\
\downarrow{f_n} & & \downarrow{f_{n-1}} \\
B_n & \xrightarrow{\partial_n^B} & B_{n-1}
\end{array}
$$

We denote by $\text{Ch}_{\geq 0}(\mathcal{A})$ the category of chain complexes in the abelian category $\mathcal{A}$ with morphisms given by chain maps. One can check that this indeed forms a category (exercise). A chain complex $(A, \partial)$ is called exact if $\text{im}(\partial_{n+1}) = \ker(\partial_n)$ for all $n$. Note that for any chain complex, $\partial^2 = 0$ implies $\text{im}(\partial_{n+1}) \subset \ker(\partial_n)$. Keeping in mind the category of abelian groups as an example, one could measure the failure of a chain complex to be exact at a particular $n$ by the group

$$H_n(A) = \ker(\partial_n)/\text{im}(\partial_{n+1}).$$

Indeed, a chain complex in the category of abelian groups is exact if and only if $H_n(A) = 0$ for all $n$. One can define these objects for chain complexes in an arbitrary abelian category as the cokernel of the inclusion $\text{im}(\partial_{n+1}) \to \ker(\partial_n)$.

**Definition 3.** For any chain complex $A.$ in an abelian category $\mathcal{A}$, the objects $H_n(A.)$ are called the homology of $A.$.

We now move on to simplicial objects. Let $\Delta$ denote the category whose objects are the finite ordered sets $\{0, 1, \ldots, n\}$ with morphisms given by non-decreasing set maps $[n] \to [m]$.

**Definition 4.** A simplicial object, $A$, in a category $\mathcal{A}$ is a presheaf $A : \Delta^{op} \to \mathcal{A}$.

Again, we want to form a category out of these objects, so we must specify morphisms between simplicial objects. Well, simplicial objects are just functors so naturally we take natural transformations as our morphisms. More concretely, one can view a simplicial object in $A$ as a sequence $(A_n)_{n=0}^\infty$ of objects of $\mathcal{A}$ with face operators $d_i : A_n \to A_{n-1}$, $i = 0, \ldots, n$ and degeneracy operators $\sigma_i : A_n \to A_{n+1}$, $i = 0, \ldots, n$ satisfying the relations:
\[
d_i d_j = d_{j-1} d_i \text{ if } i < j.
\]
\[
\sigma_i \sigma_j = \sigma_{j+1} \sigma_i \text{ if } i \leq j
\]
\[
\begin{cases}
\sigma_{j-1} d_i & i < j \\
1 & i = j, j+1 \\
\sigma_j d_{i-1} & i > j+1
\end{cases}
\]

Morphisms in this case are just sequences of morphisms \(f_n : A_n \to A'_n\) in \(\mathcal{A}\) which commute with face and degeneracy operators, that is, such that the following diagram commutes for all \(n, i, j\):

\[
\begin{array}{ccc}
A_{n+1} & \xleftarrow{\sigma_j} & A_n \\
\downarrow{f_{n+1}} & & \downarrow{f_n} \\
A'_{n+1} & \xleftarrow{\sigma'_j} & A'_n
\end{array}
\]

One last thing that we will need in order to prove the main theorem is the following characterization of morphisms in \(\Delta\). For a proof, see [Wei95].

**Proposition 1.** For any morphism \(\alpha : [m] \to [n]\) in \(\Delta\), there is a unique epi-monic factorization \(\alpha = \varepsilon \circ \eta\), where \(\varepsilon\) is uniquely a composition of face maps \(d^i : [k-1] \to [k]\), and \(\eta\) is uniquely a composition of degeneracy maps \(\sigma^i : [k+1] \to [k]\).

We let \(S\mathcal{A}\) denote the category of simplicial objects in \(\mathcal{A}\) and use the two descriptions given above interchangeably. Before moving on to the main theorem, we motivate our consideration of simplicial objects by stating a few interesting results relating these objects to homotopy theory in topology. Given a topological space \(X\), it is a topologist’s raison d’être to construct invariants \(H(X)\) for which \(H(X) = H(Y)\) if \(X\) and \(Y\) are homotopy equivalent. This is used as a means of distinguishing \(X\) from \(Y\). One of the first such invariants one encounters are the homology groups of \(X\) which can be defined as follows. Let \(\Delta^n = \{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} t_i = 1 \}\). Define the group of singular \(n\)-chains of \(X\) to be

\[
S_n(X) = \mathbb{Z}\{ \sigma : \Delta^n \to X \},
\]

the free abelian group on the set of continuous maps \(\Delta^n \to X\). One can prescribe maps

\[
d^i : S_n(X) \to S_{n-1}(X) \quad \text{and} \quad \sigma^i : S_n(X) \to S_{n+1}(X),
\]

which are roughly given by adding the \(i\)-th and \((i + 1)\)-th coordinates, and dropping the \(i\)-th coordinate. One can check that this data satisfies the conditions to be a simplicial abelian group. One then defines the homology groups of \(X\), \(H_n(X)\), to be the homology groups of the associated chain complex of the simplicial abelian group \(S_n(X)\) (this is defined at the beginning of section II). These are nice invariants because of their computability. The down-side, however, is that they tend to miss a lot of information one might be interested in. They are too coarse of an invariant. One can do better by instead looking at the homotopy groups of \(X\),

\[
\pi_n(X) = [S^n, X],
\]

where the right hand side denotes the set of homotopy classes of maps \(S^n \to X\) (here \(S^n\) denotes the \(n\)-sphere, as usual). These are in general much more difficult to compute, however. How do these fit into the picture? We went from a topological space to a simplicial abelian group by constructing
the singular chains. This assignment is in fact functorial, and has a left adjoint called geometric realization

\[ \_|_ : \text{SAb} \to \text{Top}. \]

So from any simplicial abelian group \( A : \Delta^{op} \to \text{Ab} \), I can get a topological space \(|A|\). This raises the question: is there an analogue of \( \pi_n(|A|) \) in the simplicial setting? The answer to this is yes. Given \( A \), we let

\[ Z_n(A) = \bigcap_{i=0}^{n} \ker(d_i) = \{ a \in A_n | d_i(a) = 0, \forall i = 0, \ldots, n \}, \]

and define the simplicial homotopy groups of \( A \) to be

\[ \pi_n(A) = Z_n(A)/\sim, \]

where

\[ a \sim a' \iff \exists b \in A_{n+1} \text{ such that } d_i(b) = \begin{cases} 0 & i < n \\ a & i = n \\ a' & i = n + 1 \end{cases}. \]

The equivalence relation is not too bad to digest if you’re familiar with homotopy of maps, but the main point is that

\[ \pi_n(A) \cong \pi_n(|A|). \]

It follows that we can compute \( \pi_n(|A|) \) purely in terms of the combinatorics of the simplicial set \( A \). The main theorem takes this even further and says that we can compute \( \pi_n(A) \) as the homology of a chain complex. There is a lot more to this wonderful story, but this is all we will need for this note. The intrigued reader is referred to [Jar09].

### II. Main Theorem: Dold-Kan Correspondence

The Dold-Kan correspondence is an equivalence of categories between \( \text{Ch}_{\geq 0}(\mathcal{A}) \) and \( \text{S}\mathcal{A} \). The really nice thing about this equivalence is that simplicial homotopy groups of a simplicial object correspond to the homology of the corresponding chain complex. That these categories are equivalent is not extremely surprising. Afterall, given a simplicial object \( (|A_n|, |d_i|, |\sigma_j|) \) in \( \mathcal{A} \), one can easily create a connective chain complex \( C(A) \) in \( \mathcal{A} \) whose objects are

\[ C(A)_n = \begin{cases} A_n, & n \geq 0 \\ 0, & n < 0 \end{cases}, \]

and differential given by

\[ \partial_n = \sum_{i=0}^{n} (-1)^i d_i, \quad n \geq 1, \]

and \( \partial_0 = 0 \). The differential is maybe less obvious, but one can check, using the simplicial identities, that this indeed forms a chain complex. This is called the associated chain complex of the simplicial object \( A_n \). However, it turns out that this chain complex does not give us the equivalence that we want. The problem is that it contains too much information, in particular, it contains information about the degeneracy maps \( \sigma_i : A_n \to A_{n+1} \). Since chain complexes do not contain any data that looks like maps which increase degree by one, we should expect that this information is not captured in
our equivalence map. The correct equivalence map turns out to be the normalized chain complex of 
\( \{A_n\}, N(A) \) given by
\[
N(A)_n = \bigcap_{i=1}^{n-1} \ker(d_i) \subset A_n, \quad \partial_n = (-1)^n d_n, \quad n \geq 1
\]
and \( N(A)_0 = 0, \partial_0 = 0 \). We note that this is in fact a subcomplex of the former complex. By this we
mean that \( N(A)_n \subset C(A)_n \) for all \( n \), and the differential of the normalized chain complex is simply
the restriction of the differential for the associated chain complex.

**Remark.** Here we are thinking of \( N(A)_n \) and \( C(A)_n \) as set-like objects with elements for simplicity
and exposition, but there is always a way to translate this into arbitrary abelian category theory
language. The key result here, if one is interested, is the Freyd-Mitchell Embedding Theorem.

The definition of the normalized chain complex seems arbitrary at first look. Rather than taking this
for granted, we briefly motivate this definition.

Let \( G \) be a simplicial abelian group, that is, a simplicial object in the category of abelian groups. In
section I we defined the homotopy groups of \( G \) to be a quotient of the set
\[
Z_n(G) = \{ g \in G_n | d_i(g) = 0, \forall i = 0, \ldots, n \}.
\]
Note the similarity to the \( n \)-th object in the normalized chain complex:
\[
N(G)_n = \bigcap_{i=1}^{n-1} \ker(d_i) = \{ g \in G | d_i(g) = 0, \forall i = 0, \ldots, n - 1 \}.
\]
Its not too hard to see that we have
\[
Z_n(G) = \ker(d_n : N(G)_n \to N(G)_{n-1}) = \ker((-1)^n d_n : N(G)_n \to N(G)_{n-1}) = \ker(\partial_n).
\]
The homology of \( N(G) \) at \( n \) is also the quotient of this set. As mentioned at the end of section I,
we want simplicial homotopy groups to correspond to homology under the desired equivalence. This
hints that we are on track to acheive this! We are now ready to state the main theorem:

**Theorem 2.** Dold-Kan Theorem For any abelian category \( \mathcal{A} \), the normalized chain complex functor
\( N \) is an equivalence of categories between the category \( S\mathcal{A} \) of simplical objects in \( \mathcal{A} \) and the category
\( Ch_{\geq 0}(\mathcal{A}) \) of connective chain complexes in \( \mathcal{A} \). Furthermore, under this correspondence, simplicial
homotopy corresponds to homology.

**Remark 1.** We will only prove the equivalence statment. The proof that simplicial homotopy corre-
sponds to homology follows rather easily from a continuation of the discussion preceding the theorem
statment, but there is a subtlety here that one should take note of: while the relation used to define
\( \pi_n(G) \) works for any group \( G \), in general this is not an equivalence relation if we replace \( G \) by an
arbitrary simplicial set. The simplicial sets for which this does work are called *fibrant*, and one usually
takes \( H_n(N(G)) \) as the definition of \( \pi_n(G) \) for \( G \) a non-fibrant simplicial set.

We will follow the proof in [Rak13]. The proof of this theorem will actually require us to go
back and examine more closely the reason why the associated chain complex does not give us an
equivalence.

**Definition 5.** For \( A \in Ob(S\mathcal{A}) \), we define the degenerate subcomplex of \( C(A) \) by
\[
D(A)_n = \sum_{i=0}^{n-1} \sigma_i(C_{n-1}(A)) \subset C_n(A), \quad n \geq 1,
\]
and \( D_0(A) = 0 \).
That is, the degenerate subcomplex is simply the subcomplex of things generated by degenerate objects. The following result will be key in our proof of the main theorem. For a proof, see [Wei95]

**Lemma 1.** The associated chain complex splits as the direct sum of the normalized chain complex and the degenerate subcomplex:

$$C(A) = N(A) \oplus D(A).$$

Now we are ready to prove the main theorem. We start by constructing the inverse equivalence. Let $C$ be a connective chain complex in $\mathcal{A}$. Define $\Gamma : Ch_{\geq 0}(\mathcal{A}) \to S\mathcal{A}$ by

$$C \mapsto \left( [n] \mapsto \Gamma(C)_n = \bigoplus_{\eta : [n] \to [k]} C_k[\eta] \right),$$

where the direct sum is taken over all surjections $\eta : [n] \to [k]$ in $\Delta$, and $C_k[\eta]$ denotes a copy of $C_k$ indexed by $\eta$. We note that there are only finitely many surjections for a given $n$, so that the direct sum notation is indeed appropriate. For this to be a simplicial object, we must define $\Gamma(C)(\alpha)_n : \Gamma(C)_n \to \Gamma(C)_m$

for any morphism $\alpha : [m] \to [n]$ in $\Delta$. By the universal property of direct sums, it suffices to define for each pair $(k, \eta : [n] \to [k])$, a map

$$C_k[\eta] \to \Gamma(C)_m.$$

We define this as follows. First, factor $\eta \circ \alpha$ into its epi-monic factorization

$$[m] \xrightarrow{\eta'} [j] \xrightarrow{\epsilon} [k],$$

for some $j$. We define

$$C_k[\eta] \to C_j[\eta']$$

by

$$\begin{cases} id_{C_k}, & \text{if } k = j \\ (-1)^k \partial & \text{if } j = k - 1, \epsilon = d_k \\ 0 & \text{else.} \end{cases}$$

Composing with the inclusion $C_j[\eta'] \hookrightarrow \Gamma(C)_m$ gives us the desired map. To check that this is indeed a simplicial object in $\mathcal{A}$ one must check that composition is respected. That is, if $\mu : [l] \to [m]$ and $\alpha : [m] \to [n]$ are morphisms, we must check that

$$\Gamma(C)(\alpha \circ \mu) = \Gamma(C)(\mu) \circ \Gamma(C)(\alpha).$$

This follows from commutativity of the diagram below:

$$\begin{array}{ccc} l & \xrightarrow{\mu} & m & \xrightarrow{\alpha} & n \\ \downarrow{\eta''} & & \downarrow{\eta'} & & \downarrow{\eta} \\ i & \xrightarrow{\theta} & j & \xrightarrow{\epsilon} & k \end{array}$$

Note that commutativity of the diagram implies that $(\epsilon \theta)\eta''$ is the unique epi-monic factorization of $\alpha \mu$.

Now we construct a natural isomorphism

$$\Psi : N \circ \Gamma \Rightarrow id_{Ch_{\geq 0}(\mathcal{A})}.$$
This is actually not very difficult, all we have to do is take the inclusion

\[ \Psi_n(C) : N(\Gamma(C))_n \hookrightarrow C(\Gamma(C))_n = \Gamma(C)_n = \bigoplus_{\eta : [n] \rightarrow [k]} C_k[\eta]. \]

However, this needs to be a map \( N(\Gamma(C))_n \rightarrow C_n \).

**Proposition 2.** The image of \( \Psi_n(C) \) is \( C_n[id_n] = C_n \).

**Proof.** For any surjection \( \eta : [n] \twoheadrightarrow [k] \) with \( k < n \), there is a factorization of \( \eta \) as \( [n] \overset{\sigma}{\twoheadrightarrow} [n-1] \twoheadrightarrow [k] \).

This implies that \( C_k[\eta] \subset D_n(\Gamma(C)) \), so by Lemma 1, this summand is not in the image of \( \Psi_n(C) \). It follows that the only summands in this image must correspond to \( k = n \) and the only surjection \( [n] \twoheadrightarrow [n] \) in \( \Delta \) is the identity. \( \square \)

So \( \Psi_n(C) \) is an isomorphism from \( N(\Gamma(C))_n \) to \( C_n \) for each \( n \), implying that it is indeed a natural isomorphism as desired. Now we construct a natural isomorphism

\[ \Phi : \Gamma \circ N \Rightarrow \text{id}_{S_\Delta^A} \]
as follows. For any simplicial object \( (A_k) \) in \( \mathcal{A} \) and each \( n \), we define

\[ \Phi_n(A) : N(A)_n = \bigoplus_{\eta : [n] \rightarrow [k]} N(A)_k[\eta] \rightarrow A_n \]

by letting its restriction to the \((k, \eta)\)-th factor be the composition

\[ N(A)_k[\eta] \rightarrow A_k \overset{A(\eta)}{\rightarrow} A_n. \]

This definition is a bit loaded, and there are several things to check:

1. \( \{\Phi_n(A)\} \) is a morphism of simplicial objects in \( \mathcal{A} \).

**Proof.** Let \( \alpha : [n] \rightarrow [m] \) be a morphism in \( \Delta \). For any surjection \( \eta : [m] \rightarrow [k] \), let \( \eta \alpha = \varepsilon \eta' \) be the epi-monic factorization. Suppose first that \( \varepsilon \neq d^k \). Then we have the following diagram in which the two outer squares commute:

\[
\begin{array}{ccc}
N(A)_k[\eta] & \xrightarrow{\alpha^*} & \Gamma(N(A))_m \\
\downarrow & & \downarrow \\
A_k & \overset{\Phi_n(A)}{\rightarrow} & A_m \\
\end{array}
\]

\[
\begin{array}{ccc}
\Gamma(N(A))_n & \xrightarrow{\alpha^*} & \Gamma(N(A))_n \\
\downarrow & & \downarrow \\
A_n & \overset{\Phi_n(A)}{\rightarrow} & A_n \\
\end{array}
\]

\[
\begin{array}{ccc}
N(A)_k[\varepsilon] & \xrightarrow{\alpha^*} & N(A)_k[\varepsilon'] \\
\downarrow & & \downarrow \\
A_k & \overset{\Phi_n(A)}{\rightarrow} & A_k \\
\end{array}
\]

We see that since \( A \) is functorial, \( \alpha^* A(\eta) = A(\eta \alpha) \) so that, starting at the top left corner in the diagram and taking the top path towards \( A_n \) is the same as taking the lower path. So it appears that we have a morphism of simplicial complexes so long as we restrict to the summands for which \( \varepsilon \neq d^k \). If we can establish the same result for those of which \( \varepsilon = d^k \), we are done. In this case, there are really two things to check since there are two nonzero maps out of \( N(A)_k[\eta] \), the identity

\[ N(A)_k[\eta] \rightarrow N(A)_k[\eta'] \]
and the map
\[ N(A)_k[\eta] \xrightarrow{(-1)^k \delta} N(A)_{k-1}[\eta']. \]

In the latter map, \( \delta \) denotes the differential for the chain complex \( N(A) \), which we recall is just \((-1)^k d_k\). It follows that in the diagram below in which the two outer squares commute, we again have that taking the upper path from \( N(A)_k[\eta] \) to \( A_n \) is the same as taking the lower path. In other words, \((-1)^k (-1)^k d_k = d^k\).

\[
\begin{array}{ccc}
N(A)_k[\eta] & \xrightarrow{\alpha^*} & \Gamma(N(A)_m) \\
\downarrow & & \downarrow \Phi_n(A) \\
A_k & \xrightarrow{\alpha^*} & A_m \\
\downarrow A(\eta) & & \downarrow A(\eta a) \\
A_k & \xleftarrow{\Phi_n(A)} & A_{k-1}
\end{array}
\]

2. \( \Phi \) is natural in \( A \).

**Proof.** Let \( \psi : A \to B \) be a natural transformation of simplicial objects in \( \mathcal{A} \). Naturality simply follows from the fact that, for all \( n, k \) and surjections \( \eta : [n] \to [k] \), the upper path going from the top left to \( B \) is the same as the lower path in the diagram below.

\[
\begin{array}{ccc}
N(A)_k[\eta] & \xrightarrow{\psi^*} & \Gamma(N(B))_n \\
\downarrow & & \downarrow \Phi_n(A) \\
A_k & \xrightarrow{\psi^*} & A_n \\
\downarrow A(\eta) & & \downarrow A(\eta a) \\
A_k & \xleftarrow{\Phi_n(A)} & A_{k-1}
\end{array}
\]

3. \( \{\Phi_n(A)\} \) is an isomorphism for all simplicial objects, \( A \), in \( \mathcal{A} \).

**Proof.** If \( n = 0 \), then the only surjection \([0] \to [k]\) is the identity \([0] \to [0]\). Thus

\[ \Gamma(N(A))_0 = N(A)_0 = A_0 \]

and the map \( \Phi_0(A) \) is just the identity. We proceed by induction to prove that \( \Phi_n(A) \) is an isomorphism for all \( n \). Here, we again take an element-theoretic approach. For a proof that is free of reference to elements, which by contrast requires a lot more homological algebra than introduced here, see [Wei95]. First we prove that this map is surjective for all \( n \). By Lemma 1, \( A_n = N(A)_n \oplus D(A)_n \). As in the \( n = 0 \) case, the factor \( N(A)_n[\text{id}_{[n]}] \) maps identically onto itself under \( \Phi_n(A) \) so that \( N_n(A) \subset \text{im}(\Phi_n(A)) \). Now take \( y \in D_n(A) \). By definition, there is a \( y' \) and \( i \) such that \( \sigma_i(y') = y \). By induction, there is \( x \) such that

\[ \Phi_{n-1}(A)(x) = y' \implies \sigma_i \Phi_{n-1}(A)(x) = \sigma_i(y') = y \implies \Phi_n(A)(\sigma_i(x)) = y. \]

This completes the proof of surjectivity. To show that \( \Phi_n(A) \) is injective for each \( n \), we define an ordering on the set of surjections \([n] \to [k]\) for fixed \( 0 \leq k \leq n \). For each such surjection, \( \eta \), we define a section of \( \eta \)

\[ s_\eta : [k] \hookrightarrow [n] \]
by
\[ s_\eta(i) = \max\{j \in [n] | \eta(j) = i\}. \]

Then we write \( \eta \leq \eta' \) if and only if
\[ s_\eta(i) \leq s_{\eta'}(i), \quad \forall i \in [k]. \]

Now with \( 0 \leq k \leq n \) fixed, take \( (x_\eta) \in \ker \Phi_n(A) \). Suppose, for a contradiction, that \( \hat{\eta} : [n] \to [k] \) is the maximal surjection under this ordering such that \( x_\hat{\eta} \neq 0 \). Then
\[ \pi_{id[k]} \circ \Gamma(\hat{\eta})(x_\eta) = x_\hat{\eta}, \]
and \( \Gamma(\hat{\eta})(x_\eta) \in \ker \Phi_k(A) \), for all surjections \( \eta : [n] \to [k] \). By our inductive hypothesis, \( \ker \Phi_k(A) = 0 \) so that \( x_\hat{\eta} = 0 \). Thus \( x_\eta = 0 \) for all \( \eta \neq id_{[n]} \), but we know that on the factor indexed by \( id_{[n]} \), \( \Phi_n(A) \) is the inclusion \( N(A)_n \rightarrow A_n \) so that \( x_{id[n]} = 0 \).

This completes the proof of the main theorem.

III. A quick application: Eilenberg MacLane spaces

Let \( G \) be abelian group. For any non-negative integer \( n \), we ask: Is there a space, which we will denote by \( K(G,n) \), such that
\[ \pi_k(K(G,n)) = \begin{cases} G, & k = n \\ 0, & k \neq n \end{cases}. \]

The Dold-Kan Correspondence gives a beautifully simple constructive answer to this question in the affirmative. To construct such a space, start with a chain complex \( G[n] \) with
\[ G[n]_k = \begin{cases} G, & k = n \\ 0, & k \neq n \end{cases}. \]

Then
\[ H_k(G[n]) = \begin{cases} G, & k = n \\ 0, & k \neq n \end{cases} \Rightarrow \pi_k(\Gamma(G[n])) = \begin{cases} G, & k = n \\ 0, & k \neq n \end{cases} \Rightarrow |\Gamma(G[n])| \simeq K(G,n). \]

The reason topologists care about these spaces is that they represent the cohomology of a space \( X \):
\[ H^n(X) = [X,K(G,n)]. \]

Given their importance, these spaces deserve a name: Eilenberg-MacLane spaces. We note that, as a space, \( K(G,n) \) is only well defined up to homotopy equivalence. In this case, one speaks of 'models' for \( K(G,n) \) as explicit examples of spaces which 'realize' this homotopy type. It is well-known that the circle, \( S^1 \), is a model for the Eilenberg-MacLane space \( K(Z,1) \). A fun exercise is to start with the chain complex
\[ \cdots \rightarrow 0 \rightarrow 0 \rightarrow Z \rightarrow 0 \rightarrow \cdots \]
with \( Z \) in degree 1 and go through the two functors \( \Gamma \) and \( |\_| \) to see if one can identify the result with \( S^1 \). Of course, we did not actually construct an explicit geometric realization functor here. The intrigued reader is, once again, refered to [Jar09].
REFERENCES

