1. Find a version of Bézout's theorem in $\mathbf{P}^1 \times \mathbf{P}^1$. (Hint: to predict the formula, convince yourself first that any curve can be deformed into a union of vertical and horizontal lines, in analogy to the way we predicted Bézout's theorem in $\mathbf{P}^2$.) Once you have a conjecture, can you prove it?

2. (a) You can find a line through any two points. How many points can you interpolate with a conic?

   (b) Is the conic unique? What happens to the conic if 3 points lie on a line? What if 4 points lie on a line?

   (c) Suppose that F is a regular projective quartic curve. Show that $F^3 \to \mathrm{Pic}^3(F)$ is surjective. (Hints: Show that, for all $R \in F$, if $D = P_1 + P_2 + P_3 + P_4$ is an effective divisor of degree 4 then $D \equiv E + R$ for some effective divisor of degree 3. Start by considering a conic through D. How many other points of F will it meet?)

   (d) Is the conic in the last part always unique? What does this tell you about the injectivity of $\mathrm{Eff}^3(F) \to \mathrm{Pic}^3(F)$.

   (e) Deduce that, if we fix a basepoint P_0 of F, every divisor in $\mathrm{Pic}^0(F)$ can be represented (not necessarily uniquely) as $D - 3P_0$ where D is an effective divisor of degree 3. Devise a procedure to compute $(D - 3P_0) + (E - 3P_0)$ when D and E are effective divisors of degree 3 on F.

3. Suppose that F is a regular algebraic plane curve. Show that every valuation of $K(F)$ is μ_P for some point P.

4. One approach to showing that elliptic curves can't be parameterized algebraically:

   (a) Note that if F is an elliptic curve and $\mathcal{O}_{F,p} \simeq K[t]_{(t)}$ then $K(F) \simeq K(t)$.

   (b) Show that every automorphism of $K(t)$ is a Möbius transformation.

   (c) Show that a Möbius transformation fixes at most 2 valuations of $K(t)$.

   (d) Show that, for F in Weierstraßform $y^2 z = x^3 + axz^2 + bz^3$, the map $(x, y) \mapsto (x, -y)$ fixes 4 points of F.

5. Another approach to showing that elliptic curves can't be parameterized:

   (a) Show that $\mathrm{Pic}(F)$ can be computed from $K(F)$ (without knowledge of F). (Hint: points are the same as valuations.)

   (b) Conclude that the fields of rational functions on curves with distinct Picard groups cannot be isomorphic.

6. And another (over $\mathbf{C}$): use Riemann–Hurwitz to show that there is not even a non-constant map from a projective genus 0 curve to a genus 1 curve. (Can you extend this to show that there is not any homomorphism

of K-algebras from $K(F)$ to $K(t)$ when F is a plane curve of genus 1?) (These arguments acutally work over other fields as well, once we know how to prove Riemann–Hurwitz over them.)