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Part I

Algebraic manifolds

0 Before the beginning

0.1 Recommended background

Reading 0.1. [AM69, Chapters 1–3, 7], [Vak14, Sections 1.1–1.4]

Familiarity with commutative algebra at the level of [AM69, Chapters 1–3, 7], as well as basic point set topology, are essential. While category theory is not strictly necessary, some familiarity is strongly recommended, at the level of [Vak14, Sections 1.1–1.3]. At a minimum, you should be comfortable with universal properties. I will try to recall basic facts about limits and colimits as we need them, but if you haven't already encountered limits and colimits, you may want to study [Vak14, Section 1.4] too.

Another reference for category theory is [Gro66]. This is a thorough, but relatively brief survey of the key concepts used in algebraic geometry, including the existence of sufficiently many injectives in Grothendieck abelian categories, and the theory of representable functors.

Acquaintance with differential geometry or complex analysis (especially Riemann surfaces) may be helpful, but is not essential.

0.2 References and how to use these notes

Each section of these notes is meant to correspond to one lecture, but these notes are not meant to be a complete reference for the course. Their main purpose is to help me organize the topics we will cover and to summarize what I want to say in lecture. You will need to consult other sources. In most cases, I have given a list of other references at the beginning of each section and in the table of contents.

Sometimes the references will cover more material than we do in lecture. It's always a good idea to look at this other material, but you may encounter some concepts we haven't defined in class. As a rule of thumb, you can skip parts of the reading that aren't mentioned in these notes.

I will draw a lot of the course material from Vakil's *Foundations of Algebraic Geometry* [Vak14]. This book is excellent, and if we had more time I might have attempted to follow it linearly. As it is, we are going to jump around quite a lot, which is why I am using these notes to try to keep things organized.

In many places, the presentation in the notes won't be quite the same as the presentation in Vakil's book. One of the major differences is that I am going to spend more time on the functor of points. I'm going to trust you to keep the different approaches straight, but please let me know if things get muddled.

You might want to consult some other texts in case you find their presentation more compelling. Here are some suggestions:

(i) Hartshorne's *Algebraic Geometry* [Har77] is the classic reference. It is a bit terse, and a majority of the content is in the exercises.

(ii) Mumford's *Red Book of Varieties and Schemes* [Mum99] is a very good place to look for intuition. It is less complete than other references.

(iii) Mumford has also written two textbooks [Mum76] and [MO], the latter of which was typeset and updated by Oda. The second volume is close to the spirit of this course, and may be the best auxiliary reference.

(iv) The *Stacks Project* [Sta15] is a definitive reference for an increasingly complete list of topics in algebraic geometry. Completeness and generality are often prioritized over readability, so the Stacks Project works well as a reference for specific results but less well as a textbook.

0.3 Goals for this course

The main goal for this course is to give you, the student, enough background to read a paper or advanced text in algebraic geometry, or to follow an algebraic geometry seminar. Secondarily, I hope to introduce you to enough algebraic geometry to participate in a summer research project, if you are interested.

The course is structured around a few theorems that I hope will provide motivation for the subject, which may otherwise be kind of technical.

0.4 Exercises and grading

You need to do exercises to learn algebraic geometry. You need to do a whole lot of exercises, many more than I could possibly grade.

0.5 Acknowledgements

Thanks to all of the students who discovered and corrected errors in this text. Specific acknowledgements appear with each correction. Thanks also to Shawn Burkett for fixing a frustrating error in my LaTeXcode.

Chapter 1

Introduction to algebraic geometry

1 Bézout's theorem

q:bezout

Question 1.1. How many points do two algebraic plane curves have in common?

The answer to this question is Bézout's theorem. We will discuss several formulations of this theorem and a sketch of the proof. Our first goal in the course will be to make these statements, and the proof outlined below, precise.

By an (affine) algebraic plane curve, we will mean the set of solutions to a polynomial $f(x, y)$ in two variables. We can assume the coefficients of f are real numbers and that we are looking for solutions in $\mathbf{R}^2$, although in a moment we will want to look for solutions in $\mathbf{C}^2$ (and at that point we might as well allow coefficients in $\mathbf{C}$ as well).

The first example of an algebraic plane curve is a line. A line is given by a polynomial $ax + by + c$ where a and b are not both zero. In other words a line is given by a polynomial f of degree 1. (Degree of a polynomial in x and y is measured by giving both x and y degree 1.)

Exercise 1.2. Show that any line can be parameterized algebraically as $(\xi(t), \eta(t))$.

Exercise 1.3. More generally, suppose that A is a commutative ring containing elements a, b, and c such that a and b generate A as an ideal. For any A-algebra B, let $X(B)$ be the set of pairs $(x, y) \in B^2$ such that $ax + by + c = 0$. Prove that there are polynomials $\xi, \eta \in A[t]$ such $(x, y) \in X(B)$ if and only if there is a $t \in B$ such that $(x, y) = (\xi(t), \eta(t))$.

Solution. Consider the following sequence:

$$0 \to B \xrightarrow{\binom{-b}{a}} B^2 \xrightarrow{(a\ b)} B \to 0$$

We argue it is exact.

The map $B^2 \to B$ is surjective since there are x and y in A (hence in B) such that $ax + by = 1$. The kernel certainly contains things of the form $(-bt, at)$. If (u, v) is in the kernel then $au + bv = 0$. Choose x and y such that $ax + by = 1$. Then

$$u = axu + byu = -bxv + byu = -b(xv - yu)$$
$$v = axv + byv = axv - ayu = a(xv - yu)$$

which implies that (u, v) has the form $(-bt, at)$ where $t = xv - yu$. Finally if $(-bt, at) = (0, 0)$ then $(ax + by)t = 0$ so that $t = 0$. Therefore the sequence is exact.

Now pick $(x_0, y_0) \in A^2$ such that $ax_0 + by_0 = -c$, which is possible since a and b generate A as an ideal. Then take $\xi(t) = x_0 - bt$ and $\eta(t) = y_0 + at$. □

If f and g define plane curves C and D, then $C \cap D$ is the set of points (x, y) such that $f(x, y) = g(x, y) = 0$.

We can try some examples. If C and D are both lines then $C \cap D$ almost always consists of exactly one point. If the lines are parallel then we get usually get no solutions, but if the lines are the same, we get infinitely many solutions.

If $\deg C = 1$ and $\deg D = d$ then parameterize C by $(x(t), y(t))$. The intersection points correspond to the values of t such that $g(x(t), y(t)) = 0$. This is a degree d polynomial in t, so we expect d solutions—at least if we look in $\mathbf{C}$.

However, we don't always get d solutions, even when $d = 2$:

(i) Suppose $g(x, y) = x^2 + y^2 - 1$ and $y(t) = 1$ and $x(t) = t$. Then $g(x(t), y(t)) = t^2$ has just one solution (in any field).

(ii) Suppose $g(x, y) = xy$ and $x(t) = t$ and $y(t) = 0$. Then there are infinitely many solutions (any value of t).

(iii) Suppose that $g(x, y) = xy - 1$ and $x(t) = t$ and $y(t) = 1$. Then $g(x(t), y(t)) = 1$ there are no solutions at all.

What is going on geometrically? In the first case, we have a tangency. But suppose we move the line a little. Take $x(t) = t$ and $y(t) = s$. For different values of s we get different lines, and as long as s is near to but not equal to 1, we get two points of intersection. Thus we expect that *most* curves C and D (whatever that means), won't have a tangency, and this phenomenon won't occur.

In the second example, the line is a *component* of the curve D and we get infinitely many intersections. Again, we can try moving the line. If we try something like $x(t) = t$ and $y(t) = s(t + 1)$ then for $s \neq 0$ but near 0, we have two intersection points. Once again, we will be able to say that most curves C and D, don't share a component, so this phenomenon also does not usually occur. (Technically, this example is a special case of the previous one: It is a tangency of infinite order.)

In the last example, there is only one point of intersection when we expect two. Geometrically, we can see that C is parallel to an asymptote of D. If we deform C a little, say by taking $x(t) = t$ and $y(t) = 1 + st$ then as long as s is close but not equal to zero we get two solutions. As $s \to 0$, one of the solutions escapes to infinity. Once again, we can say that most lines intersect D in two points.

In view of these observations, *we exclude pairs of curves C and D that are tangent, share components, or share asymptotes* in Question 1.1.

More subtly, we have seen that making small changes to our curves C and D does not change the number of points of intersection between them, as long as the small changes do not introduce tangencies, common components, or common asymptotes. That is, *if C_t and D_t are one-parameter families of curves such that for no value of t do C_t and D_t have a tangency, common component, or common asymptote, then $|C_t \cap D_t|$ is constant.*

The final ingredient in a proof of Bézout's theorem will be to observe that *for any curves C and D (satisfying our assumptions) there is a 1-parameter family C_t and D_t (also satisfying our assumptions for each value of t) with $C_0 = C$ and $D_0 = D$ such that C_1 consists of* $\deg C$ *lines and D_1 consists of* $\deg D_1$ *lines.*

We can compute very easily that $|C_1 \cap D_1| = \deg(C)\deg(D)$. Putting all of these observations together, we have

$$|C \cap D| = |C_1 \cap D_1| = \deg(C)\deg(D).$$

This proves Bézout's theorem:

Theorem 1.4 (Bézout's theorem in the affine plane)**.** *For most algebraic plane curves C and D we have $|C \cap D| = \deg(C)\deg(D)$.*

'Most curves' may be interpreted to include curves that are not tangent, do not have parallel asymptotes, and do not have any components in common.

1.1 Projective space

st-projective-space

If you have encountered Bézout's theorem before, you have probably seen a more precise version. The first way we can improve the statement is to consider the asymptotes more carefully.

Consider $g(x, y) = xy - 1$ and $f(x, y) = y - 1$. Then C has degree 1 and D has degree 2, so we expect their intersection to consist of two points. When we replaced C with a nearby curve C_s, this was indeed the case, but as $s \to 0$, one of those intersection points escaped to infinity *and was replaced by a common asymptote*. The asymptote really wants to be an intersection point!

If we count asymptotes as intersection points, maybe we can get a better version of Bézout's theorem. Unfortunately, this isn't quite right: Consider $g(x, y) = xy - 1$ and $f(x, y) = y$. This time there are no intersection points at all, but moving C slightly we see that *two* intersection points are escaping to the same asymptote. In fact, this means that C and D are *tangent* at infinity.

We can get a better sense of what is going on with a change of coordinates. Let $x_1 = x^{-1}$ and $y_1 = y/x$. Note that these coordinates don't make sense near $x = 0$, but they do make sense when x is very large. The asymptotic intersection occurs at $(x_1, y_1) = (1, 0)$.

In these coordinates, the equation for D is $y_1/x_1^2 - 1 = 0$ or, rearranging, $y_1 = x_1^2$. The equation for C is $y_1/x_1 = 0$, or just $y_1 = 0$ by rearranging. These two curves are indeed tangent.

Secretly, we are working in local coordinates on the projective plane. By definition, the projective plane $\mathbf{CP}^2$ consists of all 1-dimensional subspaces in the 3-dimensional complex vector space $\mathbf{C}^3$. For each point (x, y) of $\mathbf{C}^2$ we have a line in $\mathbf{C}^3$ spanned by the vector $(x, y, 1)$. Thus $\mathbf{C}^2$ is contained in $\mathbf{CP}^2$, but $\mathbf{CP}^2$ is bigger. If we let (x, y) approach infinity in $\mathbf{C}^2$, the corresponding point of $\mathbf{CP}^2$ approaches a legitimate limit. (In other words, $\mathbf{CP}^2$ is compact.)

Theorem 1.5 (Bézout's theorem in the projective plane)**.** *For most* projective *algebraic plane curves C and D, the intersection $C \cap D$ consists of* $\deg(C)\deg(D)$ *points.*

'Most curves' may be interpreted to include curves that share no components and no tangencies.

1.2 Multiplicities

The statement of Bézout's theorem can be improved even more. We've noticed that tangencies correspond to collisions of pairs intersection points. Higher order tangencies correspond to higher order collisions:

Exercise 1.6. Let C be defined by $y - x^3 = 0$ and let D be defined by $y = 0$. Show that the intersection $C \cap D$ is a single point, but that if D is deformed to a nearby line, there are 3 points of intersection.

We can see the tangency algebraically. If we intersect $y - x^3 = 0$ with $x = 0$, we get the equations $x = y = 0$. This reflects the fact that these two curves intersect transversally. On the other hand, intersecting with $y = 0$ gives $x^3 = y = 0$. This is a different equation that has the same solutions in $\mathbf{C}$ as $x = y = 0$. However, *it has different solutions in some rings that are not fields.* This precisely reflects the fact that one line is tangent (to second order) and the other is not.

In the theory of schemes, not all points are treated equally. The equation $y = x^3 = 0$ defines a fatter point than does $y = x = 0$.

Exercise 1.7. (i) Show that $y = x = 0$ and $y = x^3 = 0$ have the same set of solutions in any field.

(ii) Find a commutative ring A such that $y = x = 0$ and $y = x^3 = 0$ have different solution sets in A.

Theorem 1.8 (Bézout's theorem with multiplicities)**.** *For most* projective *algebraic plane curves C and D, the intersection $C \cap D$ consists of* $\deg(C)\deg(D)$ *points* when counted with multiplicity.

'Most curves' may be interpreted to include curves that share no components.

1.3 Intersection theory or derived algebraic geometry

The key in all of this discussion has been to consider moving our curves slightly. Intersection theory and derived algebraic geometry build this into the definition of intersection, yielding a very clean statement:

Theorem 1.9 (Bézout's theorem in intersection theory)**.** All *projective algebraic plane curves C and D intersect in* $\deg(C)\deg(D)$ *points, provided the intersection is interpreted via intersection theory.*

1.4 Some more enumerative questions

Question 1.10. If $L_1, \ldots, L_4$ are four lines in $\mathbf{C}^3$ how many lines L meet all four of them?

Question 1.11. If X is a surface in $\mathbf{C}^3$ defined by a cubic polynomial, how many lines lie on X?

2 A dictionary between algebra and geometry

In this section, we are going to investigate how geometric concepts are manifested algebraically and vice versa.

2.1 Points and functions

The most basic algebraic object we have at our disposal is an element of the ring $\mathbf{C}[x_1, \dots, x_n]$. We regard these as functions from $\mathbf{C}^n$ to $\mathbf{C}$.

The most basic geometric concept is that of a point. If $\xi \in \mathbf{C}^n$ is a point then we obtain a homomorphism

$$\mathrm{ev}_\xi : \mathbf{C}[x_1, \dots, x_n] \to \mathbf{C}$$
$$\mathrm{ev}_\xi(f) = f(\xi).$$

Exercise 2.1 (Easy)**.** Verify that this actually is a homomorphism and that it is surjective.

Since ev_ξ is surjective and its target is a field, its kernel is a maximal ideal, $\mathfrak{m}_\xi$. Hilbert's Nullstellensatz says that these are the only maximal ideals of $\mathbf{C}[x_1, \dots, x_n]$:

rst-nullstellensatz

Theorem 2.2 (Corollary to Hilbert's Nullstellensatz)**.** *Every maximal ideal of* $\mathbf{C}[x_1, \dots, x_n]$ *is of the form* $(x_1 - \xi_1, \dots, x_n - \xi_n)$ *for some* $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{C}^n$.

2.2 Algebraic subsets

def:alg-subset

Definition 2.3. If $J \subset \mathbf{C}[x_1, \dots, x_n]$ is a set of polynomials then $V(J)$ is the set of all $\xi \in \mathbf{C}^n$ such that $f(\xi) = 0$ for all $f \in J$. An *algebraic subset* of $\mathbf{C}^n$ is a subset that is equal to $V(J)$ for some set J.

If X is a subset of $\mathbf{C}^n$, we define $I(X)$ to be the set of all $f \in \mathbf{C}[x_1, \dots, x_n]$ such that $f(\xi) = 0$ for all $\xi \in X$.

Exercise 2.4 (A commutative algebra warmup)**.** (i) Let J' be the radical ideal generated by J. Show that $V(J) = V(J')$.[1]

(ii) For any $X \subset \mathbf{C}^n$ show that $I(X)$ is a radical ideal of $\mathbf{C}[x_1, \dots, x_n]$.

Theorem 2.5 (Hilbert's Nullstellensatz)**.** *If* $J \subset K[x_1, \dots, x_n]$ *is an ideal then* $I(V(J)) = \sqrt{J}$.

Exercise 2.6 (Easy, given the previous exercise)**.** Use the Nullstellensatz to prove that $V(I(X)) = X$ for any algebraic subset of $\mathbf{C}^n$.

Exercise 2.7 (Easy, given the previous exercise)**.** Give a one-to-one correspondence between algebraic subsets of $\mathbf{C}^n$ and (isomorphism classes of) surjections $\mathbf{C}[x_1, \dots, x_n] \to A$ with A reduced.[2]

The actual Nullstellensatz is often formulated in different ways. In order to make the statement, we generalize Definition 2.3 to an arbitrary field:

Definition 2.8. More generally, if K is a field with an algebraic closure $\overline{K}$, and $J \subset K[x_1, \dots, x_n]$ then we write $V(J)$ for the set of all $\xi \in \overline{K}^n$ such that $f(\xi) = 0$ for all $f \in J$. If $X \subset \overline{K}^n$, we write $I(X)$ for the set of all $f \in K[x_1, \dots, x_n]$ such that $f(\xi) = 0$ for all $\xi \in X$.

---

[1] An ideal J' is called *radical* if $f^n \in J' \implies f \in J'$. The smallest radical ideal containing J' is denoted $\sqrt{J'}$.

[2] A commutative ring is *reduced* if it has no nonzero nilpotent elements.

nss:1 **Theorem 2.9** (Nullstellensätze)**.** *(i) (Zariski's lemma) If K is a field and L is a field extension of K that is finitely generated as a commutative ring then L is finite dimensional over K.*

nss:5 *(ii) (Weak Nullstellensatz) Let K be a field. Let $J \subset K[x_1, \ldots, x_n]$ be an ideal. Then $V(J) = \varnothing$ if and only if $J = K[x_1, \ldots, x_n]$.*

nss:6 *(iii) (Hilbert's Nullstellensatz) If K is a field and $J \subset K[x_1, \ldots, x_n]$ is an ideal then $I(V(J)) = \sqrt{J}$.*

nss:2 *(iv) If K is an algebraically closed field then the maximal ideals of $K[x_1, \ldots, x_n]$ are all of the form $(x_1 - \xi_1, \ldots, x_n - \xi_n)$ for $\xi_i \in K$.*

Exercise 2.10 (Some parts of this exercise are likely to be hard)**.** Show that the different statements of Hilbert's Nullstellensatz are equivalent.

Solution. (i) $\implies$ (ii). Suppose K is a field with algebraic closure $\overline{K}$ and $J \subset K[x_1, \ldots, x_n]$ is an ideal. If $J = K[x_1, \ldots, x_n]$ then $V(J) = \varnothing$, clearly. Conversely, suppose that $J \neq K[x_1, \ldots, x_n]$. Let m be a maximal ideal containing J. Then $L = K[x_1, \ldots, x_n]/m$ is a finitely generated K-algebra and it is a field. Therefore it is a finite extension of K, by (i), so there is an embedding $L \subset \overline{K}$. Let $\xi = (\xi_1, \ldots, \xi_n)$ be the image of $(x_1, \ldots, x_n)$ in $\overline{K}$ under this identification. Then $\xi \in V(J)$.

(ii) $\implies$ (iii). (Rabinowitsch's trick) Let $\overline{K}$ be an algebraic closure of K. The inclusion $J \subset I(V(J))$ is immediate. Suppose $f \in I(V(J))$. Consider the ideal $J' = (J, yf - 1)$ of $K[x_1, \ldots, x_n, y]$. Under the projection $\overline{K}^{n+1} \to \overline{K}^n$, the set $V(J') \subset \overline{K}^{n+1}$ projects to a subset of $V(J)$ where f is non-zero. By the choice of f, this subset is empty, so $V(J') = \varnothing$. Thus $J' = K[x_1, \ldots, x_n, y]$. But then $JK[x_1, \ldots, x_n, f^{-1}] = K[x_1, \ldots, x_n]$ so there is some n with $f^n \in J$.

(iii) $\implies$ (iv). Let J be a maximal ideal. Then $V(J) \subset K^n$ is non-empty by (iii), hence contains at least one point ξ. Therefore $I(V(J)) \subset I(V(\xi)) = (x_1 - \xi_1, \ldots, x_n - \xi_n)$. But both ideals are maximal.

(iv) $\implies$ (i). Let L be a field that finitely generated as a K-algebra. Since it is finitely generated, we can find a surjection $K[x_1, \ldots, x_n] \to L$ whose kernel is a maximal ideal m. Then $\overline{K}m \subset \overline{K}[x_1, \ldots, x_n]$ is an ideal (not necessarily maximal). Choose $\overline{m} \supset \overline{K}m$ that is maximal. The quotient $\overline{K}[x_1, \ldots, x_n]/\overline{m}$ is isomorphic to $\overline{K}$ by (iv) and we get homomorphism

$$L = K[x_1, \ldots, x_n]/m \to \overline{K}[x_1, \ldots, x_n]/\overline{m} = \overline{K}$$

which shows that L is a finitely generated subfield (any homomorphism of fields is injective) of $\overline{K}$, hence is finite dimensional over K. □

We won't prove the Nullstellensatz for a while. The modern perspective on algebraic geometry treats all prime ideals as points, not just the maximal ideals, so the Nullstellensatz isn't quite as fundamental. The Nullstellensatz for prime ideals is much easier, and we will prove it in the next lecture.

2.3 Morphisms of algebraic subsets

We already know what an algebraic function from $\mathbf{C}^n$ to $\mathbf{C}$ is: It's just a polynomial in the variables $x_1, \ldots, x_n$. In other words, it's an element of $\mathbf{C}[x_1, \ldots, x_n]$. If X is an algebraic subset of $\mathbf{C}^n$ then we declare that a morphism from X to $\mathbf{C}$ is an element of $\mathbf{C}[x_1, \ldots, x_n]$ with f and g considered equivalent if $f(\xi) = g(\xi)$ for all $\xi \in \mathbf{C}^n$.

Exercise 2.11. (i) Show that the morphisms from an algebraic set $X \subset \mathbf{C}^n$ to $\mathbf{C}$ are in canonical bijection with $\mathbf{C}[x_1, \ldots, x_n]/I(X)$.

(ii) Show that the maximal ideals of $\mathbf{C}[x_1, \ldots, x_n]/I(X)$ are the same as the maximal ideals of $\mathbf{C}[x_1, \ldots, x_n]$ that contain $I(X)$ are the same as the points of X. (Hint: You will want to use the Nullstellensatz (Theorem 2.2) here.)

Suppose $X \subset \mathbf{C}^n$ and $Y \subset \mathbf{C}^m$ are algebraic subsets. What is a morphism $X \to Y$? We should certainly have a morphism $X \to \mathbf{C}^m$ in this case, which amounts to m morphisms from X to $\mathbf{C}$. That is, it means we have m elements of $A = \mathbf{C}[x_1, \ldots, x_n]/I(X)$, which we can also regard as a homomorphism

$$\mathbf{C}[y_1, \ldots, y_m] \to A.$$

Notice that the left side is the set of algebraic functions on $\mathbf{C}^m$ and the right side is the set of algebraic functions on X. If φ denotes the map $X \to \mathbf{C}^m$ then this homomorphism just sends a function $f \in \mathbf{C}[y_1, \ldots, y_m]$ to $f \circ \varphi \in A$. We usually write $\varphi^* f$ for the function $f \circ \varphi$.

What does it mean for the image of φ to lie inside Y? It means that for any $f \in I(Y)$ and $\xi \in X$ we have $f(\varphi(\xi)) = 0$. In other words, $\varphi^* f(\xi) = 0$ for all $\xi \in X$. If g is a representative for $\varphi^* f$ in $\mathbf{C}[x_1, \ldots, x_n]$ then this means $g \in I(X)$. Thus $\varphi^* f = 0$ in $A = \mathbf{C}[x_1, \ldots, x_n]/I(X)$.

Thus our condition that φ define a map $X \to Y$ is that $\varphi^* I(Y) = 0$ in A. By the universal property of the quotient ring, this means φ^* can be regarded as a homomorphism

$$B = \mathbf{C}[y_1, \ldots, y_m]/I(Y) \to \mathbf{C}[x_1, \ldots, x_n]/I(X) = A.$$

Of course, this isn't surprising when we think about B as the ring of functions on Y. If we have a map $\varphi : X \to Y$ and f is a function on Y then $f \circ \varphi$ is a function on X.

2.4 Abstract algebraic sets

In the last section, we saw that every algebraic set $X \subset \mathbf{C}^n$ gave rise to a reduced, finite type $\mathbf{C}$-algebra. Conversely, every reduced, finite type $\mathbf{C}$-algebra is the quotient of some $\mathbf{C}[x_1, \ldots, x_n]$ by a radical ideal, hence corresponds to an algebraic set. Different choices of generators give different embeddings in $\mathbf{C}[x_1, \ldots, x_n]$ give different embeddings in $\mathbf{C}^n$, but the different algebraic sets are all isomorphic, according to our definition of morphisms of algebraic sets above.

Exercise 2.12. Show that there is a contravariant equivalence between algebraic sets and reduced[3] finite type[4] $\mathbf{C}$-algebras.

2.5 Tangent vectors

Exercise 2.13. (i) Let $\xi \in \mathbf{C}$. Construct an identification $\mathbf{C}[x]/\mathfrak{m}_\xi^2 \simeq \mathbf{C}[\epsilon]/(\epsilon^2)$ sending $x - \xi$ to ϵ.

[3] This means 'has no nilpotents'.
[4] This means 'finitely generated as a commutative ring'.

(ii) Show that under this identification, the map

$$\mathbf{C}[x] \to \mathbf{C}[x]/\mathfrak{m}_\xi \simeq \mathbf{C}[\epsilon]/(\epsilon^2)$$

sends $f \in \mathbf{C}[x]$ to $f(\xi) + \epsilon f'(\xi)$. (Suggestion for how to think about this: Interpret x as $\xi + \epsilon$ and think about the Taylor series.)

Exercise 2.14. (i) Show that for any tangent vector v at a point $\xi \in \mathbf{C}^n$ the function

$$\delta : \mathbf{C}[x_1, \dots, x_n] \to \mathbf{C}[\epsilon]/(\epsilon^2)$$
$$\delta(f) = f(\xi) + \big(v \cdot \nabla f(\xi)\big)\,\epsilon$$

is a homomorphism of commutative rings (∇f denotes the gradient).

(ii) Show that the point ξ and the tangent vector v can be recovered from this homomorphism. (Hint: Write $\delta(f) = \varphi_0(f) + \epsilon\varphi_1(f)$. Set $\xi_i = \varphi_0(x_i)$ to get the point. Set $v_i = \varphi_1(x_i)$ to get the vector.)

(iii) Conclude that there is a one-to-one correspondence between pairs (ξ, v) where $\xi \in \mathbf{C}^n$ and v is a tangent vector to $\mathbf{C}^n$ at ξ and homomorphisms of commutative rings $\mathbf{C}[x_1, \dots, x_n] \to \mathbf{C}[\epsilon]/(\epsilon^2)$.

The next two exercises are not recommended. A lot of subtleties arise.

Exercise 2.15. Generalize the previous exercise to give an identification

$$TX \simeq \mathrm{Hom}_{\mathbf{C}\text{-}\mathbf{Alg}}(\mathbf{C}[x_1, \dots, x_n]/I(X), \mathbf{C}[\epsilon]/(\epsilon^2))$$

when $X = V(J) \subset \mathbf{C}^n$ is a manifold.[5]

Exercise 2.16. Show that when X is a *compact* complex manifold,

$$TX \simeq \mathrm{Hom}_{\mathbf{C}\text{-}\mathbf{Alg}}(C^\infty(X), \mathbf{C}[\epsilon]/(\epsilon^2)).$$

A similar statement holds for real manifolds.

So far, we have shown that reduced, finite type $\mathbf{C}$-algebras correspond to algebraic sets. The algebra $\mathbf{C}[\epsilon]/(\epsilon^2)$ is not reduced, but if we broaden our horizons just a little and pretend it corresponds to a space D then we have

$$TX = \mathrm{Hom}(D, X).$$

That is D *is the universal point with tangent vector*! It will turn out that D is a scheme that really does consist of one point with a little bit of infinitesimal 'fuzz' around it.

[5]This identification is *always* true once one has defined the tangent space of a singular space. We will later take this as the definition of the tangent space.

Geometry	Algebra
$\xi \in \mathbf{C}^n$	$\mathrm{ev}_\xi : \mathbf{C}[x_1, \ldots, x_n] \to \mathbf{C}$ $\mathfrak{m}_\xi \subset \mathbf{C}[x_1, \ldots, x_n]$ maximal ideal
$f : \mathbf{C}^n \to \mathbf{C}$	$f \in \mathbf{C}[x_1, \ldots, x_n]$ $\mathbf{C}[y] \to \mathbf{C}[x_1, \ldots, x_n]$
$X \subset \mathbf{C}^n$ algebraic subset	$I \subset \mathbf{C}[x_1, \ldots, x_n]$ radical ideal $\mathbf{C}[x_1, \ldots, x_n] \to A$ surjective, A reduced, finite type $\mathbf{C}$-algebra
algebraic set X $X = \mathrm{Hom}(A, \mathbf{C})$	reduced, finite type $\mathbf{C}$-algebra $A = \mathrm{Hom}(X, \mathbf{C})$
morphism of algebraic sets $f : X \to Y$	homomorphism of commutative rings $B \to A$
tangent vector $(\xi, v) \in TX$	$v \in (\mathfrak{m}_\xi / \mathfrak{m}_\xi^2)^\vee$ $A \to \mathbf{C}[\epsilon]/(\epsilon^2)$
affine scheme X	commutative ring A
morphism of affine schemes $X \to Y$	morphism of commutative rings $B \to A$

Chapter 2

Introduction to schemes

3 The prime spectrum and the Zariski topology

Reading 3.1. [MO, §I.1, pp. 1–4], [Vak14, §§3.2–3.5, 3.7], [Mum99, §II.1], [AM69, Chapter 1, Exercises 15–28], [Har77, pp. 69–70]

3.1 The Zariski topology

In the last section, we saw that the *points* of $\mathbf{C}^n$ can be recovered algebraically from the ring $\mathbf{C}[x_1, \ldots, x_n]$. However, the *topology* of $\mathbf{C}^n$ can't be recovered algebraically. In this section we will see how to get a topology that is coarser than the usual topology on $\mathbf{C}^n$. The construction is quite general, and works with $\mathbf{C}[x_1, \ldots, x_n]$ replaced by any commutative ring.

def:spec

Definition 3.2 (The prime spectrum)**.** We write $\operatorname{Spec} A$ for the set of prime ideals of a commutative ring A. For each $\mathfrak{p} \in \operatorname{Spec} A$, let

$$\mathbf{k}(\mathfrak{p}) = \operatorname{frac}(A/\mathfrak{p}).$$

This is called the *residue field* of $\mathfrak{p}$. We define

$$\mathrm{ev}_{\mathfrak{p}} : A \to A/\mathfrak{p} \to \operatorname{frac}(A/\mathfrak{p}) = \mathbf{k}(\mathfrak{p})$$

be the homomorphism that sends f to $f \bmod \mathfrak{p}$. It is convenient to write $f(\mathfrak{p})$ instead of $\mathrm{ev}_{\mathfrak{p}}(f)$, although one must take care to remember that $f(\mathfrak{p})$ and $f(\mathfrak{q})$ don't always live in the same set when $\mathfrak{p} \neq \mathfrak{q}$.

For any $J \subset A$, let

$$V(J) = \{\mathfrak{p} \in \operatorname{Spec} A \bigm| \mathrm{ev}_{\mathfrak{p}}(J) = 0\}.$$

Equivalently, $V(J)$ is the set of $\mathfrak{p} \in \operatorname{Spec} A$ such that $J \subset \mathfrak{p}$.

We write $D(J)$ for the complement of $V(J)$ in $\operatorname{Spec} A$. When we need to emphasize the ring A, we write I_A, V_A, D_A, etc. When J consists of just one element f, we write $V(f) = V(\{f\})$ and $D(f) = D(\{f\})$.

A subset $Z \subset \operatorname{Spec} A$ is called *closed* if $Z = V(J)$ for some $J \subset A$. A subset $U \subset \operatorname{Spec} A$ is called *open* if $U = D(J)$ for some ideal J. Sets of the form $D(f)$ for $f \in A$ are called *principal open subsets* or *distinguished open subsets.*

We also define

$$I(Z) = \{f \in A \mid f(Z) = 0\} = \bigcap_{\mathfrak{p} \in Z} \mathfrak{p}.$$

Question 3.3. Here is something to think about: Is every open subset principal? We will answer this question later.

Exercise 3.4. Let $\mathfrak{p}$ be a prime ideal of A. Show that $\{\mathfrak{p}\}$ is a closed subset of $\operatorname{Spec} A$ if and only if $\mathfrak{p}$ is a maximal ideal.

Solution. The closure of $\{\mathfrak{p}\}$ is $V(\mathfrak{p})$. So $\{\mathfrak{p}\}$ is closed if and only if $\mathfrak{p}$ is the only prime ideal containing $\mathfrak{p}$. In other words, if and only if $\mathfrak{p}$ is maximal. □

The exercise shows that this topology usually is not Hausdorff. It contains many points that are not closed. This sounds pathological, but it turns out to be convenient once you get used to it.

Exercise 3.5. Show that

$$V(\sum J_i) = \bigcap V(J_i)$$
$$V(JK) = V(J \cap K) = V(J) \cup V(K)$$

for any ideals J and K. Conclude that the definitions of open and closed sets in Definition 3.2 give a topology, called the *Zariski topology.*

Solution. We prove that $V(JK) = V(J \cap K) = V(J) \cup V(K)$. We have $JK \subset J \cap K \subset J$ and $J \cap K \subset K$, so $V(JK) \supset V(J \cap K) \supset V(J) \cup V(K)$ by reversal of inclusions. We prove that $V(JK) \subset V(J) \cup V(K)$.

Suppose that $\mathfrak{p} \in V(JK)$ and $\mathfrak{p} \notin V(K)$. Then there is some $g \in K$ such that $g(\mathfrak{p}) \neq 0$. But $gJ \subset JK$ so $g(\mathfrak{p})f(\mathfrak{p}) = 0$ for all $f \in J$. As $g(\mathfrak{p}) \neq 0$, this means $f(\mathfrak{p}) = 0$ for all $f \in J$, so $\mathfrak{p} \in V(J)$, as requried. □

3.2 Examples

Here are some useful facts from commutative algebra that you may want to recall for the following exercise and later ones:

Theorem 3.6. *(i) A principal ideal domain is a unique factorization domain.*

(ii) If A is a unique factorization domain then $A[x]$ is a unique factorization domain.

(iii) In a unique factorization domain, the ideal generated by an irreducible element is prime.

Proof. The hardest statement to prove is the second one. Recall that a commutative ring is a unique factorization domain if it satisfies the ascending chain condition on principal ideals and every irreducible element is prime. Suppose that we have an ascending chain of principal ideals $(f_1) \subset (f_2) \subset \cdots \subset A[x]$. Then the degrees of the polynomials f_i are decreasing, hence must eventually stabilize. Then let a_i be the leading coefficient of f_i. We also have an ascending chain of ideals $(a_i) \subset (a_{i+1}) \subset \cdots \subset A$, so this must eventually stabilize. One the degrees and leading coefficients are constant, we have $(f_i) = (f_{i+1})$ since $f_i = cf_{i+1}$ where c is a polynomial of degree zero, and $(a_i) = (a_{i+1})$ implies that c is a unit, since A is an integral domain.

Now we will show every irreducible element of $A[x]$ is prime. Let f be an irreducible element of $A[x]$. If $f \in A$ then f is clearly prime since $A[x]/fA[x] = (A/fA)[x]$ is an integral domain. We assume that $f \notin A$.

Let K be the field of fractions of A. Then $A[x] \subset K[x]$ and $K[x]$ is a principal ideal domain, hence a unique factorization domain. Suppose that f divides gh in $A[x]$. Then f divides g or f divides h in $K[x]$, since $K[x]$ is a principal ideal domain. So we need to prove that if f divides g in $K[x]$ then f divides g in $A[x]$.

In other words, under the assumption that f is irreducible in $A[x]$ and $g \in A[x]$ and $g = fh$ in $K[x]$, we want to deduce that $h \in A[x]$. Equivalently, we want to show that the power of each irreducible $t \in A$ in h is non-negative. To show this, localize at the prime (t) in A. If $u \in A_{(t)}$ is not a unit then $u = st$ for some $s \in A_{(t)}$. Consider the chain $(u) \subset (t^{-1}u) \subset (t^{-2}u) \subset \cdots$ of submodules of K. Since A satisfies the ascending chain condition on principal ideals, this chain must stabilize or eventually leave A. It cannot stabilize, for if $(t^{-i}u) = (t^{-i-1}u)$ then t would be a unit, since A is an integral domain. Therefore there is some n such smallest n that $t^{-n}u \notin (t)$. Then $t^{-n}u$ is a unit, so $(u) = (t^n)$. In particular, *every principal ideal in $A_{(t)}$ is of the form (t^n) for some integer $n \geq 0$.*

We can therefore choose k minimally so that $t^k h \in A[x]$. Reduce modulo t to get $\overline{t^k g} = \bar{f}\,\overline{t^k h}$ in $(A/tA)[x]$. Then $(A/tA)[x]$ is a principal ideal domain since A/tA is a field. In particular, its only zero divisor is zero. But $\bar{f} \neq 0$ (since f is irreducible and $f \notin A$) and $\overline{t^k h} \neq 0$ (since k was chosen minimally) so $\overline{t^k g} \neq 0$. Thus $k = 0$ and $h \in A[x]$. □

Exercise 3.7. Suppose k is a field.

(i) Show that $\operatorname{Spec} k$ is a single point.

(ii) Show that $\operatorname{Spec} k[\epsilon]/(\epsilon^2)$ is a single point.

(iii) Show that $\operatorname{Spec}(k \times k)$ consists of two points. What is the topology?

Solution. We have two maximal ideals, corresponding to the two projections to k. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$. If $\mathfrak{p}$ is a prime ideal then $e_1 e_2 = 0$ so $e_1 \in \mathfrak{p}$ or $e_2 \in \mathfrak{p}$, so $\mathfrak{p}$ contains one of the two maximal ideals (e_1) or (e_2), hence is equal to (e_1) or to (e_2). □

Exercise 3.8. Describe the points and topology of $\operatorname{Spec} \mathbf{Z}$.

Exercise 3.9. (i) Describe the points and topology of $\operatorname{Spec} \mathbf{C}[x]$.

(ii) Describe the points and topology of $\operatorname{Spec} \mathbf{R}[x]$.

(iii) Describe the points and topology of $\operatorname{Spec} \mathbf{Q}[x]$.

Exercise 3.10. Suppose A is an integral domain. Show that $\operatorname{Spec} A$ contains a point that is dense. This is called the *generic point* of $\operatorname{Spec} A$.

Exercise 3.11. Give a point of $\operatorname{Spec} \mathbf{C}[x, y]$ that is neither a point of $\mathbf{C}^2$ nor the generic point.

3.3 Basic properties

Exercise 3.12. Suppose that A is a commutative ring and $f \in A$. Show that there is a universal homomorphism $\varphi : A \to B$ such that $\varphi(f)$ is invertible. (Hint: $B = A[u]/(uf - 1)$.)

Exercise 3.13. (i) Show that $D(J) = \bigcup_{f \in J} D(f)$.

(ii) Show that the intersection of two principal open subsets is a principal open subset.

(iii) Conclude that the principal open subsets of $\operatorname{Spec} A$ form a basis of the Zariski topology.

affine-quasicompact

Exercise 3.14 (Unions of principal open affine subsets. Important!). (i) Suppose A is a commutative ring and $f_1, \ldots, f_n \in A$. Show that $\operatorname{Spec} A = \bigcup D(f_i)$ if and only if $(f_1, \ldots, f_n)A = A$.

(ii) Conclude that $\operatorname{Spec} A$ is *quasicompact*[1] for any commutative ring A.

Exercise 3.15 (The prime Nullstellensatz). This is much easier than Hilbert's Nullstellensatz, which might also be called the *maximal Nullstellensatz*.

(i) (You may want to use this one to prove the next one, or you may want to skip this part because it is a special case of the next one.) For any commutative ring A, show that $I_A(\operatorname{Spec} A)$ is the radical of A. (Hint: Let f be an element of A that is contained in every prime ideal. Consider $A[f^{-1}]$. What are its prime ideals?)

Solution. Primes of $A[f^{-1}]$ are primes of A that don't contain f. There are none. Therefore $A[f^{-1}]$ is the zero ring. But this means f is nilpotent, since the kernel of $A \to A[f^{-1}]$ consists of all elements of A annihilated by a power of f. □

(ii) For any commutative ring A and any subset $J \subset A$, show that $I(V(J))$ is the radical ideal generated by J. (Hint: Reduce to the previous part by replacing A with A/J. Or imitate the proof of the previous part.)

Solution. By definition, $I(V(J))$ is the intersection of all prime ideals containing J. This is certainly a radical ideal since all prime ideals are radical. Therefore it contains the radical ideal $\sqrt{J}$ generated by J. To see the converse, suppose $f \in I(V(J))$. Consider the ring
$$A[f^{-1}]/JA[f^{-1}] = (A/JA)[f^{-1}].$$
The prime ideals of this ring are exactly the prime ideals of A that contain J and do not contain f. By definition, there are no such primes, so $(A/JA)[f^{-1}]$ has no prime ideals—it is the zero ring. Thus there must be some n such that $f^n \mod JA = 0$, i.e., $f^n \in JA$. Thus f is contained in the radical ideal generated by J. □

(iii) Conclude that $Z \subset \operatorname{Spec} A$ is closed if and only if $Z = V(I(Z))$.

Residue fields

We give two categorical characterizations of the points of the prime spectrum.

Exercise 3.16 (Minimal homomorphisms to fields). Call a homomorphism from A to a field k *minimal* if, whenever L and K are fields and there is a commutative diagram of solid lines

$$\begin{tikzcd}
 & k \arrow[dr] \arrow[dd, dashed] & \\
A \arrow[ur] \arrow[dr] & & L \\
 & K \arrow[ur] &
\end{tikzcd}$$

[1] This is what non-algebraic geometers in North America usually call compact. It means that every open cover of $\operatorname{Spec} A$ has a finite subcover.

there is a unique dashed arrow extending the diagram. Two homomorphisms $f : A \to k$ and $g : A \to k'$ are said to be *isomorphic* if there is an isomorphism $h : k \to k'$ with $hf = g$.

Show that the points of $\operatorname{Spec} A$ correspond to isomorphism classes of *minimal homomorphisms* $A \to k$, where k is a field.

Exercise 3.17 (Epimorphisms to fields)**.** A morphism of commutative rings $f : A \to B$ is called an *epimorphism* if, for any commutative ring C, composition with f induces an injection $\operatorname{Hom}(B, C) \to \operatorname{Hom}(A, C)$. In other words, f is an epimorphism if, for any homomorphisms $g, h : B \to C$, we have $gf = hf$ if and only if $g = h$.

(i) Show that any surjective homomorphism is an epimorphism.

(ii) Not every epimorphism is a surjection: Suppose that A is an integral domain and B is its field of fractions. Show that $A \to B$ is an epimorphism. (More generally, show that any localization is an epimorphism.)

(iii) (This part may be difficult. The proof suggested below works in much greater generality, and we will see it repeatedly.) Show that a homomorphism from a commutative ring A to a field K is an epimorphism if and only if the image of A generates K as a field. (Hint: Replace A with the field generated by its image in k. Show that $k \to K$ is an epimorphism if and only if $k = K$ by taking $C = K \otimes_A K$ in the definition of an epimorphism. Let $i, j : K \to K \otimes_k K$ be given by $i(x) = 1 \otimes x$ adn $j(x) = x \otimes 1$. Prove that $i = j$ if and only if $k = K$ by proving the sequence

$$0 \to k \to K \xrightarrow{i-j} K \otimes_k K$$

is exact. Do this by proving the sequence

$$0 \to K \to K \otimes_k K \xrightarrow{i'-j'} K \otimes_k K \otimes_k K$$

with $i'(y \otimes x) = y \otimes 1 \otimes x$ and $j'(y \otimes x) = y \otimes x \otimes 1$ is exact. To verify this, show that the maps $K \otimes_k K \to K$ sending $y \otimes x$ to yx and $K \otimes_k K \otimes_k K \to K \otimes_k K$ sending $z \otimes y \otimes x$ to $zy \otimes x$ split the sequence.

Solution. Note that $(ds+sd)(y\otimes x) = d(yx)+s(y\otimes 1\otimes x - y\otimes x\otimes 1) = yx\otimes 1 + y\otimes x - yx\otimes 1 = yx\otimes 1$. So $\mathrm{id} = ds + sd$. But then suppose $d(f) = 0$. We get $f = ds(f)$ so f is in the image of d.

Thus both sequences are exact, so the equalizer of i and j is k. In particular, $k \to K$ can only be an epimorphism if $k = K$. □

3.4 Functoriality

Exercise 3.18 (Functoriality of the prime spectrum)**.** Suppose $f : A \to B$ is a homomorphism of commutative rings. Show that $p \mapsto f^{-1}p$ defines a function $\operatorname{Spec} f : \operatorname{Spec} B \to \operatorname{Spec} A$. Show that this definition respects composition of homomorphisms.

Exercise 3.19. Let $\varphi : A \to B$ be a homomorphism and let $u : \operatorname{Spec} B \to \operatorname{Spec} A$ by the induced morphism of spectra. Show that $u^{-1}D(f) = D(\varphi(f))$. Conclude that $u^{-1}D(J) = D(\varphi(J)B)$ and $u^{-1}V(J) = V(\varphi(J))$ for any ideal J of A. Conclude from this that u is continuous.

Solution. We have $\mathfrak{p} \in u^{-1}D(f)$ if and only if $\varphi^{-1}\mathfrak{p} \in D(f)$ if and only if $f \notin \varphi^{-1}\mathfrak{p}$ if and only if $\varphi(f) \notin \mathfrak{p}$ if and only if $\mathfrak{p} \in D(\varphi(f))$. □

Solution. We have $\mathfrak{p} \in u^{-1}D(f)$ if and only if $f(u(\mathfrak{p})) \neq 0$ if and only if $\varphi(f)(\mathfrak{p}) \neq 0$ if and only if $\mathfrak{p} \in D(\varphi(f))$.

We get the statement about $D(J)$ because u^{-1} preserves unions, and we get the statement about $V(J)$ because u^{-1} preserves complements. □

Exercise 3.20. Show that every point of $\operatorname{Spec} A$ corresponds to a homomorphism of commutative rings $A \to k$ for some field k.

iv-prop-open-subset **Exercise 3.21** (The universal property of an open subset)**.** Let $J \subset A$ be any subset and let $D(J) \subset \operatorname{Spec} A$ be an open subset. Show that the map $u : \operatorname{Spec} B \to \operatorname{Spec} A$ associated to $f : A \to B$ factors through $D(J)$ if and only if $f(J)B = B$.

Solution. We have $u(\operatorname{Spec} B) \subset D(J)$ if and only if $u^{-1}D(J) = \operatorname{Spec} B$ if and only if $D(\varphi(J)) = \operatorname{Spec} B$ if and only if $\varphi(J)B = B$. □

-prop-closed-subset **Exercise 3.22** (The universal property of a closed subset)**.** Show that $\varphi : A \to B$ induces a map $u : \operatorname{Spec} B \to \operatorname{Spec} A$ factoring through $V(J)$ if and only if $f(J)B$ is a nilideal (every element is nilpotent).

Solution. We have $u(\operatorname{Spec} B) \subset V(J)$ if and only if $u^{-1}V(J) = \operatorname{Spec} B$ if and only if $V(\varphi(J)) = \operatorname{Spec} B$ if and only if every element of $\varphi(J)$ is nilpotent. □

Exercise 3.23. Suppose that $\varphi : A \to B$ is a ring homomorphism and $u : \operatorname{Spec} B \to \operatorname{Spec} A$ is the corresponding morphism on affine schemes. Let $\mathfrak{p}$ be a point of $\operatorname{Spec} A$. Prove that $u^{-1}\mathfrak{p} = \operatorname{Spec} B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} = \operatorname{Spec} B \otimes_A \mathbf{k}(\mathfrak{p})$.

Solution. Note that $\{\mathfrak{p}\} = V(\mathfrak{p}) \cap D(A - \mathfrak{p})$. Therefore $u^{-1}\mathfrak{p} = u^{-1}V(\mathfrak{p}) \cap u^{-1}D(A - \mathfrak{p}) = V(u^{-1}\mathfrak{p}) \cap D(\varphi(A - \mathfrak{p}))$. By definition, these are the prime ideals of B that contain $\varphi(\mathfrak{p})$ and do not meet $\varphi(A - \mathfrak{p})$. These are the same as the prime ideals of $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$. □

Exercise 3.24 (Surjectivity of integral morphisms)**.** Suppose A is a commutative ring and f is an integral polynomial with coefficients in A. Let $B = A[t]/(f)$. Show that $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective. (Hint: Reduce to the case where A is a field.)

Solution. The fiber over $\mathfrak{p} \in \operatorname{Spec} A$ is $\operatorname{Spec}(B \otimes_A \mathbf{k}(\mathfrak{p}) = \mathfrak{k}(\mathfrak{p})[t]/(f \bmod \mathfrak{p})$. Since f is integral, $f \bmod \mathfrak{p}$ is not constant, so so $\mathfrak{k}(\mathfrak{p})[t]/(f \bmod \mathfrak{p})$ is not the zero ring. □

3.5 More examples

Exercise 3.25. If R is a discrete valuation ring then $\operatorname{Spec} R$ consists of two points, one open and dense and the other closed. (If you don't know what a discrete valuation ring is, assume $R = \mathbf{C}[t]_{(t)}$ or $R = \mathbf{Z}_{(p)}$.)

Exercise 3.26. Describe the points and topology of $\mathbf{C}[x, y]_{(x,y)}$.

Exercise 3.27. Describe the points and topology of $\mathbf{Z}[x]$.

Exercise 3.28. Let $A' \to A$ be a surjection of commutative rings whose kernel is nilpotent. Show that the map $\operatorname{Spec} A \to \operatorname{Spec} A'$ is a homeomorphism.

ffine-generic-point **Exercise 3.29.** Suppose A is a commutative ring.

(i) Let $\mathfrak{p}$ be a prime ideal of A. Show that $\{\mathfrak{p}\}$ is dense in $V(\mathfrak{p})$. Conclude that $V(\mathfrak{p})$ is *irreducible*: it is impossible to write $V(\mathfrak{p})$ as the union of two closed subsets $A \cup B$ unless at least one of them is equal to $V(\mathfrak{p})$ itself.

(ii) Suppose that $Z \subset \operatorname{Spec} A$ is an irreducible subset. Show that there is a unique prime ideal $\mathfrak{p} \subset A$ such that $V(\mathfrak{p}) = Z$.

Solution. Suppose $\mathfrak{p} \in Z \subset V(\mathfrak{p})$ and Z is closed. Then $\mathfrak{p} \subset I(\mathfrak{p}) \supset I(Z) \subset I(V(\mathfrak{p})) = \mathfrak{p}$, since $\mathfrak{p}$ is a radical ideal. Therefore $I(Z) = \mathfrak{p}$, so $Z = V(I(Z)) = V(\mathfrak{p})$.

Suppose that $Z \subset \operatorname{Spec} A$ is irreducible, and let $J = I(Z)$. Suppose $fg \in J$. That means $f(Z)g(Z) = 0$, so $V(f) \cup V(g) \supset Z$. But Z is irreducible, so either $V(f) \supset Z$ or $V(g) \supset Z$. That is $f(Z) = 0$ or $g(Z) = 0$, so $f \in J$ or $g \in J$. □

4 Sheaves

Reading 4.1. [Vak14, §§2.1–2.4, 2.7 (pp. 69–83)], [Har77, §II.1 (pp. 60–65)]

4.1 Why sheaves?

In geometry, one usually has a *ring of functions* associated to a space. For example, in differential geometry one can take the ring of C^∞ functions, valued in $\mathbf{R}$ or in $\mathbf{C}$. In topology, one has a ring of continuous functions, valued in $\mathbf{R}$ or $\mathbf{C}$ (or any topological ring).

In algebraic geometry, we turn this around and declare that every commutative ring *should be* the ring of functions on some space, which we call an affine scheme. We also allow ourselves to glue spaces together along open subsets. In a sense, schemes are the minimal collection of spaces that can be constructed from these axioms.

It is possible to proceed quite formally along these lines, and we will discuss this in Lecture **??**. For the sake of concreteness, and adherence to historical conventions, we will first give a definition in which schemes do have an underlying space. However, there will be one very strange departure: functions are not determined by their values at points.

Exercise 4.2. Give an example of a commutative ring A and two elements $f, g \in A$ such that $\mathrm{ev}_\xi(f) = \mathrm{ev}_\xi(g)$ for all $\xi \in \operatorname{Spec} A$. Interpret this as the failure of functions to be determined by their values at points.

In differential geometry, for example, one can describe the maps of differentiable manifolds $X \to Y$ as the functions on the underlying sets that have some desirable local property. Since functions in algebraic geometry are not determined by their values at points, one cannot specify morphisms between schemes this way. Instead, we need to explicitly specify the ring of functions. We describe a morphism of schemes as a morphism of topological spaces *with compatible homomorphism between their rings of functions.*

But schemes can be glued together from affine schemes in nontrivial ways. In contrast to differentiable manifolds, where the global functions always determine the local functions, schemes often do not have many global functions at all. In fact, one already sees this in complex geometry:

Exercise 4.3. Show that all holomorphic functions from a compact Riemann surface to $\mathbf{C}$ are constant.

Solution. If X is a compact Riemann surface and f is a holomorphic function on X then $|f|$ achieves a maximum value somewhere on X. But if f is not constant then f is an open map, so $|f(X)|$ cannot include an upper bound. □

Locally, a scheme may have a lot of functions, but these can fail to glue together to give global functions. When thinking about functions on a scheme, one is therefore obliged to think about functions on *all open subsets simultaneously*. In other words, one thinks about the *sheaf* of functions, not just the sheaf's ring of global sections.

4.2 The definitions of presheaves and sheaves

def:sheaf **Definition 4.4.** Let X be a topological space. A *presheaf (of sets)* on X consists of the following data and conditions:

PSH1 a set $F(U)$ for each open $U \subset X$ (one often writes $\Gamma(U, F) = F(U)$);

PSH2 a function $\rho_{UV} : F(U) \to F(V)$ whenever $V \subset U$ are open subsets of X;

PSH3 equality $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$ when $W \subset V \subset U$ are open subsets of X.

Usually one writes $\xi\big|_V$ instead of $\rho_{UV}(\xi)$ when $\xi \in F(U)$ and $V \subset U$. A presheaf is called a *sheaf* if it satisfies the following additional conditions:

def:sheaf:1

SH1 if $\xi, \eta \in F(U)$ and $\bigcup U_i = U$ and $\xi\big|_{U_i} = \eta\big|_{U_i}$ for all i then $\xi = \eta$;

def:sheaf:2

SH2 if $\bigcup U_i = U$ and $\xi_i \in F(U_i)$ for all i and $\xi_i\big|_{U_i \cap U_j} = \xi_j\big|_{U_i \cap U_j}$ then there is a $\xi \in F(U)$ such that $\xi\big|_{U_i} = \xi_i$ for all i.

We obtain sheaves of groups, abelian groups, rings, commutative rings, etc. by substituting the appropriate concept for set and the appropriate notion of homomorphism for function in the definition of a presheaf.

Exercise 4.5. (i) All presheaves on a point are sheaves.

(ii) The category of sheaves on a point is equivalent (in fact isomorphic) to the category of sets.

Exercise 4.6. Suppose F is a sheaf on a topological space. Prove that $F(\varnothing)$ is a 1-element set.

Exercise 4.7. Recognize presheaves as contravariant functors from the category of open subsets of X to the category of sets.

4.3 Examples of sheaves

Exercise 4.8 (Constant presheaf)**.** Let X be a topological space and let S be a set. Define $F(U) = S$ for all open $U \subset X$. Give an example of X for which F is not a sheaf. (Hint: Exercise 4.10 below may give a clue. For almost any space X you pick, F will not be a sheaf, but you should try to find a simple example.)

Exercise 4.9 (Subsheaves)**.** If F and G are presheaves, we say that F is a *subpresheaf* of G if $F(U) \subset G(U)$ for all U. Suppose that G is a sheaf and F is a subpresheaf of G. Prove that F is a sheaf if and only if whenever $\xi \in G(U)$ and there is an open cover of U by sets V such that $\xi\big|_V \in F(V)$ then $\xi \in F(U)$.

sheaf-of-cont-funcs

Exercises 4.10 and 4.11 are very important! They are essentially equivalent, though, so feel free to do just one of them. Later we will see that *every* sheaf arises from these constructions.

Exercise 4.10 (Sheaf of functions)**.** You should do at least one of this exercise or the next.

Let X and Y be topological spaces and define $F(U)$ to be the set of continuous functions $U \to Y$, for each $U \in \mathbf{Open}(X)$. Show that F is a sheaf.

The collection of all functions is also a sheaf. If X and Y are manifolds, the differentiable functions form a sheaf.

ex:sheaf-of-sections

Exercise 4.11 (Sheaf of sections)**.** Suppose that $\pi : E \to X$ is a continuous function. For each open $U \subset X$, let $F(U)$ be the set of continuous functions $\sigma : U \to E$ such that $\pi \circ \sigma = \mathrm{id}_U$. These are called *sections* of E over U. Prove that F is a sheaf.

4.4 Morphisms of sheaves

def:sheaf-morphism

Definition 4.12. If F and G are presheaves on a topological space X, a *morphism* $\varphi : F \to G$ consists of functions $\varphi_U : F(U) \to G(U)$ such that whenever $V \subset U$ is an open subset we have $\rho_{UV} \circ \varphi_U = \varphi_V \circ \rho_{UV}$.

A morphism of sheaves is a morphism of the underlying presheaves.

sheaf-of-sheaf-homs★ **Exercise 4.13** (The sheaf of morphisms)**.** Suppose that F and G are presheaves on X. For each open $U \subset X$, let $H(U) = \mathrm{Hom}_{\mathbf{Sh}(U)}(F\big|_U, G\big|_U)$.

(i) Show that H is a presheaf in a natural way.

(ii) Show that if G is a sheaf then H is a sheaf.

4.5 Sheaves are like sets

Virtually any definition concerning sets can be interpreted in $\mathbf{Sh}(X)$ if we interpret $\forall$ and $\exists$ as follows:

(i) $\forall\, \xi \in F$ means "for all open U and all $\xi \in F(U) \ldots$"

(ii) $\exists\, \xi \in F$ means "there is an open cover U_i and $\xi_i \in F(U_i) \ldots$

For example:

Definition 4.14. A morphism of sets $\varphi : F \to G$ is injective if for all $\xi, \eta \in F$ we have $\varphi(\xi) = \varphi(\eta)$ only if $\xi = \eta$. A morphism of sheaves $\varphi : F \to G$ is *injective* if, for all open subsets U of X and all $\xi, \eta \in F(U)$, we have $\varphi(\xi) = \varphi(\eta)$ only if $\xi = \eta$.

A morphism of sets $\varphi : F \to G$ is surjective if for all $\eta \in G$ there is some $\xi \in F$ such that $\varphi(\xi) = \eta$. A morphism of sheaves of sets $\varphi : F \to G$ is *surjective* if, for all open $U \subset X$ and all $\eta \in G(U)$, there is an open cover $U = \bigcup V_i$ and elements $\xi_i \in F(V_i)$ such that $\varphi(\xi_i) = \eta\big|_{V_i}$.

Exercise 4.15 (Axiom of choice)**.** The axiom of choice says that for every surjection $\varphi : F \to G$ there is a morphism $\sigma : G \to F$ such that $\varphi \circ \sigma = \mathrm{id}_G$. Show that the axiom of choice is false in $\mathbf{Sh}(S^1)$, where S^1 is the circle. (Hint: Let F be the sheaf of sections of the universal cover $\mathbf{R} \to S^1$ and let G be the final sheaf $G(U) = 1$ for all open $U \subset S^1$.)

Exercise 4.16 (Axiom of choice, redux)**.** On the other hand, the above example is not a counterexample if we interpret the existential quantifier in the axiom of choice according to the sheaf-theoretic translation. Here is an example where even the sheaf-theoretic interpretation fails. Let X be the Hawai'ian earring. Recall that there are projections $X \to X_n = \bigwedge_{i=1}^n S^1$ for each $n \in \mathbf{N}$ (in fact, X is the inverse limit of these projections). For each n, let Y_n be the universal cover of X_n and let $Z_n = Y_n \times_{Y_n} X$. Let $Z = \coprod_{n \in \mathbf{N}} Z_n$. We have an evident surjective map $Z \to X \times \mathbf{N}$. Let F be the sheaf of sections of Z and let G be the sheaf of sections of $X \times \mathbf{N}$. Show that $F \to G$ is surjective but does not have a section over any open cover of X.

Solution. If U is any open neighborhood of the basepoint in X then there is a number n such that U contains all but n of the circles in X. Therefore $Z_n \to X$ has no section over U, and hence $Z \to X \times \mathbf{N}$ can have no section over U. $\square$

Exercise 4.17. Show that a morphism of sheaves $\varphi : F \to G$ is an isomorphism if and only if it is *bijective* (both injective and surjective). Note that *isomorphism* means *has a two-sided inverse.*

4.6 Sheaves on a basis

Reading 4.18. [Vak14, §2.7]

Suppose X is a topological space and $\mathscr{U} \subset \mathbf{Open}(X)$ is a basis of X. The definition of a presheaf on $\mathscr{U}$ is obtained by substituting $\mathscr{U}$ for $\mathbf{Open}(X)$ in Definition 4.4. Since the intersection of two basic open subsets is not necessarily a basic open subset, the definition of a sheaf requires a small modification in condition **SH2**.

def:sheaf-basis **Definition 4.19.** Let $\mathscr{U}$ be a basis for a topological space X. A presheaf on $\mathscr{U}$ is said to be a sheaf if it satisfies **SH1** and

def:sheaf-basis:2 **SH2′** if $U = \bigcup_{i,j \in I} U_i$ is an open cover in $\mathscr{U}$, and $\xi \in F(U_i)$ are elements such that, for each $i, j \in I$, there is an open cover $U_i \cap U_j = \bigcup_{k \in K_{ij}} V_{ijk}$ with $\xi_i\big|_{V_{ijk}} = \xi_j\big|_{V_{ijk}}$ for all $k \in K_{ij}$, then there is a $\xi \in F(U)$ such that $\xi\big|_{U_i} = \xi_i$ for all $i \in I$.

Exercise 4.20. Show that if **SH1** holds for F and $\mathscr{U}$ is stable under finite intersections, conditions **SH2** and **SH2′** are equivalent.

Theorem 4.21. *Suppose $\mathscr{U} \subset$ **Open**(X) is a basis. If F is a sheaf on $\mathscr{U}$ then F extends in a unique way (up to unique isomorphism) to a sheaf on* **Open**(X).

This procedure can be viewed as an example of *sheafification*, which we will discuss in Lecture 9 using the *espace étalé*.

For each $V \in$ **Open**(X), define

$$G(V) = \varprojlim_{\substack{U \in \mathscr{U} \\ U \subset V}} F(U).$$

In case you are not familiar with limits, here is an explicit description of this limit: It is a tuple $(\xi_U)_{\substack{U \in \mathscr{U} \\ U \subset V}}$ with each $\xi_U \in F(U)$ such that whenever $U' \subset U$ we have $\xi_U\big|_{U'} = \xi_{U'}$.

Exercise 4.22. Construct a canonical isomorphism $G\big|_{\mathscr{U}} \simeq F$.

If H is any sheaf extending F to **Open**(X) then consider $\xi \in H(V)$. For every $U \subset V$ in $\mathscr{U}$, we have $\xi\big|_U \in F(U)$ and $\xi\big|_U\big|_{U \cap U'} = \xi\big|_{U'}\big|_{U \cap U'}$ so ξ determines an element of $G(V)$ (using **SH2**). This element is unique by **SH1**.

Exercise 4.23. Fill in the details from the last paragraph to show that if there is a sheaf H extending F then there is a unique isomorphism $H \to G$.

Exercise 4.24. Complete the proof by showing G is a sheaf.

Solution. Finally, we check G actually is a sheaf. Suppose $\xi, \eta \in G(V)$ agree on an open cover $\{V_i\}$. Then consider any $U \subset V$ in $\mathscr{U}$. We get $\xi_U\big|_{U \cap V_i} = \eta_U\big|_{U \cap V_i}$ for all i. The $U \cap V_i$ cover U and F is a sheaf on $\mathscr{U}$, so we get $\xi_U = \eta_U$ for all $U \subset V$ in $\mathscr{U}$. This implies $\xi = \eta$, by definition of $G(V)$. This proves **SH1**.

Now suppose $\xi_i \in V_i$ for all V_i in an open cover over V with $\xi\big|_{V_i \cap V_j} = \xi_j\big|_{V_i \cap V_j}$. For each $U \subset V$ in $\mathscr{U}$, we get $\xi_i\big|_{U \cap V_i}$ and $\xi_i\big|_{U \cap V_i}\big|_{U \cap V_i \cap V_j} = \xi_j\big|_{U \cap V_j}\big|_{U \cap V_i \cap V_j}$. Therefore the $\xi_i\big|_{U \cap V_i}$ glue together to define ξ_U.

To get an element of $G(V)$, we have to check that $\xi_U\big|_{U'} = \xi_{U'}$. By definition, $\xi_{U'}\big|_{V_i \cap U'} = \xi_i\big|_{V_i \cap U'}$ and $\xi_U\big|_{V_i \cap U} = \xi_i\big|_{V_i \cap U}$ for all i. Thus

$$\xi_U\big|_{U'}\big|_{V_i \cap U'} = \xi_U\big|_{V_i \cap U}\big|_{V_i \cap U'} = \xi_i\big|_{V_i \cap U'} = \xi_{U'}\big|_{V_i \cap U'}.$$

But the $V_i \cap U'$ cover U' and F is a sheaf on $\mathscr{U}$ so by **SH1**, we get $\xi_U\big|_{U'} = \xi_{U'}$, as desired. Thus (ξ_U) defines an element of $G(V)$. $\square$

Exercise 4.25. Show that a morphism of sheaves defined on a basis of open sets extends uniquely to a morphism defined on the whole space.

5 Ringed spaces and schemes

Reading 5.1. [Vak14, §§3.2, 3.4, 3.5, 4.1, 4.3], [Mum99, §II.1], [Har77, pp. 69–74]

Definition 5.2 (Ringed space)**.** A *ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of commutative rings on X.

We usually write X for a ringed space $(X, \mathcal{O}_X)$, effectively using the same symbol for both the underlying space and the space together with its structure sheaf. This is an abuse of terminology, but usually doesn't cause too much trouble. If we must distinguish X from its underlying topological space, we write $|X|$ for the topological space.

There are many familiar examples:

Not an important exercise to write up carefully, but it is a very good idea to think about what has to be checked in these exercises.

Exercise 5.3. The following are ringed spaces:

(i) X is a manifold and $\mathcal{O}_X$ is the sheaf of C^∞ functions on X valued in $\mathbf{R}$ or $\mathbf{C}$;

(ii) X is a topological space and $\mathcal{O}_X$ is the sheaf of continuous functions on X, valued in any topological commutative ring;

(iii) X is a complex manifold and $\mathcal{O}_X$ is the sheaf of holomorphic functions on X;

In the next exercise, we construct the structure sheaf of an affine scheme. As it will be useful later, and incurs no greater effort, we will actually construct a sheaf on $\operatorname{Spec} A$ associated to any A-module, M. The structure sheaf is the result of this construction applied to $M = A$, in which case the sheaf is denoted $\mathcal{O}_{\operatorname{Spec} A}$.

ex:structure-sheaf This exercise is very important! It illustrates all kinds of useful techniques.

Exercise 5.4. Let A be a commutative ring and let M be an A-module. Define $\widetilde{M}(D(f)) = M[f^{-1}]$ for each principal open affine $D(f) \subset \operatorname{Spec} A$.

(i) Define the restriction homomorphisms in a natural way so that this is a presheaf on the basis of principal open subsets $\operatorname{Spec} A$.

Solution. We have to construct maps $\widetilde{M}(D(f)) \to \widetilde{M}(D(g))$ whenever $D(g) \subset D(f)$. Recall that we showed a map $\operatorname{Spec} B \to \operatorname{Spec} A$ associated to $A \to B$ factors through $D(f)$ if and only if the image of f in B is invertible (Exercise 3.21). In other words, the homomorphism $A \to A[g^{-1}]$ must factor through $A[f^{-1}]$. But $D(g)$ is the image of $\operatorname{Spec} A[g^{-1}] \to \operatorname{Spec} A$ so we get a map $A[f^{-1}] \to A[g^{-1}]$. Tensoring with M gives the desired map $\widetilde{M}(D(f)) = M[f^{-1}] \to M[g^{-1}] = \widetilde{M}(D(g))$. □

ex:structure-sheaf:1a This exercise is a special case of Theorem 8.43.

(ii) Show that this presheaf is a sheaf.

def:scheme

Definition 5.5. A ringed space $(X, \mathcal{O}_X)$ is called an *affine scheme* if it is isomorphic to $(\operatorname{Spec} A, \widetilde{A})$ for some commutative ring A. A sheaf F on an affine scheme $\operatorname{Spec} A$ is called *quasicoherent* if there is an A-module M such that $F \simeq \widetilde{M}$.

A *scheme* is a ringed space that has an open cover by affine schemes.[2] A sheaf F on a scheme is called *quasicoherent* if its restriction to each affine open subscheme in a cover is quasicoherent.

[2]Warning: In older literature, what is today called a 'scheme' was called a 'prescheme'. The word 'scheme' was reserved for what is today called a 'separated scheme'.

5.1 Quasicoherent sheaves

Definition 5.6. Let X be a topological space with a sheaf of rings $\mathcal{O}_X$. A sheaf of $\mathcal{O}_X$-modules is a sheaf of abelian groups $\mathscr{F}$ on X along with a map $\mathcal{O}_X \times \mathscr{F} \to \mathscr{F}$ such that, for every open $U \subset X$, the maps $\mathcal{O}_X(U) \times \mathscr{F}(U) \to \mathscr{F}(U)$ make $\mathscr{F}(U)$ into a $\mathcal{O}_X(U)$-module.

Definition 5.7. Suppose that X is a scheme. A sheaf of $\mathcal{O}_X$ is said to be *quasicoherent* if, for each open affine subscheme $U = \operatorname{Spec} A$ of X, the sheaf $\mathscr{F}\big|_U$ on U is isomorphic to $\widetilde{M}$ for some A-module M.

Exercise 5.8. Prove that a sheaf of $\mathcal{O}_X$-modules is quasicoherent if and only if its restriction to every open subset in an open cover is quasicoherent.

Definition 5.9. More generally, a sheaf $\mathscr{F}$ on a ringed space $(X, \mathcal{O}_X)$ is said to be *quasicoherent* if can be given locally by generators and relations. That is, if there is a cover of X by open subsets U such that there are exact sequences of $\mathcal{O}_U$-modules:

$$\mathcal{O}_U^{\oplus r} \to \mathcal{O}_U^{\oplus s} \to \mathscr{F} \to 0$$

The direct sums are not necessarily finitely indexed.

5.2 Descent

The following exercises guide you through one solution to Exercise 5.4 (ii). While this solution is longer than the others that follow, it is the simplest conceptually and has the added benefit to generalize to the faithfully flat topology.

ex:subcover-1

Exercise 5.10. Let F be a presheaf on a basis $\mathscr{U}$ for a topological space X.

(i) Show that F satisfies **SH1** if and only if

$$F(U) \to \prod_{i \in I} F(U_i) \tag{5.1}$$ eqn:8

is injective whenever $U = \bigcup U_i$ is an open cover of U in $\mathscr{U}$.

(ii) Show that a particular instance of (8.2) is injective if and only if there is a subcollection $J \subset I$ such that $\bigcup_{i \in J} U_i = U$ and

$$F(U) \to \prod_{i \in J} F(U_i) \tag{5.2}$$ eqn:11

is injective.

ex:subcover-2

Exercise 5.11. Assume that F is a separated presheaf (this means F satisfies **SH1**) on a basis $\mathscr{U}$ for a topological space X that is closed under intersections.

(i) Show that F satisfies **SH2** if and only if

$$F(U) \to \prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j \in I} F(U_i \cap U_j) \tag{5.3}$$ eqn:9

is exact[3] whenever $U = \bigcup U_i$ is an open cover of U in $\mathscr{U}$. (Note: Make sure you understand what all of the maps are in this diagram!)

[3]A diagram of sets $A \xrightarrow{f} B \overset{g}{\underset{h}{\rightrightarrows}} C$ is said to be *exact* if f is injective and the image of f is exactly the collection of all $b \in B$ such that $g(b) = h(b)$. This condition is equivalent to exactness of the sequence $0 \to A \xrightarrow{f} B \xrightarrow{g-h} C$ when the objects are abelian groups.

(ii) Show that a particular instance of (∗) is exact if and only if there is a subcollection $J \subset I$ such that $\bigcup_{i \in J} U_i = U$ and the sequence

$$F(U) \to \prod_{i \in J} F(U_i) \rightrightarrows \prod_{i,j \in J} F(U_i \cap U_j) \tag{5.4}$$

eqn:10

is exact.

Solution. Suppose that $J \subset I$ and (5.4) is exact. Injectivity of the first arrow in (∗) is immediate from the assumption that F be a separated presheaf. Suppose $x_i \in F(U_i)$, $i \in I$ is a collection of objects such that $x_i\big|_{U_i \cap U_j} = x_j\big|_{U_i \cap U_j}$ for all $i, j \in I$. Then in particular, $x_i \in F(U_i)$ for all $i \in J$ so by the exactness of (5.4), there is an $x \in F(U)$ such that $x\big|_{U_i} = x_i$ *for all* $i \in J$. We need to show that $x\big|_{U_i} = x_i$ for all $i \in I$.

Fix some $i \in I$. Note that $\bigcup_{j \in J}(U_i \cap U_j) = U_i$. Therefore by **SH1**, to check that $x\big|_{U_i} = x_i$, it is sufficient to check that $x\big|_{U_i}\big|_{U_i \cap U_j} = x_i\big|_{U_i \cap U_j}$ for all $j \in J$. But we have

$$x\big|_{U_i}\big|_{U_i \cap U_j} = x\big|_{U_j}\big|_{U_i \cap U_j} = x_j\big|_{U_i \cap U_j} = x_i\big|_{U_i \cap U_j}$$

as desired. □

Exercise 5.12. Combine the previous two exercises to show that a presheaf on a basis $\mathscr{U}$ of quasicompact open subsets that is closed under intersection is a sheaf if and only if the sequences (∗) are exact whenever $U = \bigcup U_i$ is a finite cover in $\mathscr{U}$.

Exercise 5.13. Generalize the last two exercises to apply to all bases, not just those closed under intersections.

ion-faithfully-flat

Exercise 5.14. Let A be a commutative ring. Let J be a subset of A that generates a A as an ideal. Show that a sequence of A-modules

$$M' \to M \to M''$$

is exact if and only if the sequence of localized modules[4]

$$M'_f \to M_f \to M''_f$$

is exact for all $f \in J$.

structure-sheaf-bis

Exercise 5.15. Let A be a commutative ring and let M be an A-module. Prove that $\widetilde{M}$ is a sheaf on the basis $\mathscr{U}$ of principal open affine subsets of Spec A:

(i) Reduce the problem to showing that the sequence

$$M \longrightarrow \prod_{i \in I} M[f_i^{-1}] \Longrightarrow \prod_{i,j \in I} M[f_i^{-1}, f_j^{-1}] \tag{5.5}$$

eqn:13

is exact whenever I is a subset of A such that $IA = A$.

[4]Recall that $M_f = A[f^{-1}] \otimes_A M$ is the module over $A[f^{-1}]$ induced by M. It can be constructed explicitly as the set of symbols $f^{-n}x$ with $x \in M$, subject to the relation $f^{-n}x = f^{-m}y$ if there is some k such that $f^k(f^m x - f^m y)$. It can also be constructed as the direct limit $\varinjlim_{n \in \mathbf{N}} f^{-n}M$, where $f^{-n}M = M$ for all M and the map $f^{-n}M \to f^{-m}M$ for $n < m$, sends $f^{-n}x \in f^{-n}M$ to $f^{-m}f^{m-n}x \in f^{-m}M$.

(ii) Prove that the exactness of (5.5) is equivalent to the exactness of the sequences

$$M_f \longrightarrow \prod_{i\in I} M_f[f_i^{-1}] \rightrightarrows \prod_{i,j\in I} M_f[f_i^{-1}, f_j^{-1}] \qquad (**_I)$$ eqn:12

for all $f \in I$. (Note: $A_f = A[f^{-1}]$. The mixed notation is just to make the equation look prettier.) (Warning: Be careful about commuting localization with products.)

Solution. Since $\bigcup D(f_i) = \operatorname{Spec} A$, we can find a finite subcover $J \subset I$. By Exercises 5.10 and 5.11, the exactness of (5.5) is equivalent to the exactness of $(*_J)$, the sequence associated to the cover J. Note that in this case the products in (5.5) are *finite*, so they commute with localization. Therefore by Exercise 5.14, the exactness of $(*_J)$ is equivalent to the exactness of $(**_J)$. Then by Exercise 5.10 and 5.11, again, the exactness of $(**_J)$ is equivalent to the exactness of $(**_I)$. □

(iii) Prove that the sequences $(**_I)$ are exact. (Hint: You can do this by explicitly splitting the sequence or by using a chain homotopy. There is a way to do this that doesn't require any messy algebra, using Exercises 5.10 and 5.11.)

Solution. The map $M_f \to \prod M_f[f_i^{-1}]$ can be interpreted as

$$\widetilde{M}(D(f)) \to \prod_{i\in I} \widetilde{M}(D(f)\cap D(f_i)).$$

By Exercise 5.10, to show the injectivity, it is sufficient to prove the injectivity after passage to a subcollection of I. Just take $\{f\} \subset I$, in which case the map is an isomorphism!

This proves the injectivity part of $(**_I)$, from which it follows that $\widetilde{M}$ is a separated presheaf. We can now apply Exercise 5.11 to the sequence

$$\widetilde{M}(D(f)) \longrightarrow \prod_{i\in I} \widetilde{M}(D(f)\cap D(f_i)) \rightrightarrows \widetilde{M}(D(f)\cap D(f_i)\cap D(f_j)).$$

Replace I with the subcollection $\{f\}$ and the sequence becomes

$$\widetilde{M}(D(f)) \longrightarrow \prod_{i\in I} \widetilde{M}(D(f)) \rightrightarrows \widetilde{M}(D(f)),$$

which is obviously exact! □

Exercise 5.14 showed that if $\operatorname{Spec} A = \bigcup D(f_i)$ then the rings $A[f_i^{-1}]$ are a faithfully flat collection of A-algebras:

Definition 5.16. Suppose that A is a commutative ring. A collection of A-algebras B_i, indexed by a set I, is said to be *faithfully flat* if it is equivalent for a sequence (5.6)

$$M' \to M \to M'' \qquad (5.6)$$ eqn:17

of A-modules to be exact or for the sequences (30.4)

$$B_i \otimes_A M' \to B_i \otimes_A M \to B_i \otimes_A M'' \qquad (5.7)$$ eqn:18

to be exact for all i.

Exercise 5.17. Verify that the proof outlined in Exercise ?? used nothing more than that the collection of A-algebras $A[f_i^{-1}]$ had a refinement by a finite, faithfully flat subcollection.

5.3 Partitions of unity

This section guides you through another solution to Exercise (ii) that shares some spiritual similarity with the partition of unity arguments that appear in differential geometry.

Exercise 5.18. Reduce the problem to showing that the sequence

$$0 \to M \to \prod_{i \in I} M[f_i^{-1}] \to \prod_{i,j \in I} M[f_i^{-1}, f_j^{-1}] \tag{5.8}$$ eqn:14

is exact for whenever $I \subset A$ and $IA = A$. Make sure you know what the maps in this sequence are before you try to prove anything!

Exercise 5.19. (i) If $x \in M$ and x restricts to zero in $M[f_i^{-1}]$ then $f_i^n x = 0$ for some $n \geq 0$.

(ii) If $(f_1, \ldots, f_k) = A$ then $(f_1^{n_1}, \ldots, f_k^{n_k}) = A$ as well.

Solution. The radical of $(f_1^{n_1}, \ldots, f_k^{n_k})$ is the same as the radical of $(f_1, \ldots, f_k)$. But $1 \in (f_1, \ldots, f_k)$ so $1^n \in (f_1^{n_1}, \ldots, f_k^{n_k})$, i.e., $(f_1^{n_1}, \ldots, f_k^{n_k}) = A$. □

Exercise 5.20. (i) Prove the exactness of (5.8) at M.

Solution. In the sheaf conditions, we can assume $U = \operatorname{Spec} A$. Suppose $g, h \in \widetilde{M}(\operatorname{Spec} A)$ and $g\big|_{D(f_i)} = h\big|_{D(f_i)}$. Then $f_i^{k_i}(g-h) = 0$ so there is a single n such that $f_i^n(g-h) = 0$. But $\sum a_i f_i^n = 1$ for some a_i. Thus $\sum a_i f_i^n (g-h) = g - h = 0$, i.e., $g = h$. □

(ii) Prove the exactness of (5.8) at $\prod_{i \in I} A[f_i^{-1}]$.

Solution. Suppose $g_i \in \widetilde{M}(D(f_i))$. For each i, choose $h_i \in \widetilde{M}(U)$ such that $h_i\big|_{D(f_i)} = f_i^n g_i$. (One can assume there are finitely many f_i.) Then $h_i\big|_{D(f_i f_j)} = h_j\big|_{D(f_i f_j)}$ for all i and j. Choose m such that $f_j^m h_i\big|_{D(f_i)} = f_j^m h_j\big|_{D(f_i)}$ for all i, j. (Again, using that there are finitely many indices.) Then take a_i such that $\sum a_i f_i^{n+m} = 1$. Take $g = \sum a_i f_i^m h_i$. Then

$$g\big|_{D(f_i)} = \sum a_j f_j^m h_j\big|_{D(f_i)} = \sum a_j f_j^m h_i\big|_{D(f_i)} = \sum a_j f_j^{n+m} g_i = g_i$$

as desired. □

5.4 Chain homotopies

We give a third solution of Exercise 5.4 (ii).

Exercise 5.21. Suppose that $K^\bullet$ is a complex of A-modules with differential $d_i : K^i \to K^{i+1}$ and that there are morphisms $s_i : K^i \to K^{i-1}$ for all i satisfying $d_{i-1}s_i + (-1)^i s_{i+1} d_i$. Prove that $K^\bullet$ is exact.

Exercise 5.22. Prove that the sequences

$$0 \to M[f_k^{-1}] \otimes_k \mathcal{O}(U) \to M[f_k^{-1}] \otimes_M \prod_i \mathcal{O}(U_i) \to M[f_k^{-1}] \otimes_M \prod_{i,j} \mathcal{O}(U_i \cap U_j)$$

are exact using the method from the previous exercise and use this to complete the proof of Exercise 5.4 (ii).

Solution. We want to prove the sequence

$$0 \to \mathcal{O}(U) \to \prod_i \mathcal{O}(U_i) \to \prod_{i,j} \mathcal{O}(U_i \cap U_j)$$

is exact. This can be verified locally: It is sufficient to check that the sequence

$$0 \to M[f_k^{-1}] \otimes_k \mathcal{O}(U) \to M[f_k^{-1}] \otimes_M \prod_i \mathcal{O}(U_i) \to M[f_k^{-1}] \otimes_M \prod_{i,j} \mathcal{O}(U_i \cap U_j)$$

is exact. This sequence simplifies (using the finiteness of the products):

$$0 \to M[f_k^{-1}] \to \prod_i M[f_k^{-1}, f_i^{-1}] \to \prod_{i,j} M[f_k^{-1}, f_i^{-1} f_j^{-1}]$$

Now we construct a chain homotopy between the identity and the zero map. Define:

$$h_0 : \prod_i M[f_k^{-1}, f_i^{-1}] \to M[f_k^{-1}] : (a_i) \mapsto a_k$$
$$h_1 : \prod_{i,j} M[f_k^{-1}, f_i^{-1}] \to \prod_i M[f_k^{-1}, f_i^{-1} : (a_{i,j})_{i,j} \mapsto (a_{i,k})_i.$$

Then we check:

$$\begin{aligned}
(dh_{-1} + h_0 d)(a) &= h_0 d(a) = h_0(a\big|_{D(f_k f_i)})_i = a \\
(dh_0 + h_1 d)((a_i)_i) &= d(a_k) + h_1((a_i\big|_{D(f_k f_i f_j)} - a_j\big|_{D(f_k f_i f_j)})_{i,j}) \\
&= (a_k\big|_{D(f_k f_i)})_i + (a_i\big|_{D(f_k f_i)} - a_k\big|_{D(f_k f_i)})_i \\
&= (a_i)_i.
\end{aligned}$$

□

Chapter 3

First properties of schemes

6 Examples

Reading 6.1. [Vak14, §§4.4]

6.1 Open subschemes

Exercise 6.2. Show that an open subset of a scheme is equipped with the structure of a scheme in a natural way. (Hint: Restriction of a sheaf is a sheaf.)

6.2 Affine space

Definition 6.3 (Affine space)**.** The scheme $\operatorname{Spec} \mathbf{Z}[x_1, \ldots, x_n]$ is denoted $\mathbf{A}^n$ and is called *n-dimensional affine space.*

6.3 Gluing two affine schemes

Exercise 6.4. Suppose that X and Y are two schemes and $U \subset X$ and $V \subset Y$ are open subsets such that $U \simeq V$ and under this isomorphism $\mathcal{O}_U \simeq \mathcal{O}_V$. Construct a scheme Z whose underlying topological space is the union of X with Y along $U \simeq V$ and for which $\mathcal{O}_Z\big|_X = \mathcal{O}_X$ and $\mathcal{O}_Z\big|_Y = \mathcal{O}_Y$. (The statement of this exercise is deliberately vague in several ways. Part of your job is to make it precise.)

Exercise 6.5. Using the notation of the last exercise, let k be a field (you may find it easier to assume k is algebraically closed) and let $X = \mathbf{A}^1_k = \operatorname{Spec} k[x]$ and let $Y = \mathbf{A}^1_k = \operatorname{Spec} k[y]$. There is an open subset $U = D(x) \subset X$ and $V = D(y) \subset Y$.

(i) Construct two distinct homeomorphisms $U \simeq V$ and corresponding identifications $\mathcal{O}_U \simeq \mathcal{O}_V$. (Hint: One should correspond to $x = y$ and one should correspond to $x = y^{-1}$.)

(ii) Apply the last exercise to obtain a scheme Z for each of these two isomorphisms. Describe these spaces qualitatively and explain why they are different. (Hint: Consider a point $\xi \in X$. Move ξ so that $x(\xi)$ approaches 0. Move ξ so that $x(\xi)$ approaches ∞.)

6.4 Gluing more than two affine schemes

Exercise 6.6. How should this construction be modified when gluing 3 or more affine schemes along open subsets?

6.5 Projective space

In topology, $\mathbf{CP}^n$ is the set of 1-dimensional subspaces of $\mathbf{C}^{n+1}$. It is topologized as the quotient of $\mathbf{C}^{n+1} \smallsetminus \{0\}$ by $\mathbf{C}^*$. For each $i = 0, \ldots, n$, let $V_i \subset \mathbf{C}^{n+1}$ be the span of the n coordinate vectors excluding e_i. Then $e_i + V_i$ consists of all vectors whose i-th coordinate is 1. The image of $e_i + V_i$ is an open subset U_i of $\mathbf{CP}^n$ and this gives a system of charts for $\mathbf{CP}^n$ as a complex manifold.

We can't imitate all of this algebraically, at least not yet. However, we can imitate the charts.

For each i, let $U_i = \operatorname{Spec} A_i$ where

$$A_i = \mathbf{Z}[x_{0/i}, \ldots, x_{n/i}]/(x_{i/i} - 1).$$

The choice of notation makes the gluing that is about to happen easier. Later on, we will see that there is a way to think about $x_{k/i}$ as x_k/x_i in some bigger ring, but introducing this notation now would probably be misleading. Let

$$A_{ij} = A[x_{j/i}^{-1}].$$

Exercise 6.7. Verify that there is an identification $A_{ij} \simeq A_{ji}$ sending

$$x_{k/i} \mapsto x_{k/j} x_{i/j}^{-1}.$$

Let

$$A_{ijk} = A[x_{j/i}^{-1}, x_{k/i}^{-1}].$$

Note that $A_{ijk} = A_{ikj}$ and that the exercise above gives induces an isomorphism $A_{ijk} \to A_{jik}$ for any i, j, k.

Exercise 6.8. The previous exercise gives *two* identifications between A_{ijk} and A_{kji}:

$$A_{ijk} \to A_{jik} = A_{jki} \to A_{kji}$$
$$A_{ijk} = A_{ikj} \to A_{kij} = A_{kji}$$

Show that these two maps are the same.

Exercise 6.9. Use these identifications to glue the $U_i = \operatorname{Spec} A_i$ together into a scheme, $\mathbf{P}^n$.

Projective space is extremely important because almost every scheme one encounters in practice can be constructed as the intersection of an open subscheme and a closed subscheme of projective space.

6.6 The Grassmannian

There is also a space whose points correspond to the k-dimensional subspaces of $\mathbf{C}^n$, called the *Grassmannian*, $\mathbf{Grass}(k,n)$. One way to construct the Grassmannian is to take the set $V \subset \mathbf{C}^{n\times k}$ of all $n \times k$ matrices whose columns are linearly independent.

This exercise may be skipped. We will see later that V is open in the Zariski topology.

Exercise 6.10. Check that the set $V \subset \mathbf{C}^{n\times k}$ is open.

The group $\mathrm{GL}(k)$ of invertible $k \times k$ matrices acts on the V by multiplication on the right. The orbits of this group action correspond to k-dimensional subspaces of $\mathbf{C}^n$, so $\mathbf{Grass}(k,n) = V/\operatorname{GL}(k)$.

This constructs $\mathbf{Grass}(k,n)$ as a topological space. In order to get the structure of a complex manifold, we construct charts. For each subset $I \subset \{1,\ldots,n\}$, let $V_I \subset V$ be the subset whose $I \times k$ submatrix is the identity.

Exercise 6.11. For each $I \subset \{1,\ldots,n\}$, construct a bijection $V_I \simeq \mathbf{C}^{(n-k)\times k}$.

The maps $V_I \to \mathbf{Grass}(k,n)$ are injective, and we can use them as charts for the structure of a complex manifold.

We can't yet imitate the quotient construction for the Grassmannian but we can imitate the charts.

For each $I \subset \{1,\ldots,n\}$, we construct a ring:

$$A_I = \mathbf{Z}\big[x_{ij/I} \big| 1 \le i \le n, 1 \le j \le k\big] \big/ \big(x_{ii/I} - 1, x_{ij} \big| i,j \in I, i \ne j\big)$$

If $J \subset \{1,\ldots,n\}$ is another k-element subset (potentially the same as I). Let M_{IJ} be the $J \times k$ matrix whose entries are the variables $x_{ij/I}$ with $i \in J$. Let

$$A_{J/I} = A_I\big[\det(M_{J/I})^{-1}\big].$$

Exercise 6.12. Verify that there is an isomorphism of commutative rings $\phi_{JI} : A_{I/J} \to A_{J/I}$ by the map

$$\big(x_{ij/J}\big)_{\substack{1\le i\le n\\ 1\le j\le k}} \mapsto \big(x_{ij/I}\big)_{\substack{1\le i\le n\\ 1\le j\le k}} M_{J/I}^{-1}.$$

If $K \subset \{1,\ldots,n\}$ is yet another k-element subset, let $A_{JK/I}$ be

$$A_{JK/I} = A_I\big[\det(M_{J/I})^{-1}, \det(M_{K/I})^{-1}\big].$$

Note that $A_{JK/I} = A_{KJ/I}$. The formula for ϕ_{JI} also gives an isomorphism $A_{IK/J} \to A_{JK/I}$ and we now have two isomorphisms:

$$A_{JK/I} \xrightarrow{\phi_{IJ}} A_{IK/J} \xrightarrow{\phi_{JK}} A_{IJ/K}$$
$$A_{JK/I} \xrightarrow{\phi_{IK}} A_{IJ/K}$$

Exercise 6.13. Verify that these two isomorphisms are the same.

Exercise 6.14. Use these isomorphisms to glue the $U_I = \operatorname{Spec} A_I$ into a scheme, $\mathbf{Grass}(k,n)$.

7 Absolute properties of schemes

Most useful properties of schemes are *relative*, meaning they may be applied to families of schemes. We can't talk about relative properties yet since we haven't yet defined morphisms of schemes, so we'll only introduce a limited array of definitions.

Reading 7.1. [Vak14, §§3.3, 3.6], [Har77, §II.3]

7.1 Connectedness

Definition 7.2. A scheme X is *connected* if its underlying topological space is connected.

Worth being aware of, not necessarily important to write up.

Exercise 7.3.

(i) Show that a scheme X is disconnected if and only if $\Gamma(X, \mathcal{O}_X)$ contains an idempotent element other than 0 and 1. (Hint: Let e be an idempotent. Then $D(e)$ and $D(1-e)$ are disjoint open subsets whose union is X.)

(ii) Show that $\operatorname{Spec}(A \times B) = \operatorname{Spec}(A) \amalg \operatorname{Spec}(B)$.

Solution. Let $e_1 = (1,0)$ and $e_2 = (0,1)$ inside $C = A \times B$. Then $(e_1, e_2)C = C$, so $D(e_1) \cup D(e_2) = \operatorname{Spec} C$. Also, $V(e_1, e_2) = V(C) = \varnothing$, so $D(e_1) \cap D(e_2) = \varnothing$. This shows $\operatorname{Spec} C = \operatorname{Spec} A \amalg \operatorname{Spec} B$ as a topological space. The open subset $D(f) \subset \operatorname{Spec} A$ corresponds to $D(f, 0) \subset \operatorname{Spec} C$ and $D(g) \subset \operatorname{Spec} B$ corresponds to $D(0, g) \subset \operatorname{Spec} C$.

To check that this is an identification of schemes, we need to check that the structure sheaves agree. We have

$$\mathcal{O}_{\operatorname{Spec} C}(D(f,0)) = C[(f,0)^{-1}] = C[e_1^{-1}][(f,0)^{-1}] = A[f^{-1}] = \mathcal{O}_{\operatorname{Spec} A}(D(f))$$
$$\mathcal{O}_{\operatorname{Spec} C}(D(0,g)) = C[(0,g)^{-1}] = C[e_2^{-1}][(0,g)^{-1}] = B[g^{-1}] = \mathcal{O}_{\operatorname{Spec} A}(D(g)).$$

□

An exercise for the logically oriented. The first 4 parts should be easy. The last is pretty hard.

Exercise 7.4. Let A_i be a collection of commutative rings, indexed by i in a set I.

(i) Construct a map $\coprod \operatorname{Spec} A_i \to \operatorname{Spec} \prod A_i$.

Solution. We have $\prod A_i \to A_i$ for all i, giving $\operatorname{Spec} A_i \to \operatorname{Spec} \prod A_i$ for all i. This gives the desired map by the universal property of the coproduct. □

(ii) Show that this map is an isomorphism if I is a finite set. (Hint: Use an earlier exercise.)

(iii) Show that this map is always injective.

Solution. Let p and q be points of $\coprod \operatorname{Spec} A_i$ with the same image. Then $p \in \operatorname{Spec} A_i$ for some i and $q \in \operatorname{Spec} A_j$ for some j. If p and q have the same image then the two maps $\prod \operatorname{Spec} A_i \to k(p)$ and $\prod \operatorname{Spec} A_i \to k(q)$ have the same kernel. In particular, $i = j$. Then the map $\prod \operatorname{Spec} A_i \to k(p) = k(q)$ factors through the projection on A_i, which implies $p = q$ in $\operatorname{Spec} A_i$. □

(iv) Show that this is *not* an isomorphism if I is infinite. (Hint: Affine schemes are quasicompact.)

(v) Construct an element of $\operatorname{Spec} \prod A_i$ that is not in the image of $\coprod \operatorname{Spec} A_i$. (Hint: I don't suggest attempting this problem unless you know what an ultrafilter is.)

7.2 Quasicompactness

Definition 7.5. A scheme X is said to be *quasicompact* if every open cover of X has a finite subcover.

This is a repeat of Exercise 3.14.

Exercise 7.6. Show that every affine scheme is quasicompact.

Not very important, should be easy.

Exercise 7.7.

(i) Construct an example of a scheme that is not quasicompact.

Solution. An infinite disjoint union of nonempty affine schemes. □

(ii) Construct an example of a connected scheme that is not quasicompact.

Solution. The complement of the origin in $\mathbf{A}^\infty$. □

Exercise 7.8. (i) Show that a scheme with a finite cover by affine open subschemes is quasicompact.

(ii) Conclude that projective space is quasicompact.

7.3 Quasiseparatedness

Definition 7.9. A scheme X is said to be *quasiseparated* if the intersection of any two quasicompact open subsets of X is quasicompact.

Exercise 7.10. Give an example of a scheme that is not quasiseparated.

Solution. Join $\mathbf{A}^\infty = \operatorname{Spec} k[x_1, x_2, x_3, \ldots]$ to itself along the complement of $V(x_1, x_2, x_3, \ldots)$. The intersection of the two copies of $\mathbf{A}^\infty$ is $\mathbf{A}^\infty \smallsetminus \{0\}$. This is not quasicompact. □

7.4 Nilpotents

Definition 7.11. Recall that a commutative ring is said to be *reduced* if it contains no nonzero nilpotent elements. A scheme X is said to be *reduced* if $\mathcal{O}_X(U)$ is a reduced ring for all open subsets U of X.

ex:reduced-scheme

Exercise 7.12. Let X be any scheme. Construct a reduced scheme X_{red} with the same underlying topological space as X by replacing $\mathcal{O}_X(U)$ by its associated reduced ring.

Definition 7.13. A scheme is said to be *integral* if it is reduced and irreducible.

7.5 Irreducibility

Definition 7.14. A scheme X is *reducible* if its underlying topological space is the union of two closed subsets, neither of which is equal to X. Otherwise it is *irreducible*.

Important, but should be easy.

Exercise 7.15. Show a scheme X is irreducible if and only if every pair of open subsets of X have non-empty intersection.

Solution. If X is reducible, say $X = Y \cup Z$. Let $U = X \smallsetminus Y$ and $V = X \smallsetminus Z$. Then $U \cap V = X \smallsetminus (Y \cup Z)$ is empty if and only if $Y \cup Z = X$. □

Important, but should be easy.

Exercise 7.16. Show that an affine scheme $X = \operatorname{Spec} A$ is reducible if and only if A contains a non-nilpotent divisor of zero.

Solution. Suppose $|X| = |V(I)| \cup |V(J)|$ and neither $|V(I)|$ nor $|V(J)|$ is contained in the other. We can assume I and J are radical ideals. Then neither I nor J is contained in the other. Choose $f \in I \smallsetminus J$ and $g \in J \smallsetminus I$. Then $fg \in I(|V(I)|) \cap I(|V(J)|)$ so fg is nilpotent. Say $(fg)^n = 0$. Then $f^n g^n = 0$. But neither f^n nor g^n is nilpotent because all nilpotents appear in both I and J.

Conversely, if $fg = 0$ then $|X| = |V(f)| \cup |V(g)|$. If f and g are not nilpotent then $|V(f)| \neq |X|$ and $|V(g)| \neq |X|$. □

This is a good one to do. It's not extremely important, but it requires putting together a few different ideas without being too difficult.

Exercise 7.17 ([Har77, Proposition II.3.1])**.** Prove that a reduced scheme X is irreducible if and only if $\mathcal{O}_X(U)$ is an integral domain for all open $U \subset X$.

Solution. Assume X is integral. Let U be an open subset of X. Then U is irreducible. Suppose $fg = 0$ in $\mathcal{O}_X(U)$. Then $V_U(f) \cup V_U(g) = U$ so $V_U(f) = U$ or $V_U(g) = U$. Without loss of generality, assume the former. Then for every open affine subset $U' \subset U$, we have $V_{U'}(f\big|_{U'}) = U'$. Hence $f\big|_{U'}$ is nilpotent in $\mathcal{O}_X(U')$ and X is reduced so $f\big|_{U'} = 0$. Thus $f = 0$, since $\mathcal{O}_X$ is a sheaf. This proves $\mathcal{O}_X(U)$ is an integral domain.

Conversely, suppose that $X = Y \cup Z$ for two closed subsets Y and Z, neither containing the other. Let $U = V \amalg W$ where V is a nonempty open affine subset of $Y \smallsetminus Z$ and W is an open affine subset of $Z \smallsetminus Y$. Then $\mathcal{O}_X(U) = \mathcal{O}_X(V) \times \mathcal{O}_X(W)$ is not an integral domain. □

7.6 Noetherian and locally noetherian schemes

Definition 7.18. A scheme that has an open cover by spectra of noetherian rings is called *locally noetherian*. If the cover can be chosen to be finite then the scheme is said to be *noetherian*.

A useful fact to know. The first part should be easy. The second part uses a trick you should become familiar with if you aren't already.

Exercise 7.19.

(i) Show noetherian is equivalent to the conjunction of locally noetherian and quasi-compact.

(ii) Show that $\operatorname{Spec} A$ is locally noetherian if and only if A is a noetherian ring. (Hint: Two ideals that are locally the same are the same.)

Solution. Choose $f_1, \ldots, f_n \in A$ such that $A[f_i^{-1}]$ is noetherian for all i. Let I_j be an ascending collection of ideals in A. The chains $I_j A[f_i^{-1}]$ all stabilize for sufficiently large j. We can choose j to work for all i, since there are only finitely many f_i. Thus there is a j_0 such that $I_j A[f_i^{-1}]$ is constant for all i and all $j \geq j_0$.

We argue this means I_j is constant for $j \geq j_0$. This reduces to the following assertion: If I and J are ideals and $IA[f_i^{-1}] = JA[f_i^{-1}]$ for all i then $I = J$. Choose $x \in I$. Then for each i we can find n_i such that $f_i^{n_i} x \in J$. Choose a_i such that $\sum a_i f_i^{n_i} = 1$. Then $x = \sum a_i f_i^{n_i} x \in J$. Thus $I \subset J$ and the same argument shows that $J \subset I$ as well. □

This exercise and the next are essentially the same. Think about both, but don't write up more than one of them.

Exercise 7.20. If X is a noetherian scheme then every open subset of X is quasi-compact. (In fact, every subset whatsoever is quasicompact, and the proof isn't any harder.)

Solution. Choose a finite cover of X by affine opens U, each noetherian. Consider an ascending collection of open subsets of X. This stabilizes in each affine open U in the cover. Since there are only finitely many in the cover, the whole chain stabilizes. □

Exercise 7.21. Show that the underlying topological space of a noetherian scheme is noetherian: *any* increasing union of open subsets stabilizes.

Solution. This is true in an open affine subscheme, and there is a finite cover by open affines. □

Important fact!

Exercise 7.22 (Irreducible components)**.** Prove that a noetherian scheme is the union of finitely many irreducible closed subsets. Conclude that a noetherian ring has finitely many minimal prime ideals.

Solution. We prove there is at least one irreducible closed subset of X with nonempty interior. If X is irreducible this is obvious. Otherwise, $X = Y \cup Z$, with Y and Z both closed, neither equal to X. We can assume that $Y \cap Z$ does not meet the interior of either Y or Z. By noetherian induction, at least one of these has an irreducible closed subset with nonempty interior, which is not contained in the intersection $Y \cap Z$, hence is a closed subset of X with nonempty interior.

Now pick an irreducible closed subset Z of X with nonempty interior. Let X_1 be the closure of its complement. Repeat with X replaced by X_1 to get X_2. We get a descending chain of closed subsets of X, which must stabilize. But it can't stabilize to something other than the empty set, which completes the proof. □

Exercise 7.23 (Noetherian induction (cf. [Har77, Exercise 3.16]))**.** Let X be a noetherian scheme and S a collection of closed subsets of X. Assume that whenever $Z \subset X$ is closed and S contains all proper closed subsets of Z, the set Z also appears in S. Prove that X appears in S.

Solution. Let T be the complement of S among closed subsets of X. If T is nonempty, we can choose a maximal descending collection in T. Since X is noetherian, this descending chain has a smallest element. Call it Z. Then every closed subset of Z other than Z itself is in S. Therefore Z is in S. This is a contradiction, so T must have been empty. Thus S contains all closed subsets of X, in particular X itself. □

Exercise 7.24. Find equations defining the union of the 3 coordinate axes in $\mathbf{A}^3$.

7.7 Generic points

Important fact. The argument is fairly straightforward topology, hence not too important to write up.

Exercise 7.25. Each irreducible closed subset Z of a scheme X has a unique point that is dense in Z. This is called the *generic point* of Z.

Solution. Let Z be an irreducible closed subset of X and let $U \subset X$ be an affine open subset whose intersection with Z is nonempty. Then $Z \cap U$ is dense in U because Z is irreducible. If $Z \cap U = A \cup B$ for two closed subsets A and B then $Z = \overline{A \cup B} = \bar{A} \cup \bar{B}$ (closure commutes with finite unions) so $Z = \bar{A}$ or $Z = \bar{B}$, say the former. Then A is dense in Z, so A is dense in U, and as A is closed in U, this means $A = U$. This shows U is irreducible.

Each irreducible closed subset of an affine scheme has a unique generic point (Exercise 3.29). Then the generic point of $U \cap Z$ is dense in $U \cap Z$, hence in Z.

We still have to prove that the generic point is unique. Suppose that Z contains two generic points ξ and η. Let $U \subset Z$ be an affine open subset. Then ξ and η are both contained in $U \cap Z$ because ξ and η are contained in *every* open subset of Z. Then $\xi = \eta$ because irreducible closed subsets of affine schemes have *unique* generic points (Exercise 3.29 again). □

7.8 Specialization and generization

Definition 7.26. Suppose x and y are points of a scheme X. If y lies in the closure of x then we say x *specializes* to y and y *generizes* to x. We often write $x \rightsquigarrow y$ for this.

ex:specialization Important fact, not important to write up.

Exercise 7.27.

(i) Show that closed subsets are stable under specialization and open subsets are stable under generization.

(ii) Give an example of a subset of a scheme that is stable under generization but not open and an example of a subset that is stable under specialization but not closed.

Solution. The set of all closed points of $\mathbf{A}^1_{\mathbf{C}}$ is stable under specialization (closed points only specialize to themselves) but not closed (the only infinite closed subset of $\mathbf{A}^1_{\mathbf{C}}$ is the whole space). □

7.9 Constructible sets

Definition 7.28. A subset of an affine scheme $\operatorname{Spec} A$ is called *constructible* if it can be desribed using a finite sentence involving elements of A and the logical operations *and*, *or*, and *not*.

A subset of a scheme X is called *locally constructible* if its intersection with every affine open subscheme of X is constructible.

Exercise 7.29. On a noetherian scheme, a constructible set is a finite union of locally closed subsets.

Exercise 7.30. Let X be a scheme and $Z \subset X$ a locally constructible subset. If Z is stable under generization then Z is open. If Z is stable under specialization then Z is closed.

Solution. It is sufficient to assume $X = \operatorname{Spec} A$. Assume $Z \subset X$ is stable under specialization. Then Z is a finite union of $D(f) \cap V(g_1, \ldots, g_n)$ □

8 Faithfully flat descent

Reading 8.1. [MO, §I.5], [Har77, §II.5]

The object of this section is a rather abstract reconstruction theorem for modules under a commutative rings. It is known as *fpqc descent*, or *faithfully flat descent*. Two important corollaries are that quasicoherent sheaves are indeed sheaves in the Zariski topology, and that the category of quasicoherent sheaves on an affine scheme $\operatorname{Spec} A$ is equivalent to the category of A-modules.

8.1 Modules and diagrams

Definition 8.2. Let $\mathscr{B}$ be a category and let A be a commutative ring. A *diagram* of A-algebras is a functor $B : \mathscr{B} \to A\text{-}\mathbf{Alg}$.

Definition 8.3. Let $B : \mathscr{B} \to A\text{-}\mathbf{Alg}$ be a diagram of A-algebras. A B-*module* is a system of

(i) a $B(\alpha)$-module $\mathscr{M}(\alpha)$, for each $\alpha \in \mathscr{B}$, and

(ii) a $B(u) : B(\alpha) \to B(\beta)$ homomorphism $\mathscr{M}(u) : \mathscr{M}(\alpha) \to \mathscr{M}(\beta)$ (also notated u_*) for each $u : \alpha \to \beta$ in $\mathscr{B}$, such that

(iii) the composition $\mathscr{M}(u) \circ \mathscr{M}(v) = \mathscr{M}(uv)$ whenever u and v are composable morphisms of $\mathscr{B}$.

The diagram is called *cartesian* or *quasicoherent* if

(iv) for every $u : \alpha \to \beta$ in $\mathscr{B}$, the map

$$B(\beta) \otimes_{B(\alpha)} \mathscr{M}(\alpha) \to \mathscr{M}(\beta) : \lambda \otimes x \mapsto \lambda.u_*(x)$$

is an isomorphism.

A morphism of B-modules $\mathscr{M} \to \mathscr{N}$ is a collection of morphisms $\varphi_\alpha : \mathscr{M}(\alpha) \to \mathscr{N}(\alpha)$ such that, for every $u : \alpha \to \beta$ in $\mathscr{B}$, the diagram below commutes:

$$\begin{tikzcd} \mathscr{M}(\alpha) \arrow[r, "\varphi_\alpha"] \arrow[d, "u_*"'] & \mathscr{N}(\alpha) \arrow[d, "u_*"] \\ \mathscr{M}(\beta) \arrow[r, "\varphi_\beta"] & \mathscr{N}(\beta) \end{tikzcd}$$

The category of $\mathscr{B}$-modules is denoted $\mathscr{B}$-**Mod**. The full subcategory of quasicoherent $\mathscr{B}$-modules is denoted $\mathbf{QCoh}(\mathscr{B})$.

Exercise 8.4. (i) Suppose that A is a commutative ring and $\mathscr{M}$ is a quasicoherent module over A-**Alg**. Then $\mathscr{M}(A)$ is an A-module. Show that this gives a functor:

$$\mathbf{QCoh}(A\text{-}\mathbf{Alg}) \to A\text{-}\mathbf{Mod}$$

(ii) Show that this functor is an equivalence of categories.

Exercise 8.5. Generalizing the first part of the last exercise, suppose that $F : \mathscr{B} \to \mathscr{B}'$ is a functor, that $B' \to A$-**Alg** makes $\mathscr{B}'$ into a diagram of A-algebras, and that $B = B' \circ F$. Construct a functor

$$\mathscr{B}'\text{-}\mathbf{Mod} \to \mathscr{B}\text{-}\mathbf{Mod}$$

sending $\mathscr{M}$ to $\mathscr{M} \circ F$. Show that this takes quasicoherent modules to quasicoherent modules.

8.2 Galois theory

Definition 8.6. Let L be a finite extension field of a field K. We call L a *Galois extension* if $L \otimes_K L \simeq L^n$ for some positive integer n. The Galois group of a Galois extension is its automorphism group as a K-algebra.

Suppose that L is a finite dimensional Galois extension of K and let G be its Galois group. A G-L-module is an L-vector space M with an L-semilinear action of G. That is, we have

$$g.(x + y) = (g.x) + (g.y)$$
$$g.(\lambda x) = (g.\lambda)(g.x)$$

for all $x, y \in M$ and $g \in G$.

Exercise 8.7. Let $\mathscr{L}$ be the full subcategory of K-**Alg** spanned by L. Show that the category of G-L-modules is equivalent to the category of $\mathscr{L}$-modules (all of which are automatically quasicoherent).

If M is a K-vector space, we can get a G-L-module by taking $N = L \otimes_K M$ and defining

$$g.(\lambda \otimes x) = (g.\lambda) \otimes x.$$

Theorem 8.8 (Fundamental theorem of Galois theory)**.** *The functor* K-**Mod** $\to$ G-L-**Mod** *is an equivalence of categories.*

Exercise 8.9. Use the following steps to deduce a more familiar formulation of the fundamental theorem:

(i) Note that K-**Mod** $\to$ G-L-**Mod** preserves tensor products.

(ii) Define a (commutative) G-L-algebra to be a G-L-module M with a G-L-module homomorphism $M \otimes_L M \to M$ such that the induced map $M \times M \to M$ makes M into a commutative ring. Conclude that K-**Alg** $\simeq$ G-L-**Alg**.

(iii) Call an L-algebra *split* if it is isomorphic to $\mathrm{Hom}_{\mathbf{Sets}}(S, L) = L^S$ for some finite set S. Prove that the full subcategory of split L-algebras in L-**Alg** is equivalent to the category of sets via this functor. (Hint: the inverse functor sends A to $\mathrm{Hom}_{L\text{-}\mathbf{Alg}}(A, S)$.)

(iv) Call a G-L-algebra L*-split* if its underlying L-algebra is split. Deduce that the category of L-split G-L-algebras is equivalent to the category of G-sets.

(v) Define an L-split K-algebra to be a K-algebra A such that $L \otimes_K A$ is split as an L-algebra. Prove that the category of K-split L-algebras is equivalent to the category of L-split G-L-algebras.

(vi) Conclude that the category of L-split K-algebras is contravariantly equivalent to the category of finite G-sets.

(vii) Prove that under your equivalence, the action of G on itself corresponds to the K-algebra L and the (unique) action of G on a 1-element set corresponds to the K-algebra K. Deduce that subextensions of K correspond to transitive G-sets with a distinguished basepoint. Remark that transitive G-sets with a basepoint correspond to subgroups of G.

8.3 Quasicoherent sheaves on affine schemes

Let A be a commutative ring. Suppose that $\mathscr{U}$ is a collection of affine open subsets of $\operatorname{Spec} A$. Each affine open subset U of $\operatorname{Spec} A$ corresponds to an A-algebra, so we can think about $\mathscr{U}$ as a diagram of A-algebras:

$$\mathscr{O} : \mathscr{U} \to A\text{-}\mathbf{Alg}.$$

Theorem 8.10. *Suppose that* $\mathscr{U}$ *is a collection of affine open subsets of* $\operatorname{Spec} A$ *such that, for any finite collection* $\mathscr{V} \subset \mathscr{U}$*, the open set* $\bigcap_{V \in \mathscr{V}} V$ *can be covered by elements of* $\mathscr{U}$*. Then the functor*

$$A\text{-}\mathbf{Mod} \to \mathbf{QCoh}(\mathscr{U})$$

is an equivalence of categories. More precisely:

(i) if $\mathscr{U}$ *covers* $\operatorname{Spec} A$ *then the functor is faithful;*

(ii) if $\mathscr{U}$ *covers* $\operatorname{Spec} A$ *and contains all pairwise intersections of a subcover, then the functor is fully faithful;*

(iii) if $\mathscr{U}$ *covers* $\operatorname{Spec} A$ *and contains all triple intersections of a subcover, then the functor is an equivalence of categories.*

Exercise 8.11. Suppose that M is an A-module. Prove that $\widetilde{M}$ is a quasicoherent sheaf on the category of all principal open affine subsets of $\operatorname{Spec} A$.

Solution. Suppose that $\operatorname{Spec} A = \bigcup D(f_i)$. Let $\mathscr{U}$ consist of all pairwise intersections of the $D(f_i)$. Then

$$M = \operatorname{Hom}_{A\text{-}\mathbf{Mod}}(A, M) = \operatorname{Hom}_{\mathbf{QCoh}(\mathscr{U})}(\widetilde{A}, \widetilde{M}) = \Gamma(\mathscr{U}, \mathscr{M})$$

is the sheaf condition. □

Exercise 8.12. Suppose that $\mathscr{U}$ is a basis for $\operatorname{Spec} A$. Then $\mathbf{QCoh}(\mathscr{U}) = \mathbf{QCoh}(\operatorname{Spec} A)$.

8.4 Flatness

Definition 8.13. Let A be a commutative ring. An A-module M is called *flat* if, whenever

$$N' \to N \to N''$$

is an exact sequence of A-modules, the sequence

$$M \otimes_A N' \to M \otimes_A N \to M \otimes_A N''$$

is also exact. An A-algebra is called flat if it is flat as a module.

Exercise 8.14. Prove that an A-module M is flat if and only if, for every injection $N' \to N$ of A-modules, the map $M \otimes_A N' \to M \otimes_A N$ is an injection.

Exercise 8.15. Prove that $A[f^{-1}]$ is a flat A-algebra.

Exercise 8.16. Suppose that M_i, $i \in I$ is a flat family of A-modules. Prove that the direct sum $\bigoplus M_i$ is a flat A-module.

Exercise 8.17. (i) Suppose that M is a free A-module. Prove that M is a flat A-module.

(ii) Suppose that M is a *projective* A-module. Prove that M is a flat A-module.

Definition 8.18. Let $\mathscr{B}$ be a family of A-algebras. We say that $\mathscr{B}$ is *faithfully flat* if, for a sequence of A-modules

$$N' \to N \to N''$$

is exact *if and only if* the sequences

$$B \otimes_A N' \to B \otimes_A N \to B \otimes_A N''$$

are exact for all $B \in \mathscr{B}$.

ex:fp **Exercise 8.19.** Show that if $\mathscr{B}$ is a faithfully flat family of A-algebras then

(i) if M is an A-module and $\widetilde{M} = 0$ then $M = 0$;

(ii) the map $M \to \Gamma(\mathscr{B}, \widetilde{M})$ is injective;

(iii) if $\varphi : M \to N$ is an A-module homomorphism and $\widetilde{\varphi} = 0$ then $\varphi = 0$;

(iv) if $\varphi : M \to N$ is an A-module homomorphism and $\widetilde{\varphi}$ is an isomorphism then φ is an isomorphism.

8.5 Faithfully flat descent

Definition 8.20. Suppose that $B : \mathscr{B} \to A\text{-}\mathbf{Alg}$ is a diagram A-algebras. A morphism $u : \alpha \to \beta$ in $\mathscr{B}$ is called *cartesian* if, for every morphism $v : \alpha \to \gamma$ and every morphism $\varphi : B(\beta) \to B(\gamma)$ such that $\varphi \circ B(u) = B(v)$, there is a unique morphism $w : \beta \to \gamma$ such that $wu = v$.

We call B a *(covariantly) cartesian diagram* if every $\alpha \in B$ is the source of a cartesian arrow.

There is a simple way of extending any diagram B of A-algebras to a fibered diagram. We define $\bar{\mathscr{B}}$ to be the category of pairs (α, C, φ) where $\alpha \in \mathscr{B}$ and $\varphi : B(\alpha) \to C$ is a morphism of A-algebras. Morphisms $(\alpha, C, \varphi) \to (\beta, D, \psi)$ in $\mathscr{B}$ consist of an isomorphism $u : \alpha \to \beta$ and $\rho : C \to D$ such that the diagram below commutes:

$$\begin{tikzcd} B(\alpha) \arrow[r, "\varphi"] \arrow[d, "B(u)"'] & C \arrow[d, "\rho"] \\ B(\beta) \arrow[r, "\psi"] & D \end{tikzcd}$$

The projection $\bar{B} : \bar{\mathscr{B}} \to A\text{-}\mathbf{Alg}$ sends (α, C, φ) to C.

Definition 8.21. Suppose that B is a diagram of A-algebras (in other words, a functor $B : \mathscr{B} \to A\text{-}\mathbf{Alg}$). We call it a *presieve* if it is fibered and faithful; we will call it a *sieve* if it is fibered and fully faithful.

Exercise 8.22. Suppose that $\mathscr{B}$ is a cartesian diagram of A-algebras. There is a universal way to make $\mathscr{B}$ a presieve or a sieve:

(i) define $\mathscr{B}^*$ to consist of the same objects as $\mathscr{B}$ but define $\mathrm{Hom}_{\mathscr{B}^*}(\alpha, \beta)$ to be the image of $\mathrm{Hom}_{\mathscr{B}}(\alpha, \beta)$ in $\mathrm{Hom}_{A\text{-}\mathbf{Alg}}(\mathcal{O}(\alpha), \mathcal{O}(\beta))$;

(ii) define $\mathscr{B}^{**}$ to consist of the same objects as $\mathscr{B}$ but define $\mathrm{Hom}_{\mathscr{B}^{**}}(\alpha, \beta) = \mathrm{Hom}_{A\text{-}\mathbf{Alg}}(\mathcal{O}(\alpha), \mathcal{O}(\beta))$.

Our goal in this section is to identify some conditions on a diagram $\mathscr{B}$ of A-algebras under which

$$A\text{-}\mathbf{Mod} \simeq \mathbf{QCoh}(\mathscr{B}).$$

The strategy will be to relate these in via series of functors:

$$A\text{-}\mathbf{Mod} \xleftarrow{\sim} \mathbf{QCoh}(A\text{-}\mathbf{Alg}) \to \mathbf{QCoh}(\bar{\mathscr{B}}^{**}) \to \mathbf{QCoh}(\bar{\mathscr{B}}^{*}) \to \mathbf{QCoh}(\bar{\mathscr{B}}) \xrightarrow{\sim} \mathbf{QCoh}(\mathscr{B})$$

The basic lemma

Exercise 8.23. Suppose that $\mathscr{B}$ contains a faithfully flat collection of A-algebras.

(i) Then the functor

$$\mathbf{QCoh}(A\text{-}\mathbf{Alg}) \to \mathbf{QCoh}(\mathscr{B})$$

is faithful and conservative.

(ii) The map $M \to \Gamma(\mathscr{B}, \widetilde{M})$ is injective for all A-modules M.

From diagrams to cartesian diagrams

Exercise 8.24. (i) Show that, for any diagram $\mathscr{B}$ of A-algebras, the functor

$$\mathbf{QCoh}(\bar{\mathscr{B}}) \to \mathbf{QCoh}(\mathscr{B})$$

is an equivalence.

(ii) Conclude that

$$\mathbf{QCoh}(A\text{-}\mathbf{Alg}) \to A\text{-}\mathbf{Mod}$$

is an equivalence.

From cartesian diagrams to presieves

Exercise 8.25. Suppose that $\mathscr{B}$ is a cartesian diagram of A-algebras containing two arrows $f, g : B \to B'$ whose underlying ring homomorhisms are the same. Suppose that $\mathscr{M}$ is a quasicoherent module on $\mathscr{B}$. Show that $f_*, g_* : \mathscr{M}(B) \to \mathscr{M}(B')$ coincide if the following condition holds:

> There is a faithfully flat collection of maps $h : B' \to C$ in $\mathscr{B}$ such that $h_* f_* = h_* g_*$.

Solution. Let $\mathscr{C}$ be the diagram of B'-algebras consisting of all B'-algebras $h : B' \to C$ such that $h_* f_* = h_* g_*$ as maps $\mathscr{M}(B) \to \mathscr{M}(C)$. We wish to prove that the maps $f_*, g_* : \mathscr{M}(B) \to \mathscr{M}(B')$ coincide, for which it is equivalent to show that the isomorphisms $\mathrm{id}_{B'} f_*, \mathrm{id}_{B'} g_* : B' \otimes_B \mathscr{M}(B) \to \mathscr{M}(B')$ are the same.

By faithfully flat descent for identities between homomorphisms, it is sufficient to show that the maps $\mathrm{id}_C \otimes_{B'} \mathrm{id}_{B'} f_*$ and $\mathrm{id}_C \otimes_{B'} \mathrm{id}_{B'} g_*$ coincide (these are maps $C \otimes_{B'} B' \otimes_B \mathscr{M}(B) = C \otimes_B \mathscr{M}(B) \to C \otimes_{B'} \mathscr{M}(B')$. □

Exercise 8.26. Suppose that $\mathscr{B}$ is a digram of A-algebras. Prove that the functor

$$\mathbf{QCoh}(\mathscr{B}^*) \to \mathbf{QCoh}(\mathscr{B})$$

is an equivalence of categories if the following condition is satisfied:

> Whenever $f, g : B \to C$ are morphisms in $\mathscr{B}$ whose images in A-**Alg** are the same, there is a faithfully flat collection of morphisms $h : C \to D$ such that $h_* f_* = h_* g_*$.

From presieves to sieves

Now suppose that $\mathscr{B}$ is a diagram of A-algebras. If C is an A-algebra, we write C-$\mathbf{Alg}(\mathscr{B})$ for the category of pairs (B, φ) where $B \in \mathscr{B}$ and $\varphi : C \to B$ is an A-algebra homomorphism. A morphism from (B, φ) to (B', φ') is a commutative diagram

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[dr, "\varphi'"'] & B \arrow[d, "\psi"] \\ & B' \end{tikzcd}$$

where $\psi : B \to B'$ is in $\mathscr{B}$.

Exercise 8.27. Suppose that $\mathscr{U}$ is cartesian, with a finite, cofinal subdiagram, and that B is a flat A-algebra. Then the map

$$B \otimes_A \varprojlim_{U \in \mathscr{U}} \mathscr{M}(U) \to \varprojlim_{V \in B\text{-}\mathbf{Alg}/\mathcal{O}(\mathscr{U})} \mathscr{M}(V)$$

is a bijection.

Solution. Choose a finite, cofinal $\mathscr{U}_0 \to \mathscr{U}$. We have

$$\begin{aligned} B \otimes_A \varprojlim_{U \in \mathscr{U}} \mathscr{M}(U) &= B \otimes_A \varprojlim_{U \in \mathscr{U}_0} \mathscr{M}(U) \\ &= \varprojlim_{U \in \mathscr{U}_0} B \otimes_A \mathscr{M}(U) \\ &= \varprojlim_{U \in \mathscr{U}} B \otimes_A \mathscr{M}(U) \\ &= \varprojlim_{V \in B\text{-}\mathbf{Alg}/\mathcal{O}(\mathscr{U})} \mathscr{M}(V) \end{aligned}$$

as was required. □

Exercise 8.28. Suppose that $\bar{\mathscr{B}}$ is cartesian, with a finite, cofinal subdiagram, and contains a faithfully flat collection of A-algebras. Then the map

$$M \to \varprojlim_{B \in \mathscr{B}} B \otimes_A M$$

is a bijection, for all A-modules M.

Solution. Suppose that $C \in \mathscr{B}$ is flat. Observe that the $\mathscr{B}$-module $\mathscr{M}(B) = B \otimes_A M$ is quasicoherent, so we can use the previous exercise on it. Also note that $C\text{-}\mathbf{Alg}/\mathscr{B} \simeq B\text{-}\mathbf{Alg}$. Combining these, the map

$$C \otimes_A M \xleftarrow{\sim} \varprojlim_{B \in C\text{-}\mathbf{Alg}} B \otimes_A M \xrightarrow{\sim} \varprojlim_{B \in C\text{-}\mathbf{Alg}/\mathscr{B}} B \otimes_A M \xrightarrow{\sim} C \otimes_A \varprojlim_{B \in \mathscr{B}} B \otimes_A M$$

is a bijection. But $\mathscr{B}$ contains a faithfully flat collection of A-algebras, so this means

$$M \to \varprojlim_{B \in \mathscr{B}} B \otimes_A M$$

is a bijection. □

Exercise 8.29. Let $\mathscr{M}$ be a quasicoherent module on a faithful sieve $\mathscr{B}$ of A-algebras. Let $\mathscr{B}^*$ be the associated sieve. If $\mathscr{B}$ has the following property then

$$\mathbf{QCoh}(\mathscr{B}^*) \to \mathbf{QCoh}(\mathscr{B})$$

is an equivalence of categories:

> For every pair B and C in $\mathscr{B}$, and every A-algebra homomorphism $\varphi : B \to C$ (not necessarily on $\mathscr{B}$), let $\mathscr{C} \subset C\text{-}\mathbf{Alg}$ be the category of all C-algebras C' such that the map $B \xrightarrow{\varphi} C \to C'$ appears in $\mathscr{B}$. Then $\mathscr{C}$ contains a finite, cofinal subcategory and a faithfully flat collection of C-algebras.

Solution. In order to make the extension, we suppose that $\varphi : B \to C$ is a homomorphism between two algebras B and C in $\mathscr{B}$. We need to construct $\varphi_* : \mathscr{M}(B) \to \mathscr{M}(C)$ and argue that it is unique. It is the same to produce $C \otimes_B \mathscr{M}(B) \to \mathscr{M}(C)$. Let $\mathscr{C}$ be as in the condition and note that the functor

$$\mathbf{QCoh}(C\text{-}\mathbf{Alg}) \to \mathbf{QCoh}(\mathscr{C})$$

is an equivalence of categories. Now, for each C-algebra D, let $\mathscr{N}(D) = D \otimes_B \mathscr{M}(B)$. Then the isomorphisms

$$D \otimes_B \mathscr{M}(B) \simeq \mathscr{M}(D)$$

give us a map $\mathscr{N} \to \mathscr{M}\big|_{\mathscr{C}}$. By virtue of the equivalence of categories, this comes from a unique morphism

$$C \otimes_B \mathscr{M}(B) \to \mathscr{M}(C),$$

as required. □

This explains how to extend a module to a sieve. Now we wish to extend from a sieve to all A-modules.

Exercise 8.30. Suppose that $\mathscr{B}$ is a sieve of A-algebras has a finite cofinal diagram and contains a faithfully flat collection of A-algebras. Then

$$\mathbf{QCoh}(A\text{-}\mathbf{Alg}) \to \mathbf{QCoh}(\mathscr{B})$$

is an equivalence of categories.

Proof. We have a functor in the reverse direction sending $\mathscr{M}$ to $\Gamma(\mathscr{B}, \mathscr{M})^\sim$. There are canonical maps:

$$M \to \Gamma(\mathscr{B}, \widetilde{M})$$
$$\Gamma(\mathscr{B}, \mathscr{M})^\sim \to \mathscr{M}$$

We argue that these are isomorphisms. In fact, we can argue that both are isomorphisms at the same time. The first map is a homomorphism of A-modules, and the second is a collection of homomorphism

$$C \otimes_B \Gamma(\mathscr{B}, \mathscr{M}) \to \mathscr{M}(C)$$

that we regard as maps of A-modules. Suppose that D is a flat A-algebra in $\mathscr{B}$. Let $\mathscr{B}_0 \to \mathscr{B}$ be a finite coinitial diagram. Then tensoring with D we get:

$$\begin{aligned} D \otimes_A M \to D \otimes_A \Gamma(\mathscr{B}, \widetilde{M}) &= D \otimes_A \varprojlim_{B \in \mathscr{B}} \widetilde{M}(B) \\ &\xrightarrow{\sim} D \otimes_A \varprojlim_{B \in \mathscr{B}_0} \widetilde{M}(B) \\ &\sim \varprojlim_{B \in \mathscr{B}_0} D \otimes_A \widetilde{M}(B) \\ &\xleftarrow{\sim} \varprojlim_{B \in D/\mathscr{B}} \widetilde{M}(D) \\ &\xrightarrow{\sim} \widetilde{M}(D) \end{aligned}$$

This is certainly an isomorphism by definition of $\widetilde{M}$. Similarly:

$$\begin{aligned}
D \otimes_A \Gamma(\mathscr{B}, \mathscr{M})^{\sim}(C) &= D \otimes_A C \otimes_A \varprojlim_{B \in \mathscr{B}} \mathscr{M}(B) \\
&\simeq D \otimes_A C \otimes_A \varprojlim_{B \in \mathscr{B}_0} \mathscr{M}(B) \\
&\simeq C \otimes_A \varprojlim_{B \in \mathscr{B}_0} D \otimes_A \mathscr{M}(B) \\
&\simeq C \otimes_A \varprojlim_{B \in \mathscr{B}} \mathscr{M}(D \otimes_A B) \\
&\simeq C \otimes_A \varprojlim_{B \in D/\mathscr{B}} \mathscr{M}(B) \\
&\simeq C \otimes_A \mathscr{M}(D) \\
&\simeq \mathscr{M}(C)
\end{aligned}$$

□

Exercise 8.31. Suppose that B is a flat A-algebra and that $\mathscr{U}$ has a finite cofinal subdiagram. Prove that

$$B \otimes_A \mathscr{N}(U) \to$$

Exercise 8.32. Assume that $\mathscr{U}$ is cartesian. Construct a functor:

$$\varprojlim_{(U,\varphi) \in B\text{-}\mathbf{Alg}/\mathscr{O}(\mathscr{U})} \mathscr{M}(U) \to \varprojlim_{V \in \mathscr{U}} B \otimes_A \mathscr{M}(V)$$

Show this functor is a bijection.

Exercise 8.33. Suppose that $\mathscr{B}$ contains a faithfully flat collection of A-algebras and M is an A-module. Then

$$M \to \Gamma(\mathscr{B}, \widetilde{M})$$

is injective.

Exercise 8.34.

Suppose that A is a commutative ring, $B : \mathscr{B} \to A\text{-}\mathbf{Alg}$ is a diagram of A-algebras, and A' is an A-algebra. We write $B' = A' \otimes_A B$ for the diagram whose shape is $\mathscr{B}$, with $B'(\alpha) = A' \otimes_A B(\alpha)$ for all $\alpha \in \mathscr{B}$. We define A'/B to be the category of pairs (α, φ) where $\alpha \in \mathscr{B}$ and $\varphi : A' \to B(\alpha)$ is an A-algebra homomorphism. A morphism $(\alpha, \varphi) \to (\beta, \psi)$ consists of a morphism $u : \alpha \to \beta$ of $\mathscr{B}$ such that $B(u)$ is an A'-algebra homomorphism with respect to the A'-algebra structures inherited from φ, ψ. In other words, the diagram below commutes:

$$\begin{array}{ccc}
A' & \xrightarrow{\varphi} & B(\alpha) \\
\Vert & & \downarrow{\scriptstyle B(u)} \\
A' & \longrightarrow & B(\beta)
\end{array}$$

Observe that if B is cartesian then there is a map $A' \otimes_A B \to A'/B$, since if B is cartesian then $B(\alpha) \to A' \otimes_A B(\alpha)$ appears in B whenever $\alpha \in \mathscr{B}$.

Definition 8.35. Suppose that B_0 and B are diagrams of A-algebras and that B_0 is induced from a functor $\mathscr{B}_0 \to \mathscr{B}$. We say that B_0 is *n-cofinal* in B if, for every quasicoherent B-module $\mathscr{M}$ restricting to $\mathscr{M}_0$ on B_0, the map

$$\Gamma(\mathscr{B}, \mathscr{M}) \to \Gamma(\mathscr{B}_0, \mathscr{M}_0)$$

is an n-equivalence (i.e., is injective if $n = -1$ and bijective if $n = 0$). We say it is 1-cofinal if the map

$$\mathbf{QCoh}(B) \to \mathbf{QCoh}(B_0)$$

is an equivalence of categories.

We say that B_0 is *universally n-cofinal* in B if $A' \otimes_A B_0$ is cofinal in A/B for all A-algebras A'.

Theorem 8.36. *Suppose that $\mathscr{B}$ is a diagram of A-algebras.*

(i) Suppose that $\mathscr{B}$ contains a faithfully flat collection of A-algebras. Then, for any A-module M, the map

$$M \to \Gamma(\mathscr{B}, \widetilde{M})$$

is injective.

(ii) Suppose that $\mathscr{B}$ is a -1-sieve that contains a faithfully flat collection of A-algebras and has a finite 0-cofinal diagram. Then, for any A-module M, the map

$$M \to \Gamma(\mathscr{B}, \widetilde{M})$$

is bijective.

(iii) Suppose that $\mathscr{B}$ contains a faithfully flat collection of A-algebras and has a finite 1-cofinal diagram. Then $\sim$ defines an equivalence of categories

$$A\text{-}\mathbf{Mod} \to \mathbf{QCoh}(B).$$

Proof. Suppose that $x \in M$. View this as a homomorphism $A \to M$ and let K be its kernel. For each $\alpha \in \mathscr{B}$, let $x(\alpha)$ be its image in $\widetilde{M}(\alpha) = B(\alpha) \otimes_A M$. For each $B(\alpha)$ in a faithfully flat collection of A-algebras, we have

$$0 \to B(\alpha) \otimes_A K \to B(\alpha) \xrightarrow{x(\alpha)} B(\alpha) \otimes_A M$$

is exact. If each $x(\alpha)$ is zero then

$$0 \to K \to A \to 0$$

is exact by faithful flatness, so $Ax = 0$, and therefore $x = 0$. This proves the first claim.

For the second claim, let $\mathscr{B}_0$ be a 0-cofinal diagram and suppose that $A' = B(\alpha)$ is a flat A-algebra in the diagram B. Note that we have a map of diagrams $A' \otimes B_0 \to A'/\mathscr{B}$. We have a commutative square:

$$\begin{array}{ccc} A' \otimes_A \Gamma(\mathscr{B}, \widetilde{M}) & \longrightarrow & A' \otimes_A \Gamma(\mathscr{B}_0, \widetilde{M}_0) \\ \downarrow & & \downarrow \\ \Gamma(A'/B, \widetilde{M}) & \longrightarrow & \Gamma(\mathscr{B}_0, A' \otimes_A \widetilde{M}) \end{array}$$

The vertical arrow on the right is an isomorphism since $\mathscr{B}_0$ is finite and A' is flat. The horizontal arrow on top is an isomorphism because $\mathscr{B}_0$ is 0-cofinal in $\mathscr{B}$. The bottom arrow is an isomorphism because $A' \otimes_A B_0$ is cofinal in A'/B. But, by assumption, $A' \in A'/B$, so that $\Gamma(A'/B, \widetilde{M}) = \mathscr{M}(A')$. Following the diagram around, we deduce an isomorphism

For the third claim, we argue that $\sim$ and Γ are inverse equivalences of categories. We have already seen that M is isomorphic to $\Gamma(\mathscr{B}, \widetilde{M})$ under weaker hypothesis, so we need to consider a quasicoherent B-module $\mathscr{M}$ and show that the map

$$\Gamma(\mathscr{B}, \mathscr{M})^{\sim} \to \mathscr{M}$$

is an isomorphism. This means that we have to show the maps

$$B(\alpha) \otimes_A \Gamma(\mathscr{B}, \mathscr{M}) \to \mathscr{M}(\alpha)$$

are isomorphisms for every $\alpha \in \mathscr{B}$. Pick a finite, 1-cofinal diagram B_0 of B. Then the □

8.6 Descent

Definition 8.37. Suppose that $\mathscr{B}$ is a diagram of commutative rings. A $\mathscr{B}$-module is

(i) a B-module $\mathscr{M}(B)$ for each $B \in \mathscr{B}$,

(ii) a B-module homomorphism $\mathscr{M}(B) \to \mathscr{M}(C)$ for each morphism $B \to C$ in $\mathscr{B}$, such that

(iii) the map $\mathscr{M}(B) \to \mathscr{M}(D)$ associated to a composition $B \to C \to D$ in $\mathscr{B}$ is the composition of the maps $\mathscr{M}(B) \to \mathscr{M}(C) \to \mathscr{M}(D)$.

It is called *quasicoherent* if

(iv) the maps $C \otimes_B \mathscr{M}(B) \to \mathscr{M}(C)$ induced from $\mathscr{M}(B) \to \mathscr{M}(C)$ is an isomorphism.

A morphism of $\mathscr{B}$-modules $\mathscr{M} \to \mathscr{N}$ is a family of morphisms $\mathscr{M}(B) \to \mathscr{N}(B)$ such that, for every morphism $B \to C$ in $\mathscr{B}$, the diagram

$$\begin{tikzcd} \mathscr{M}(B) \arrow[r] \arrow[d] & \mathscr{M}(C) \arrow[d] \\ \mathscr{N}(B) \arrow[r] & \mathscr{N}(C) \end{tikzcd}$$

is commutative.

This exercise is not important and requires definitions that we have not given here.

Exercise 8.38. Verify that a $\mathscr{B}$-module is a commutative diagram of functors:

$$\begin{tikzcd} & \mathbf{AlgMod} \arrow[d] \\ \mathscr{B} \arrow[ur] \arrow[r] & \mathbf{ComRing} \end{tikzcd}$$

where **AlgMod** is the category of pairs (A, M) in which A is a commutative ring and M is an A-module. (It is up to you to figure out what the morphisms are.)

Verify as well that the module is quasicoherent if and only if the map $\mathscr{B} \to$ **AlgMod** is *covariantly cartesian* over **ComRing**.

Suppose that $\mathscr{B}$ is actually a diagram of A-modules, where A is a commutative ring. There are two natural functors

$$(-)^{\sim} : A\text{-}\mathbf{Mod} \to \mathbf{QCoh}(\mathscr{B})$$
$$\Gamma(\mathscr{B}, -) : \mathscr{B}\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$$

with the following definitions:

$$\widetilde{M}(B) = B \otimes_A M$$
$$\Gamma(\mathscr{B}, \mathscr{M}) = \varprojlim_{B \in \mathscr{B}} \mathscr{M}(B)$$

Exercise 8.39. Complete the description of these functors and verify that they are *adjoints*:

$$\mathrm{Hom}_{\mathscr{B}\text{-}\mathbf{Mod}}(\widetilde{M}, \mathscr{N}) = \mathrm{Hom}_{A\text{-}\mathbf{Mod}}(M, \Gamma(\mathscr{B}, \mathscr{N}))$$

for any A-module M and any $\mathscr{B}$-module $\mathscr{N}$.

def:coinitial **Definition 8.40.** Suppose that $\mathscr{B}$ is a digram. A coinitial subdiagram of $\mathscr{B}$ is a subdiagram $\mathscr{B}_0$ such that

cond:coinitial-1 (i) if $B \in \mathscr{B}$ there is some $\mathscr{B}_0 \in \mathscr{B}_0$ and a map $B_0 \to B$ in $\mathscr{B}$;

cond:coinitial-2 (ii) if there are two morphisms $i_0 : B_0 \to B$ and $i_1 : B_1 \to B$ in $\mathscr{B}$, with $B_i \in \mathscr{B}_0$ and $B \in \mathscr{B}$, then there are morphisms $j_0 : B_0 \to B_2$ and $j_1 : B_1 \to B_2$ and a map $i_2 : B_2 \to B$ in $\mathscr{B}$ such that the diagram below commutes:

$$\begin{tikzcd}
B_0 \arrow[d, "j_0"'] \arrow[dr, "i"] & \\
B_2 \arrow[r, "i_2"] \arrow[ur, phantom, "0"] & B \\
B_1 \arrow[u, "j_1"] \arrow[ur, "i_1"'] &
\end{tikzcd} \tag{8.1}$$
eqn:24

Exercise 8.41. Suppose that $\mathscr{B}$ is a diagram of sets and $\mathscr{B}_0$ is a subdigram. Show that the map

$$\varprojlim_{B \in \mathscr{B}} B \to \varprojlim_{B \in \mathscr{B}_0} B$$

(i) is an injection if $\mathscr{B}_0$ satisfies **??** of Definition 8.40, and

(ii) is a bijection if $\mathscr{B}_0$ satisfies both (i) and (ii) of Definition 8.40.

Solution. Let $A = \varprojlim_{B \in \mathscr{B}} B$ and let $A_0 = \varprojlim_{B \in \mathscr{B}_0} B$. We argue that the map $A \to A_0$ is injective and surjective. First, suppose that $x \in A$. We must show that x is determined by $x(B_0)$ for all $B_0 \in \mathscr{B}_0$. Indeed, if $B \in \mathscr{B}$ then there is a map $\varphi : B_0 \to B$, where $\varphi \in \mathscr{B}$. Then $x(B) = \varphi_* x(B_0)$, which means that $x(B_0)$ determines $x(B)$, as asserted.

For the surjectivity, we need to show that each $x \in A_0$ extends to A. Pick $B \in \mathscr{B}$. By assumption, there is a map $i_0 : B_0 \to B$ with $B_0 \in \mathscr{B}_0$. Define $x(B) = {i_0}_* x(B_0)$. We must verify that this does not depend on the choice of B_0 and i_0. If we also had $i_1 : B_1 \to B$ then there would be a diagram (8.1), which means:

$$i_{0*}x(B_0) = i_{2*}j_{0*}x(B_0) = i_{2*}x(B_2) = i_{2*}j_{1*}x(B_1) = i_{1*}x(B_1)$$

Now we have to verify that if $\varphi : B \to B'$ is a morphism in $\mathscr{B}$ then $\varphi_* x(B) = x(B')$. Choose $i : B_0 \to B$ as before. Then by definition, $x(B) = i_{0*}x(B_0)$ and $x(B') = \varphi_* i_{0*} x(B_0)$, so $\varphi_* x(B) = x(B')$, as required. □

Exercise 8.42. Show that, if $\mathscr{B}$ has a finite, coinitial diagram then, for any flat A-algebra A', then for any $\mathscr{B}$-module, the map

$$A' \otimes_A \Gamma(\mathscr{B}, \mathscr{M}) \to \Gamma(A' \otimes_A \mathscr{B}, \mathscr{M})$$

is an isomorphism.

thm:fpqc-descent

Theorem 8.43. *Let $\mathscr{B}$ be a diagram of A-algebras.*

(i) Suppose that there is a faithfully flat collection of A-algebras. Then $\sim$ is faithful and the map $M \to \Gamma(\mathscr{B}, \widetilde{M})$ is injective.

(ii) Suppose that $\mathscr{B}$ has a finite, coinitial subdiagram and a faithfully flat collection of A-algebras B such that, for every $C \in \mathscr{B}$, the maps $B \to B \otimes_A C$ and $C \to \otimes_A C$ appear in $\mathscr{B}$. Then $\sim$ is fully faithful and $M \to \Gamma(\mathscr{B}, \widetilde{M})$ is bijective.

(iii) With the same conditions as in the last part, if $\mathscr{M}$ is a quasicoherent $\mathscr{B}$-module then the map $\Gamma(\mathscr{B}, \mathscr{M})^\sim \to \mathscr{M}$ is an isomorphism and $\sim$ is an equivalence of categories.

Definition 8.44. Suppose that $A \to A'$ is a homomorphism of commutative rings. We call the homomorphism *flat* if the functor $A' \otimes_A M$ is an exact functor of the A-module M.

We say that a family of homomorphisms $A \to A'$ is *faithfully flat* if a sequence of A-modules E is exact if and only if $A' \otimes_A E$ is exact.

Proof of Theorem 8.43. The first case is Exercise 8.19.

We consider the second claim. Let M be an A-module. We argue that $M \to \Gamma(\mathscr{B}, \widetilde{M})$ is an isomorphism. It is already known to be injective by the previous case. Suppose $\varphi \in \Gamma(\mathscr{B}, \widetilde{M})$. We argue that φ is induced from an element of M. By Exercise 8.19, it is sufficient to demonstrate that the map

$$A' \otimes_A M \to A' \otimes_A \Gamma(\mathscr{B}, \widetilde{M})$$

is an isomorphism for all A' in a faithfully flat collection of A-algebras. But we have a commutative diagram:

$$\begin{tikzcd} A' \otimes_A M \arrow[r] \arrow[d, equal] & A' \otimes_A \Gamma(\mathscr{B}, \widetilde{M}) \arrow[d, "\wr"] \\ A' \otimes_A M \arrow[r] & \Gamma(A' \otimes_A \mathscr{B}, (A' \otimes_A M)^\sim) \end{tikzcd}$$

The vertical arrow on the right is an isomorphism by Exercise **??**. But all of the properties in the statement of the theorem are stable under tensor product with A', so we can replace A with A', M with $A' \otimes_A M$, and $\mathscr{B}$ with $A' \otimes_A \mathscr{B}$.

For this to be useful, we need a convenient faithfully flat collection of A-algebras A'. We choose the collection $\mathscr{B}_0$ guaranteed by the hypothesis of the theorem. Then A is replaced with A' and $A' \in \mathscr{B}_0$ is replaced with $B = A' \otimes_A A'$. Notice that the inclusion $A' \to A' \otimes_A A'$ has a retraction, by the multiplication map $a \otimes b \mapsto ab : A' \otimes_A A' \to A'$. This means that when we replace A with A', there is an algebra $B \in A' \otimes_A \mathscr{B}_0$ with a retraction onto A'.

All of this means it is sufficient to prove the second part of the theorem under the assumption that there is a $B \in \mathscr{B}_0$ with a retraction of the inclusion $A \to B$.

Now, we already know that $M \to \Gamma(\mathscr{B}, \widetilde{M})$ is injective, so we prove the surjectivity. Suppose that $x \in \Gamma(\mathscr{B}, \widetilde{M})$ and let y be the image of $x(B) \in \widetilde{M}(B) = M \otimes_A B$ under the homomorphism

$$\widetilde{M}(B) = M \otimes_A B \to M \otimes_A B \otimes_B A = M.$$

We argue that $\widetilde{y} = x$.

For this, we must show that $\widetilde{y}(C) = x(C)$ for every $C \in \mathscr{B}$. By assumption, there is a commutative diagram:

$$\begin{tikzcd}
C \arrow[r] & C \otimes_A B \arrow[r] & A \otimes_B B \otimes_A C \arrow[r, dash, "\sim"] & C \\
A \arrow[r] \arrow[u] & B \arrow[r] \arrow[u] & A \arrow[u] &
\end{tikzcd}$$

And we can chase the elements around:

$$\begin{tikzcd}
x(C) \arrow[r] & x(C \otimes_A B) \arrow[r] & x(C) \arrow[r, equal] & \widetilde{y}(C) \\
& x(B) \arrow[rr] \arrow[u] & & y \arrow[u]
\end{tikzcd} \tag{8.2}$$ eqn:8

Since the diagram commutes, we get $\widetilde{y}(C) = x(C)$; this holds for all C, so $\widetilde{x} = y$, as required.

The proof of the third part is a very similar diagram chase. Once again, we are trying to show that the maps

$$\mathscr{M}(C) \to \Gamma(\mathscr{B}, \mathscr{M})^{\sim}$$

are isomorphisms. We regard these as maps of A-modules, so we can tensor with the faithfully flat family of A-algebras $A' \in \mathscr{B}_0$ to simplify matters. This means that we can assume, as before, that there is a $B \in \mathscr{B}_0$ and a retraction $B \to A$.

Define $M = \mathscr{M}(B) \otimes_B A$. We argue that $\widetilde{M} \simeq \mathscr{M}$. It follows from this that $\widetilde{M} \simeq \Gamma(\mathscr{B}, \mathscr{M})$, by the second part. First we construct maps $\widetilde{M}(C) \to \mathscr{M}(C)$, similar to what we did above. Chasing modules in diagram (8.2), we get:

$$\begin{tikzcd}
\mathscr{M}(C) \arrow[r] & \mathscr{M}(C \otimes_A B) \arrow[r] & \mathscr{M}(C) \arrow[r, dash, "\sim"] & \widetilde{M}(C) \\
& \mathscr{M}(B) \arrow[r] \arrow[u] & A \otimes_B \mathscr{M}(B) \arrow[r] & M \arrow[u]
\end{tikzcd}$$

Since the diagram commutes, and the target of each arrow is characterized by a universal property, we get a unique isomorphism $\mathscr{M}(C) \simeq \widetilde{M}(C)$ making the diagram commute.

We are not done because we must still prove the isomorphism is natural, which is to say that for every morphism $C \to C'$ in $\mathscr{B}$, we must check the squares

$$\begin{array}{ccc} \mathscr{M}(C) & \longrightarrow & \mathscr{M}(C') \\ \wr\big| & & \big|\wr \\ \widetilde{M}(C) & \longrightarrow & \widetilde{M}(C') \end{array}$$

are commutative. Applying our construction of the isomorphism $\mathscr{M}(C) \simeq \widetilde{M}(C)$ to both C and C', we get a commutative diagram:

$$\begin{array}{ccccccc} \mathscr{M}(C') & \longrightarrow & \mathscr{M}(C' \otimes_A B) & \longrightarrow & \mathscr{M}(C') & \overset{\sim}{\longrightarrow} & \widetilde{M}(C') \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathscr{M}(C) & \longrightarrow & \mathscr{M}(C \otimes_A B) & \longrightarrow & \mathscr{M}(C) & \overset{\sim}{\longrightarrow} & \widetilde{M}(C) \\ & & \uparrow & & & & \uparrow \\ & & \mathscr{M}(B) & \longrightarrow & A \otimes_B \mathscr{M}(B) & \longrightarrow & M \end{array}$$

The square we want is the outer right.

Now we have shown that $\mathscr{M} = \widetilde{M}$, but we still need to show that the map $\Gamma(\mathscr{B}, \mathscr{M})^{\sim} \to \mathscr{M}$ is an isomorphism. The isomorphism $\widetilde{M} \to \mathscr{M}$ induces a composition of maps:

$$M \to \Gamma(\mathscr{B}, \widetilde{M}) \to \Gamma(\mathscr{B}, \mathscr{M})$$

The first of these was shown to be an isomorphism, in the second part of the proof of the theorem, and the second is an isomorphism because it is obtained by applying Γ to an isomorphism. The composition is the isomorphism we want. $\square$

8.7 Applications to quasicoherent sheaves

Corollary 8.44.1. *Let A be a commutative ring and let M be an A-module. Let $\mathscr{U}$ be the category of open sets $D(f) \subset \operatorname{Spec} A$. Then $\widetilde{M}$ is a sheaf on $\mathscr{U}$.*

Proof. Suppose that $U = \operatorname{Spec} A[f^{-1}] \in \mathscr{U}$ and $U = \bigcup_{g \in S} D(g)$. Let $\mathscr{V}$ be the category of all finite intersections of the $D(g)$. Note the sheaf condition for this cover says that

$$\widetilde{M}(U) = \varprojlim_{V \in \mathscr{V}} \widetilde{M}(V).$$

But if we take $\mathscr{B}$ to be the diagram of $A[f^{-1}]$-algebras $A[g^{-1}]$, for $g \in S$, then this is saying that the map

$$M_f \to \varprojlim_{B \in \mathscr{B}} \widetilde{M}_f(B).$$

Now, localizations are flat, and $\mathscr{V}$ contains a cover, so $\mathscr{B}$ is a faithfully flat collection of algebras. It has a finite coinitial subdiagram because U is quasicompact. Therefore the theorem applies. $\square$

Corollary 8.44.2. *Let X be a scheme and let $\mathscr{U}$ be any basis of open affine subschemes. Then*

$$\mathbf{QCoh}(X) \to \mathbf{QCoh}(\mathscr{U})$$

is an equivalence of categories.

Proof. First we prove that the functor is fathful. Suppose that $f, g : \mathscr{M} \to \mathscr{N}$ are homomorphisms that agree when restricted to $\mathscr{U}$. For each open affine $V = \operatorname{Spec} A$ of X, we have two maps of modules, $f_V, g_V : \mathscr{M}(V) \to \mathscr{N}(V)$. If $U = \operatorname{Spec} B \in \mathscr{U}$ is contained in V then $f_U, g_U : B \otimes_A \mathscr{M}(U) \to B \otimes_A \mathscr{N}(U)$ agree. But the $U \in \mathscr{U}$ contained in V cover V, so the corresponding rings are faithfully flat over A. Therefore $f_V = g_V$. This holds for all open affine $V \subset X$, so $f = g$, by definition.

Now we prove the functor is full. Suppose that $\mathscr{M}$ and $\mathscr{N}$ are quasicoherent modules on X and $f : \mathscr{M}\big|_{\mathscr{U}} \to \mathscr{N}\big|_{\mathscr{U}}$ is a homomorphism. Let $\mathscr{V}$ be the collection of all open affines $V \subset X$ that are contained in some $U \in \mathscr{U}$. If $V \in \mathscr{V}$ is contained in $U \in \mathscr{U}$, define $f_V = f_U\big|_V$. We must verify that this is well-defined. But if $V \subset U'$ then $f_{U'}\big|_V$ and $f_U\big|_V$ give two maps between the quasicoherent modules $\mathscr{M}\big|_V$ and $\mathscr{N}\big|_V$ that agree, by assumption, on the basis $\mathbf{Open}(V) \cap \mathscr{U}$. Therefore $f_{U'}\big|_V = f_U\big|_V$.

Now we have extended f to a *covering sieve* $\mathscr{V}$ of open affine subsets of X. If $W \subset X$ is an open affine subset then $\mathscr{V} \cap \mathbf{Open}(W)$ is a covering sieve of open affine subsets of W. Therefore by faithfully flat descent, there is a unique extension of $f\big|_{\mathscr{V} \cap \mathbf{Open}(W)}$ to $f_W : \mathscr{M}\big|_W \to \mathscr{N}\big|_W$.

Finally, we prove that the functor is essentially surjective. Let $\mathscr{V}$ be the category of pairs (V, U) where $U \in \mathscr{U}$ and $V \subset U$ is open an open affine subscheme. There is a map $(V, U) \to (V', U')$ if $V \subset V'$ and $U \subset U'$. For any pair $(V, U) \in \mathscr{V}$, with $V = \operatorname{Spec} B$ and $U = \operatorname{Spec} A$, we define

$$\mathscr{M}'(V, U) = B \otimes_A \mathscr{M}(U).$$

Now let $\mathscr{V}^*$ be the category with the same objects as $\mathscr{V}$, but there is a map $(V, U) \to (V', U')$ if $V \subset V'$, with no condition on U. We argue that there is a unique extension of $\mathscr{M}'$ from $\mathscr{V}$ to $\mathscr{V}^*$. To do this, we construct an isomorphism $\mathscr{M}'(V, U') \simeq \mathscr{M}'(V, U)$ for all (V, U) and (V, U') in $\mathscr{V}$. To get this, let $\mathscr{W}$ be the collection of all $W \in \mathscr{U}$ such that $W \subset V$. Then $\mathscr{M}'(-, U)\big|_{\mathscr{W}}$ and $\mathscr{M}'(-, U')\big|_{\mathscr{W}}$ can both be identified (canonically) with $\mathscr{M}$. By the full faithfullness proved above, this gives us an isomorphism $\mathscr{M}'(V, U') \simeq \mathscr{M}'(V, U)$. We omit the various naturality verifications. ∎

Chapter 4

The category of schemes

9 Relating sheaves on different spaces

sec:sheaves-2

Reading 9.1. [Vak14, §§2.3, 2.6]

9.1 Pushforward

Definition 9.2 (Pushforward of presheaves)**.** Let $f : X \to Y$ be a continuous function. If F is a presheaf on X then f_*F is the presheaf on Y whose value on and open subset $U \subset Y$ is $f_*F(U) = F(f^{-1}U)$.

Exercise 9.3 (Pushforward of sheaves)**.** The pushforward of a sheaf is a sheaf.

Exercise 9.4 (Pushforward to a point)**.** Let F be a sheaf on a topological space X and let $\pi : X \to (\text{point})$ be the projection to a point. Show that $\pi_*F = \Gamma(X, F)$ when sheaves on a point are regarded as sets.

9.2 Sheaf of sections

Definition 9.5. Let $\pi : Y \to X$ be a continuous function. A *section* of π over a map $f : Z \to X$ is a map $s : Z \to Y$ such that $\pi s = f$. In particular, a section over X is a section over the identity map $\mathrm{id} : X \to X$. We write $\Gamma(Z, Y)$ for the set of sections of $\pi : Y \to X$ over $f : Z \to X$ (leaving the names of the maps implicit).

We define a presheaf Y^{sh} on X by $Y^{\mathrm{sh}}(U) = \Gamma(U, Y)$ for all open $U \subset X$.

Exercise 9.6 (The sheaf of sections)**.** Show that Y^{sh}, as defined above, is a sheaf.

9.3 Espace étalé

Definition 9.7. A function $\pi : E \to X$ is called a *local homeomorphism* or *étale* if there is a cover of E by open subsets U such that $\pi\big|_U : U \to X$ is an open embedding.[1]

A morphism of étale spaces $\pi : E \to X$ and $\pi' : E' \to X$ is a continuous map $f : E \to E'$ such that $\pi' f = \pi$. We write $\text{ét}(X)$ for the category of all étale spaces over X.[2]

[1] The word *étale* is also applied to certain morphisms of schemes, but the definition is different.

[2] This will cause a technical, but not moral or spiritual, conflict of notation when we study étale morphisms of schemes later.

ex:map-between-etale

Exercise 9.8. Show that any map between étale spaces over X is a local homeomorphism.

If F is a presheaf over X, construct a diagram $\mathbf{Open}(X)/F$ whose objects are pairs (U, ξ) where $U \in \mathbf{Open}(X)$ and $\xi \in F(U)$. There is exactly one arrow $(U, \xi) \to (V, \eta)$ whenever $U \subset V$ and $\eta\big|_U = \xi$. Define

$$F^{\text{ét}} = \varinjlim_{(U,\xi) \in \mathbf{Open}(X)/F} U.$$

There is a projection $\pi : F^{\text{ét}} \to X$ by the universal property of the direct limit, setting $U \to X$ to be the inclusion on the (U, ξ) entry.

Exercise 9.9 (The espace étalé)**.** Show that $\pi : F^{\text{ét}} \to X$ is a local homeomorphism. (Hint: Show that the map $U \to F^{\text{ét}}$ associated to $\xi \in F(U)$ is an open embedding using Exercise 9.8 and the fact that such a map is a section.)

Exercise 9.10 (Sheaves and étale spaces are equivalent)**.** Show that the constructions $E \mapsto E^{\text{sh}}$ and $F \mapsto F^{\text{ét}}$ are inverse equivalences between ét(X) and $\mathbf{Sh}(X)$ for any topological space X.

Solution. We construct a natural identification between $\Gamma(U, F^{\text{ét}})$ and U. Given $\xi \in F(U)$, we get $U \to \varinjlim_{(U,\xi)} U$. This gives the desired map $F(U) \to \Gamma(U, F^{\text{ét}})$.

To get the inverse we need the sheaf condition on F. For each $\xi \in F(V)$ we have an open subset $U_{(V,\xi)} \subset F^{\text{ét}}$. Consider the preimages $s^{-1}U_{(V,\xi)} \subset U$. Over each such $s^{-1}U_{(V,\xi)}$ we have a section $\omega_{(V,\xi)} \in \Gamma(U, F^{\text{ét}})$. We have $s^{-1}U_{(V,\xi)} \cap s^{-1}U_{(W,\eta)} = s^{-1}(U_{(V,\xi)} \cap U_{(W,\eta)})$ (since s is injective). On the other hand, $U_{(V,\xi)} \cap U_{(W,\eta)} \subset F^{\text{ét}}$ is $U_{(T,\xi|_T)} = U_{(T,\eta|_T)}$ where T is the largest open subset of $V \cap W$ on which ξ and η agree. Thus $\omega_{(V,\xi)}\big|_{U_{(V,\xi)} \cap U_{(W,\eta)}} = \omega_{(W,\eta)}\big|_{U_{(V,\xi)} \cap U_{(W,\eta)}}$ for all (V, ξ). Therefore these sections glue together to give a section $\omega \in F(U)$.

The verification that these constructions are inverses is omitted. $\square$

9.4 Associated sheaf

Definition 9.11. If F is any presheaf then $(F^{\text{ét}})^{\text{sh}}$ is a sheaf, called the *associated sheaf* of F. We write $F^{\text{sh}} = (F^{\text{ét}})^{\text{sh}}$ for brevity.

Very important. Writing it up can get technical, so it might be more valuable to think it through than to write your proof down carefully.

Exercise 9.12 (Universal property of the associated sheaf)**.** Let F be a presheaf on a topological space X.

(i) Construct a map $F \to F^{\text{sh}}$ and show that it is universal among maps from F to sheaves.

(ii) Prove that for any sheaf G,

$$\operatorname{Hom}_{\mathbf{Psh}(X)}(F, G) \simeq \operatorname{Hom}_{\mathbf{Sh}(X)}(F^{\text{sh}}, G)$$

in a natural way. (This is really a restatement of the first part.)

sec:sheaf-pullback

9.5 Pullback of sheaves

def:fiber-product

Definition 9.13 (Fiber product)**.** If $f : X' \to X$ and $p : E \to X$ are continuous functions, a fiber product is a universal topological space E' fitting into a commutative diagram[3]

$$\begin{tikzcd} E' \arrow[r, "f'"] \arrow[d, "p'"'] & E \arrow[d] \\ X' \arrow[r] & X. \end{tikzcd}$$

We often write $E' = f^{-1}E$ and call E' the pullback of E.

Exercise 9.14 (Existence of fiber product in topological spaces)**.** Show that a fiber product can be constructed in topological as the set of all pairs $(x, e) \in X' \times E$ such that $f(x) = p(e)$, topologized as a subspace of the product.

Exercise 9.15 (Pullback of local homeomorphisms)**.** In the notation of Definition 9.13, suppose that $p : E \to X$ is a local homeomorphism. Show that $p' : E' \to X'$ is also a local homeomorphism.

Definition 9.16 (Pullback of sheaves)**.** If $f : X \to Y$ is a continuous map and G is a sheaf on Y then define

$$f^{-1}G = f^{-1}(G^{\text{ét}})^{\text{sh}}$$

Exercise 9.17. Let $f : X \to Y$ be a continuous map of topological spaces, let F be a sheaf on X, and let G be a sheaf on Y. Construct a natural bijection

$$\mathrm{Hom}_{\mathbf{Sh}(X)}(f^{-1}G, F) \simeq \mathrm{Hom}_{\mathbf{Sh}(Y)}(G, f_*F).$$

Solution. Consider two presheaves on $G^{\text{ét}}$:

$$\begin{aligned} P(U) &= \mathrm{Hom}_{\mathbf{Sh}(Y)}(U^{\text{sh}}, f_*F) = \mathrm{Hom}_{\text{ét}(Y)}(U, (f_*F)^{\text{ét}}) \\ Q(U) &= \mathrm{Hom}_{\mathbf{Sh}(X)}(f^{-1}U^{\text{sh}}, F) = \mathrm{Hom}_{\text{ét}(X)}(f^{-1}U, F^{\text{ét}}). \end{aligned}$$

(Note that an open subset of $G^{\text{ét}}$ is an étale space over Y, so it has a corresponding sheaf of sections.)

Both P and Q are sheaves on $G^{\text{ét}}$. We will show P and Q are isomorphic. It's enough to do this on a basis $\mathscr{U}$ of $G^{\text{ét}}$. We choose the basis of all open subsets of $G^{\text{ét}}$ that project homeomorphically to their images in Y. If U is such a subset then

$$P(U) = \mathrm{Hom}_{\mathbf{Sh}(Y)}(U^{\text{sh}}, f_*F) = f_*F(U) = F(f^{-1}U) = \mathrm{Hom}_{\mathbf{Sh}(X)}(f^{-1}U^{\text{sh}}, F) = Q(U),$$

exactly as required.

Now take global sections of P and Q. We discover:

$$\begin{aligned} \Gamma(G^{\text{ét}}, P) &= \mathrm{Hom}_{\mathbf{Sh}(Y)}(G, f_*F) \\ \Gamma(G^{\text{ét}}, Q) &= \mathrm{Hom}_{\mathbf{Sh}(X)}(f^{-1}G, F) \end{aligned}$$

$\square$

[3] In fact this definition applies in any category.

9.6 Stalks

Definition 9.18. If F is a presheaf over X, write $\pi : F^{\text{ét}} \to X$ for the espace étalé of F. The fiber $\pi^{-1}x$ of $F^{\text{ét}}$ over $x \in X$ is called the *stalk* of F at X and is often denoted F_x.

Exercise 9.19. If F is a presheaf on X, construct a natural isomorphism

$$F_x = \varinjlim_{\substack{U \in \mathbf{Open}(X) \\ x \in U}} F(U).$$

(Hint: One proof of this uses the universal property of f^{-1}, proved in Section 9.5.)

Solution. We use the fact that colimits commute with base change for étale spaces. (To prove this, you can use the fact that the underlying set of a colimit of topological spaces is the colimit of the underlying sets of the topological spaces and a bijection of étale spaces is a homeomorphism.) By definition, we have

$$\begin{aligned} F_x &= \{x\} \times_X F^{\text{ét}} \\ &= \{x\} \times_X \varinjlim_{(U,\xi) \in \mathbf{Open}(X)/F} U \\ &= \varinjlim_{\substack{(U,\xi) \in \mathbf{Open}(X)/F \\ x \in U}} \{x\} \\ &= \varinjlim_{x \in U \in \mathbf{Open}(X)} F(U) \end{aligned}$$

as desired. □

Solution. We do an explicit calculation using adjunction. Let G be any sheaf on $\{x\}$ (i.e., a set) and let $i : x \to X$ be the inclusion. Then $i_* G(U) = G$ if $x \in U$ and $i_* G(U) = 1$ otherwise. To give a map $F \to i_* G$ we must give maps $F(U) \to i_* G(U)$ for all $U \in \mathbf{Open}(X)$. This is trivial except when $x \in U$, so we have to give compatible maps $F(U) \to G$ for all open neighborhoods U of x. In other words, we have to give a map

$$\varinjlim_{\substack{U \in \mathbf{Open}(X) \\ x \in U}} F(U)$$

as desired. □

heaf-locally-ringed

Exercise 9.20. Prove that the stalks of the structure sheaf of a scheme are local rings. (Hint: Reduce immediately to the case of an affine scheme.)

Exercise 9.21. Let η be a point of a topological space X and let ξ be a point of X in the closure of η. Fix a set S and let F be the skyscraper sheaf at η associated to S. (If $j : \eta \to X$ is the inclusion then $F = j_* S$.) Compute the stalk of F at ξ. (If $i : \xi \to X$ is the inclusion then you are computing $i^{-1} j_* S$.)

10 Morphisms of schemes

Reading 10.1. [Vak14, §§6.1–6.3,7.1,8.1], [MO, § I.3]

10.1 Morphisms of ringed spaces

morph-ringed-spaces

Definition 10.2 (Morphisms of ringed spaces)**.** A *morphism of ringed spaces* from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a continuous function $\varphi : X \to Y$ and a homomorphism of sheaves of rings $\varphi^* : \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$.[4]

Exercise 10.3. Show that we could equivalently have specified a morphism of ringed spaces as a continuous function $\varphi : X \to Y$ and a homomorphism of sheaves of rings $\varphi^* : \varphi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$.

Exercise 10.4. (i) Show that a homomorphism of commutative rings $\varphi : B \to A$ induces a morphism of *ringed spaces* $f : \operatorname{Spec} A \to \operatorname{Spec} B$.

Solution. We have already gotten the map of topological spaces. We need to get the map $\mathcal{O}_{\operatorname{Spec} B} \to f_* \mathcal{O}_{\operatorname{Spec} A}$. It's sufficient to specify this on a basis. Suppose $g \in B$. Then $\mathcal{O}_{\operatorname{Spec} B}(D(g)) = B[g^{-1}]$ and $f_* \mathcal{O}_{\operatorname{Spec} A}(D(g)) = \mathcal{O}_{\operatorname{Spec} A}(D(\varphi(g))) = A[\varphi(g)^{-1}]$. There is a canonical map

$$B[g^{-1}] \to A[\varphi(g)^{-1}]$$

induced by the universal property of $B[g^{-1}]$ and the map $B \to A \to A[\varphi(g)^{-1}]$. □

(ii) Show that $f^{-1}D(g) = D(\varphi(g))$ for any $g \in A$.

Not difficult, but not important either. Could be good practice with sheaves if you are new to sheaves.

Exercise 10.5. Construct a morphism of ringed spaces $\operatorname{Spec} B \to \operatorname{Spec} A$ that is does not come from a morphism of rings $A \to B$.

Solution. Let $A = \mathbf{Z}$ and let $B = \mathbf{Q}$. Let ξ be a nonzero prime ideal of $\mathbf{Z}$. We get a map $f : \operatorname{Spec} \mathbf{Q} \to \operatorname{Spec} \mathbf{Z}$ sending the unique point of $\operatorname{Spec} \mathbf{Q}$ to ξ. Then $f^{-1}\mathcal{O}_{\operatorname{Spec} \mathbf{Z}} = \mathbf{Z}_\xi$ is the localization of $\mathbf{Z}$ at ξ. Let $f^* : \mathbf{Z}_\xi \to \mathbf{Q}$ be the inclusion (in fact, this is the only such map). This is a map of ringed spaces but there is no map $\mathbf{Z} \to \mathbf{Q}$ such that the preimage of (0) is ξ. □

10.2 Locally ringed spaces

Suppose that $(X, \mathcal{O}_X)$ is a ringed space and $f \in \Gamma(U, \mathcal{O}_X)$ for some open $U \subset X$. Let $D(f)$ be the largest open subset U of X such that $f\big|_U$ has a multiplicative inverse.

Important to know, less important to do.

Exercise 10.6. Use the sheaf conditions to prove that $D(f)$ exists. (One approach: Let $F(U) = \{g \in \mathcal{O}_X(U) \mid gf = 1\}$. Show that F is a sheaf and that $F(U)$ is either empty or a 1-element set for all U. Conclude that there is an open $V \subset X$ such that $F(U) = 1$ if and only if $V \subset U$.)

Definition 10.7 (Locally ringed space [AGV 3, Exercise IV.13.9])**.** A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that if $f_1, \ldots, f_n \in \Gamma(U, \mathcal{O}_X)$ and $(f_1, \ldots, f_n) = \Gamma(U, \mathcal{O}_X)$ then $D(f_1) \cup \cdots \cup D(f_n) = U$.

Exercise 10.8. (i) Prove that any ringed space with an open cover by locally ringed spaces is a locally ringed space.

[4] Warning: Other authors often use $\varphi^\sharp$ instead of φ^* here.

Solution. Suppose $f_1, \ldots, f_n \in \Gamma(U, \mathcal{O})$ generate the unit ideal. We argue that $D(f_1) \cup \cdots \cup D(f_n) = U$. Choose a cover of U by open subsets V_i that are locally ringed spaces. Then $D_{V_i}(f_j) = D(f_j) \cap V_i$. Then

$$V_i \cap (D(f_1) \cup \cdots \cup D(f_n)) = (V_i \cap D(f_1)) \cup \cdots \cup (V_i \cap D(f_n)) = V_i$$

since V_i is a locally ringed space. But then taking the union over i, we get

$$D(f_1) \cup \cdots \cup D(f_n) = \bigcup_{i=1}^n V_i \cap (D(f_1) \cup \cdots \cup D(f_n)) = \bigcup_{i=1}^n V_i = U$$

as we wanted. □

(ii) Prove that affine schemes are locally ringed spaces.

Solution. It's the definition of the Zariski topology! If $(f_1, \ldots, f_n) = A$ then we have $V(f_1, \ldots, f_n) = \varnothing$ so its complement $D(f_1, \ldots, f_n) = D(f_1) \cup \cdots \cup D(f_n)$ is all of $\operatorname{Spec} A$. □

(iii) Conclude that all schemes are locally ringed spaces.

This is important to know for schemes (Exercise 9.20), much less important to know for locally ringed spaces. For schemes, the verification should be easy.

Exercise 10.9. (i) Prove that $x \in U$ is in $D(f)$ if and only if the germ of f at x is invertible.

Solution. Suppose $x \in D(f)$. Then $(f\big|_U)_x^{-1}$ is the inverse of f in $\mathcal{O}_{X,x}$.

Suppose f_x is invertible in $\mathcal{O}_{X,x}$. Let $g \in \Gamma(V, \mathcal{O}_X)$ represent the inverse. We can assume $V \subset U$. Then $(f\big|_V g)_x = 1$. Therefore there is some open $W \subset V$ with $x \in W$ such that $f\big|_W g\big|_W = 1$. Therefore $x \in W \subset D(f)$. □

This is the usual definition of a locally ringed space, so this is important to know for the sake of communication. It's not an important exercise.

(ii) Prove that a ringed space $(X, \mathcal{O}_X)$ is a locally ringed space if and only if all of the stalks of $\mathcal{O}_X$ are local rings. (Hint: A commutative ring is local if and only if its non-unit elements form an ideal.)

Solution. Suppose X is a locally ringed space and $x \in X$. We want to show that $\mathcal{O}_{X,x}$ is a local ring. Equivalently, we want to show that the nonunits form an ideal. Suppose $f, g \in \mathcal{O}_X(U)$ represent non-units. Then $D(f) \cup D(g) = U$. Therefore $x \in D(f)$ or $x \in D(g)$. Thus either f or g is a unit in $\mathcal{O}_{X,x}$.

Conversely, suppose that each $\mathcal{O}_{X,x}$ is a local ring. Let $f_1, \ldots, f_n \in \mathcal{O}_X(U)$. If they generate the unit ideal then they generate the unit ideal in $\mathcal{O}_{X,x}$. Therefore at least one f_i is a unit in $\mathcal{O}_{X,x}$. So $x \in D(f_i)$. So $\bigcup D(f_i) = U$. □

If $f \in \Gamma(X, \mathcal{O}_X)$ then we can regard $\mathcal{O}_X$ as a sheaf of 'functions' on X: Restrict f to the stalk $\mathcal{O}_{X,\xi}$. Let $\mathfrak{m}_\xi$ be the maximal ideal of $\mathcal{O}_{X,\xi}$. Then the residue of f in the *residue field* $\mathbf{k}(\xi) = \mathcal{O}_{X,\xi}/\mathfrak{m}_\xi$ is the *value* of f at ξ.

Exercise 10.10. Show that this definition coincides with the more familiar notion of value for $\xi \in \mathbf{C}^n \subset \operatorname{Spec} \mathbf{C}[x_1, \ldots, x_n]$.

Exercise 10.11. Show that when $f(\xi)$ is interpreted as the value of ξ in the residue field of ξ that $D(f)$ is the set of points ξ where $f(\xi) \neq 0$.

10.3 Morphisms of locally ringed spaces

ef:schemes-morphism

Definition 10.12 (Morphisms of schemes)**.** If $\varphi : X \to Y$ is a morphism of ringed spaces and both X and Y are locally ringed spaces and $\varphi^{-1}(D_U(f)) = D_{\varphi^{-1}U}(\varphi^* f)$ for any open $U \subset Y$ and any $f \in \Gamma(U, \mathcal{O}_Y)$ then we say φ is a *morphism of locally ringed spaces.*

A *morphism of schemes* $\varphi : X \to Y$ is a morphism of the underlying locally ringed spaces.

Exercise 10.13. Suppose that $f : X \to Y$ is a morphism of ringed spaces.

(i) Suppose there is a cover of Y by open subsets U such that $f^{-1}U \to U$ is a morphism of locally ringed spaces. Show that f is a morphism of locally ringed spaces.

Solution. Cover X by open subsets U_i such that $f^{-1}U_i \to U_i$ is a morphism of locally ringed spaces. Suppose $V \subset X$ is open and $x \in \Gamma(V, \mathcal{O}_X)$. Then

$$f^{-1}D(x) = f^{-1}\bigcup_i U_i \cap D(x) = \bigcup_i D_{f^{-1}U_i}(f^* x\big|_{f^{-1}U_i}) = D_{\bigcup f^{-1}U_i}(f^* x) = D_{f^{-1}V}(f^* x)$$

as desired. □

(ii) Suppose that there is a cover of X by open subsets U such that $U \to Y$ is a morphism of locally ringed spaces. Show that f is a morphism of locally ringed spaces.

Solution. Suppose $x \in \Gamma(V, \mathcal{O}_Y)$ for some open $V \subset Y$. Let $U_i \subset X$ be a cover of X by open subsets such that each map $U_i \to Y$ is local morphism. Then

$$f^{-1}D(x) = \bigcup_i f\big|_{U_i}^{-1} D(x) = \bigcup_i D_{U_i}(f^* x) = D(f^* x)$$

as desired. □

Solution. Both parts of the exercise can be solved simultaneously using Exercise 10.14, below: The map f is a local morphism if and only if the maps $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ are local homomorphisms of local rings for all $x \in X$. This condition only depends on an open neighborhood of $x \in X$, hence can be verified on a cover. □

In other words, if X and Y are schemes then a morphism of ringed spaces $f : X \to Y$ is a morphism of schemes if for each point $x \in X$, there is an open affine neighborhood $U = \operatorname{Spec} A$ of X and an open affine neighborhood $V = \operatorname{Spec} B$ of $f(x)$ such that $f(U) \subset V$ and the map $\operatorname{Spec} A \to \operatorname{Spec} B$ is induced from a homomorphism $B \to A$.

ringed-space-stalks

Exercise 10.14 (The usual definition of morphisms of locally ringed spaces [AGV 3, Exercise IV.13.9 c)])**.** With notation as in Definition 10.12, show φ is a morphism of locally ringed spaces if and only if for every point x of X, the map $\varphi^* : \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism of local rings. (Recall that this means $\varphi^* \mathfrak{m}_{\varphi(x)} \subset \mathfrak{m}_x$.)

Solution. First we show that Definition 10.12 implies the morphisms on stalks are local. Suppose $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of locally ringed spaces. Choose $x \in X$. Suppse $f \in \mathfrak{m}_{\varphi(x)}$. Represent f as a local section in $\Gamma(U, \mathcal{O}_Y)$. Then $\varphi(x) \notin D_U(f)$. Thus $x \notin \varphi^{-1} D_U(f) = D_{\varphi^{-1}U}(\varphi^* f)$. Thus $\varphi^* f$ lies in $\mathfrak{m}_x$, by definition of $D_{\varphi^{-1}U}(\varphi^* f)$.

Conversely, suppose that the homomorphisms of local rings are local homomorphisms. Fix $U \subset Y$ open and $f \in \Gamma(U, \mathcal{O}_Y)$. We want to show that $x \in D_{\varphi^{-1}U}(\varphi^* f)$ if and only if $\varphi(x) \in D_U(f)$. But recall that $x \in D_{\varphi^{-1}U}(\varphi^* f)$ if and only if $\varphi^* f \notin \mathfrak{m}_x$ and $\varphi(x) \in D_U(f)$ if and only if $f \in \mathfrak{m}_{\varphi(x)}$. Thus we have

$$\begin{aligned} x \in D_{\varphi^{-1}U}(\varphi^* f) &\iff \varphi^* f \notin \mathfrak{m}_x \\ &\iff f \notin \mathfrak{m}_{\varphi(x)} \\ &\iff \varphi(x) \in D_U(f) \end{aligned}$$

as desired. □

Since $\varphi^* : \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism, it induces a homomorphism:

$$\mathbf{k}(\varphi(x)) = \mathcal{O}_{Y,\varphi(x)}/\mathfrak{m}_{\varphi(x)} \to \mathcal{O}_{X,x}/\mathfrak{m}_x = \mathbf{k}(x)$$

We also write φ^* for this homomorphism.

Conversely, if $\varphi : X \to Y$ is a morphism of locally ringed spaces, then the condition in Exercise 10.14 is equivalent to saying that there is a commutative diagram (10.1):

$$\begin{tikzcd} \mathcal{O}_{Y,\varphi(x)} \arrow[r] \arrow[d] & \mathcal{O}_{X,x} \arrow[d] \\ \mathbf{k}(\varphi(x)) \arrow[r] & \mathbf{k}(x) \end{tikzcd} \tag{10.1}$$

`eqn:alt-lrs`

We can summarize this in a motto:

> *A locally ringed space is a ringed space where functions have values at points. A morphism of locally ringed spaces is a morphism of ringed spaces $\varphi : X \to Y$ such that the formula $\varphi^* f(x) = f(\varphi(x))$ makes sense.*

10.4 Morphisms to affine schemes

Reading 10.15. [MO, Theorem I.3.7]

Theorem 10.16. *If $(X, \mathcal{O}_X)$ is a locally ringed space then* $\mathrm{Hom}(X, \operatorname{Spec} A) = \mathrm{Hom}(A, \Gamma(X, \mathcal{O}_X))$ *in a natural way.*

Proof. Note that a map $f : X \to \operatorname{Spec} A$ gives $\mathcal{O}_{\operatorname{Spec} A} \to f_* \mathcal{O}_X$. Taking global sections, we get

$$A \to \Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \to \Gamma(\operatorname{Spec} A, f_* \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X).$$

Conversely, suppose we have a homomorphism $\psi : A \to \Gamma(X, \mathcal{O}_X)$. For any $x \in X$, we get a map

$$A \to \Gamma(X, \mathcal{O}_X) \to \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/\mathfrak{m}_x. \tag{10.2}$$

`eqn:1`

The kernel of this map is a prime ideal, so this gives a map of topological spaces $\varphi : X \to \operatorname{Spec} A$. To see it is continuous, suppose that $f \in A$. Then $\varphi^{-1} D(f)$ is the collection of all $x \in X$ such that the map (10.2) does not contain f in its kernel. This is precisely $D(\psi f)$

To specify the map of sheaves of rings, it is enough to specify it on a basis of open sets. We need to specify maps

$$\begin{array}{ccc} \Gamma(D(f), \mathcal{O}_{\operatorname{Spec} A}(D(f))) & \xrightarrow{\varphi^*} & \Gamma(D(f), \varphi_* \mathcal{O}_X) \\ \| & & \| \\ A[f^{-1}] & & \Gamma(D(\psi f), \mathcal{O}_X). \end{array}$$

But by definition, the restriction of ψf to $\Gamma(D(\psi f), \mathcal{O}_X)$ has an inverse, so the map $A \to \Gamma(D(\psi f), \mathcal{O}_X)$ factors through $A[f^{-1}]$, by the universal property of localization.

Exercise 10.17. Complete the proof by verifying these two constructions are inverse to one another.

□

Exercise 10.18. If A and B are commutative rings then

$$\operatorname{Hom}_{\mathbf{Sch}}(\operatorname{Spec} B, \operatorname{Spec} A) = \operatorname{Hom}_{\mathbf{ComRng}}(A, B).$$

Solution. We have

$$\operatorname{Hom}(\operatorname{Spec} B, \operatorname{Spec} A) = \operatorname{Hom}(A, \Gamma(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B})) = \operatorname{Hom}(A, B).$$

□

Exercise 10.19. (i) Show that for any affine scheme, $\operatorname{Hom}(\operatorname{Spec} A, \mathbf{A}^1) = A$ in a natural way.

(ii) Show that for any scheme, $\operatorname{Hom}(X, \mathbf{A}^1) = \Gamma(X, \mathcal{O}_X)$.

10.5 Morphisms from affine schemes

Reading 10.20. [MO, Proposition I.3.10]

In general, it is difficult to characterize the morphisms *from* an affine scheme *to* an arbitrary locally ringed space. The reason for this is that we can only desribe what morphisms look like locally, and affine schemes generally have many open covers. However, there are some affice schemes with no nontrivial open covers.

def:llrs **Definition 10.21.** A locally ringed space is called *local* if it contains a point that is in the closure of all other points.

Exercise 10.22. Prove that a scheme is local if and only if it is the spectrum of a local ring.

Proof. Let x be a point of X that is in the closure of all other points. Since X is a scheme, x has an affine open neighborhood U. But U must be all of X: if $y \in X$ then x is in the closure of $\{y\}$, so every open neighborhood of x contains y. This means that X is affine.

Say $X = \operatorname{Spec} A$. Then the points of X correspond to the prime ideals of A and $\mathfrak{p}$ is in the closure of $\mathfrak{q}$, if and only if $f(\mathfrak{q}) = 0$ implies $f(\mathfrak{p}) = 0$, if and only if $\mathfrak{q} \supset \mathfrak{p}$. Thus every prime ideal is contained in $\mathfrak{m}_x$, which means A is a local ring. □

Exercise 10.23. Let X be a local locally ringed space (Definition 10.21) with closed point x and let Y be a scheme. Then to give a local homomorphism $X \to Y$ is the same as to give a point y of Y and a local homomorphism of local rings $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$.

Proof. Certainly $\varphi : X \to Y$ gives $y = \varphi(x)$ and $\varphi^* : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. Conversely, suppose that we are given $y \in Y$ and $\psi : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. Pick an affine open neighborhood $U = \operatorname{Spec} A$ of y in Y. We have canonical morphisms:

$$X \to \operatorname{Spec} \Gamma(X, \mathcal{O}_X) \xleftarrow{\sim} \operatorname{Spec} \mathcal{O}_{X,x} \xrightarrow{\operatorname{Spec} \psi} \operatorname{Spec} \mathcal{O}_{Y,y} \to \operatorname{Spec} A = U \subset Y$$

Note that the map $\Gamma(X, \mathcal{O}_X) \to \mathcal{O}_{X,x}$ is an isomorphism since X is local. The composition gives us the desired map $X \to Y$. □

11 Fiber products of schemes

Reading 11.1. [MO, § I.4]

Suppose that we have a diagram

$$\begin{tikzcd} & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd} \tag{11.1}$$

eqn:fib-prod-proble

in some category $\mathscr{C}$. A fiber product of X and Y over Z is a universal (final) object $X \times_Z Y$ that completes the diagram to a commutative square. That means that we have a commutative square

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

and that any commutative diagram of solid lines (11.2) can be completed by a *unique* dashed arrow:

$$\begin{tikzcd} W \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d] \\ & Y \arrow[r] & Z \end{tikzcd} \tag{11.2}$$

eqn:fib-prod

When Z is the final object of the category $\mathscr{C}$ then the fiber product is called the product, and it is denoted $X \times Y$.

Exercise 11.2. Construct the fiber product in the category of sets. Use this construction to construct the fiber product in the categories of groups, rings, and topological spaces.

Exercise 11.3. Show that fiber products do not always exist in the category of manifolds. However, show that diagram (11.1) does have a fiber product in the category of manifolds if one or the other of the maps $X \to Z$ or $Y \to Z$ is *submersive*.

Exercise 11.4. Let $\mathbf{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$. Show that $\mathbf{A}^2_k \simeq \mathbf{A}^1_k \times_{\operatorname{Spec} k} \mathbf{A}^1_k$ but that this isomorphism does not identify $|\mathbf{A}^2|$ with $|\mathbf{A}^1| \times |\mathbf{A}^1|$.

Exercise 11.5. Generalizing the previous exercise, suppose in diagram 11.1 that $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} C$, and $Z = \operatorname{Spec} A$. Prove that $\operatorname{Spec}(B \otimes_A C)$ is a fiber product of diagram 11.1 in the category of locally ringed spaces.

ex:fib-prod-points

Exercise 11.6. Give an example of a diagram (11.1) in the category of schemes (or locally ringed spaces) where $|X|$, $|Y|$, and $|Z|$ all consist of exactly one point, and $X \times_Z Y$ exists but $|X \times_Z Y|$ is not just one point.

Exercise 11.7. Show that if diagram (11.1) is a diagram in a category $\mathscr{C}$ with fiber product $X \times_Z Y$ then, for any $W \in \mathscr{C}$, we have a canonical bijection

$$\operatorname{Hom}_{\mathscr{C}}(W, X \times_Z Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathscr{C}}(W, X) \times_{\operatorname{Hom}_{\mathscr{C}}(W,Z)} \operatorname{Hom}_{\mathscr{C}}(W, Y)$$

where the fiber product on the right side is formed in the category of sets.

In this lecture, we want to see that there are fiber products in the category of schemes. In fact, we will prove that fiber products exist in the category of locally ringed spaces and show that the fiber product of schemes is a scheme.

If we want to construct the fiber product, we will need to construct its underlying topological space. In all of the examples considered above, the category in question had a 'point object' P such that the underlying set of X was $\operatorname{Hom}(P, X)$. In view of the exercise above, this meant that the underlying topological space of the fiber product *had* to be the fiber product of the underlying topological spaces. We have no 'point object' in the category of schemes, so we will have to be more careful.

However, we can *figure out* what the underlying set of $X \times_Z Y$ must be with a similar sort of reasoning. In fact, Exercise 11.6 tells us exactly what to do:

ts-of-fiber-product

Exercise 11.8. Suppose that $X \times_Z Y$ exists.

(i) For each $x \in X$ and $y \in Y$ with common image $z \in Z$, construct a map

$$x \times_z y \to X \times_Y Z.$$

(ii) Observe that $x \times_z y = \operatorname{Spec}\big(\mathbf{k}(x) \otimes_{\mathbf{k}(z)} \mathbf{k}(y)\big)$.

(iii) Suppose that $X \times_Y Z$ exists. Use the map from the first part to construct a bijection

$$\bigcup_{\substack{x \in |X|, y \in |Y|, z \in |Z| \\ f(x) = z = g(y)}} \big|\operatorname{Spec}\big(\mathbf{k}(x) \otimes_{\mathbf{k}(z)} \mathbf{k}(y)\big)\big| = \bigcup_{\substack{x \in |X|, y \in |Y|, z \in |Z| \\ f(x) = z = g(y)}} |x \times_z y| \xrightarrow{\sim} |X \times_Z Y|.$$

Solution. We construct an inverse. Suppose $t \in |X \times_Z Y|$. Let $x \in X$, $y \in Y$, and $z \in Z$ be the projections. Then we get a commutative diagram of fields:

$$\begin{tikzcd} \mathbf{k}(z) \arrow[r] \arrow[d] & \mathbf{k}(x) \arrow[d] \\ \mathbf{k}(y) \arrow[r] & \mathbf{k}(t) \end{tikzcd} \qquad (11.3)$$ eqn:fib-prod-res

This induces a homomorphism $\mathbf{k}(x) \otimes_{\mathbf{k}(z)} \mathbf{k}(y) \to \mathbf{k}(t)$. Let $\mathfrak{p}$ be the kernel. This is a point of $|\operatorname{Spec}(\mathbf{k}(x) \otimes_{\mathbf{k}(z)} \mathbf{k}(y))| \subset |T|$. This gives the required function.

Now, suppose that t is a point of $x \times_z y$. Let t' be its image in $X \times_Z Y$. Since $f(t') = x$ and $g(t') = y$ we get a unique map $t' \to x \times_z y$ and we let t'' be its image. Now we have

$$t \to t' \to t'' \to x \times_y z.$$

But therefore t and t' represent the same point of $|x \times_y z|$.

Now suppose that t is a point of $X \times_Y Z$ with images x and y and z in X and Y and Z. Then let t' be the image of t in $x \times_y z$ and let t'' be the image of t' in $X \times_Y Z$. Then we have

$$t \to t' \to t'' \to X \times_Y Z$$

so that t and t'' represent the same point of $X \times_Y Z$, as required. $\square$

We can turn this around and use it to *construct* $|X \times_Z Y|$. Let

$$|X \times_Z Y| = \bigcup_{\substack{x \in X, y \in Y, z \in Z \\ f(x) = z = g(y)}} x \times_z y.$$

Exercise 11.9. Show that there is a commutative diagram of functions:

$$\begin{tikzcd}
{|X \times_Z Y|} \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, "r"] & {|X|} \arrow[d] \\
{|Y|} \arrow[r] & {|Z|}
\end{tikzcd}$$

Solution. The commutative diagram of functions is immediate from the construction of $|X \times_Z Y|$. We only need to verify the continuity of p and q. Suppose that $U \subset |X|$ is open. $\square$

Suppose that $U \subset X$, $V \subset Y$, and $W \subset Z$ are open subsets such that $f(U) \subset W$ and $g(V) \subset W$. Let $A = \Gamma(W, \mathcal{O}_W)$, let $B = \Gamma(U, \mathcal{O}_U)$, and let $C = \Gamma(V, \mathcal{O}_V)$. Then we have a function

$$p^{-1}|U| \cap q^{-1}|V| = |U \times_W V| \to |\operatorname{Spec} B \otimes_A C|.$$

Give $|X \times_Z Y|$ the coarsest topology so that all of these maps are continuous, for all open subsets $U \subset X$, $V \subset Y$, and $W \subset Z$, and the maps p and q are continuous.

Define:

$$\mathcal{O}_{X \times_Z Y} = p^{-1}\mathcal{O}_X \otimes_{r^{-1}\mathcal{O}_Z} q^{-1}\mathcal{O}_Y$$

Exercise 11.10. Let $\mathscr{A}$ be a sheaf of commutative rings on a topological space X, and let $\mathscr{F}$ and $\mathscr{G}$ be $\mathscr{A}$-modules. Define $\mathscr{F} \otimes_{\mathscr{A}} \mathscr{G}$ to be the sheaf associated to the presheaf whose value on $U \subset X$ is $\mathscr{F}(U) \otimes_{\mathscr{A}(U)} \mathscr{G}(U)$.

(i) Show that $\mathscr{F} \otimes_{\mathscr{A}} \mathscr{G}$ has the following universal property: any bilinear map of sheaves of $\mathscr{A}$-modules

$$\mathscr{F} \times \mathscr{G} \to \mathscr{H}$$

factors uniquely through a linear map

$$\mathscr{F} \otimes_{\mathscr{A}} \mathscr{G} \to \mathscr{H}.$$

(ii) Now suppose that $\mathscr{B}$ and $\mathscr{C}$ are $\mathscr{A}$-algebras. Show that $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{C}$ has the following universal property: any pair of $\mathscr{A}$-algebra homomorphisms $\mathscr{B} \to \mathscr{D}$ and $\mathscr{C} \to \mathscr{D}$ factor uniquely through an $\mathscr{A}$-algebra homomorphism

$$\mathscr{B} \otimes_{\mathscr{A}} \mathscr{C} \to \mathscr{D}.$$

Exercise 11.11. Verify that $X \times_Z Y$, as constructed above, has the universal property of a fiber product in the category of locally ringed spaces.

Proof. Suppose that we have a commutative diagram of locally ringed spaces:

$$\begin{tikzcd} W \arrow[r, "p_1"] \arrow[d, "q_1"'] \arrow[dr, "r_1"] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

Suppose $w \in W$. Let $x = p_1(w)$ and let $y = q_1(w)$ and let $z = f(p_1(w)) = g(q_1(w))$. Then we get

$$w \to x \times_z y$$

and hence a well-defined point of $|X \times_Z Y|$. This gives a function $|W| \to |X \times_Z Y|$. We check it is continuous. Since the topology on $|X \times_Z Y|$ is given universally, we need to check p_1 and q_1 are continuous (which is given) and that if $U \subset X$, $V \subset Y$, and $W \subset Z$, with $A = \Gamma(W, \mathcal{O}_W)$, $B = \Gamma(U, \mathcal{O}_U)$, and $C = \Gamma(V, \mathcal{O}_V)$ open subsets then the map $p_1^{-1}(U) \cap q_1^{-1}(V) \to \operatorname{Spec} B \otimes_A C$ is continuous. But the maps

$$\begin{tikzcd} p_1^{-1}U \cap q_1^{-1}V \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\ \operatorname{Spec} C \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

are continuous precisely because $p^{-1}U \to X$ and $q^{-1}V \to Y$ are morphisms of locally ringed spaces. Then use the universal property of $\operatorname{Spec}(B \otimes_A C) = \operatorname{Spec} B \times_{\operatorname{Spec} A} \operatorname{Spec} C$ as a fiber product.

Now that we have a continuous map $h : |W| \to |X \times_Z Y|$ we need to augment it to a morphism of ringed spaces. We certainly have homomorphisms of commutative rings:

$$h^{-1}p^{-1}\mathcal{O}_X = p_1^{-1}\mathcal{O}_X \to \mathcal{O}_W$$
$$h^{-1}q^{-1}\mathcal{O}_Y = q_1^{-1}\mathcal{O}_Y \to \mathcal{O}_W$$

By the universal property of tensor product, we have

$$\begin{aligned} h^{-1}\mathcal{O}_{X \times_Z Y} &= h^{-1}\big(p^{-1}\mathcal{O}_X \otimes_{r^{-1}\mathcal{O}_Z} q^{-1}\mathcal{O}_Y\big) = h^{-1}p^{-1}\mathcal{O}_X \otimes_{h^{-1}r^{-1}\mathcal{O}_Z} h^{-1}q^{-1}\mathcal{O}_Y \\ &= p_1^{-1}\mathcal{O}_X \otimes_{r_1^{-1}\mathcal{O}_Z} q_1^{-1}\mathcal{O}_Y \\ &\to \mathcal{O}_W \end{aligned}$$

as required.

Now we must check it is a morphism of locally ringed spaces. Suppose that a is a local section of $\mathcal{O}_{X \times_Y Z}$. We want to show that $h^{-1}D(a) = D(h^*a)$. This is a local problem on

$|X \times_Z Y|$, so we can replace by an open cover. In particular, we can assume that a is a global section of $\mathcal{O}_{X\times_Z Y}$. Restricting to a smaller open cover, it is even possible to represent a locally as a finite linear combination $\sum b_i \otimes c_i$ where the b_i and c_i are global sections of $\mathcal{O}_X$ and $\mathcal{O}_Y$, respectively. Then $\sum b_i \otimes c_i$ is an element of $R = \Gamma(X, \mathcal{O}_X) \otimes_{\Gamma(Z,\mathcal{O}_Z)} \Gamma(Y, \mathcal{O}_Y)$, and a is pulled back from $X \times_Z Y \to \operatorname{Spec} R$. But $W \to \operatorname{Spec} R$ is a morphism of locally ringed spaces, so we may conclude. □

Exercise 11.12. Show that if X, Y, and Z are all schemes then $X \times_Z Y$ is a scheme.

12 Properties of morphisms

12.1 Nilpotents

Exercise 12.1. Let $A = \operatorname{Spec} \mathbf{C}[\epsilon]/(\epsilon^2)$. Show A has only one point but that there are nonzero functions $\operatorname{Spec} A \to \mathbf{A}^1$ that take the value 0 at this point.

12.2 Open embeddings

Exercise 12.2 (Open subschemes)**.** If X is a scheme $U \subset X$ is an open subset, define $\mathcal{O}_U$ to be the restriction of $\mathcal{O}_X$ to U. Show that U is a scheme.

Definition 12.3. A morphism of schemes $U \to X$ is said to be an *open embedding* if it can be factored as $U \to V \to X$ where $U \to V$ is an isomorphism and $V \to X$ is the inclusion of an open subscheme.

12.3 Affine morphisms

Definition 12.4 (Affine morphism)**.** A morphism of schemes $f : X \to Y$ is said to be affine if, whenever $U \subset Y$ is an affine open subset, $f^{-1}U \subset X$ is an affine open subset.

Exercise 12.5. Show that any morphism between affine schemes is affine.

12.4 Closed embeddings

Definition 12.6. A morphism of schemes $f : Z \to X$ is called a *closed embedding* if it is affine and for all affine open subsets $U \subset X$ with $U = \operatorname{Spec} A$ and $f^{-1}U = \operatorname{Spec} B$, the map $A \to B$ is a surjection.

Exercise 12.7. Show that $f : Z \to X$ is a closed embedding if and only if it is the inclusion of a closed subset and the map $f^* : \mathcal{O}_X \to f_*\mathcal{O}_Z$ is surjective.

Exercise 12.8. Show that a morphism of schemes $f : Z \to X$ is a closed embedding if and only if there is a cover of X by open affine subschemes $U = \operatorname{Spec} A$ such that $f^{-1}U \to U$ is isomorphic to $\operatorname{Spec} A/I \to \operatorname{Spec} A$ for some ideal I of A.

12.5 Locally closed embeddings

def:locally-closed

Definition 12.9. A *locally closed subscheme* is a closed subscheme of an open subscheme. A morphism of schemes $f : Z \to X$ is called a *locally closed embedding* if it can be factored as a closed embedding followed by an open embedding.

It's valuable to know pathologies like this can exist, less important to have actually seen them.

Exercise 12.10.

(i) Let $X = \mathbf{A}^\infty = \operatorname{Spec} \mathbf{C}[x_1, x_2, \ldots]$ and let $U \subset X$ be the complement of the origin $(0, 0, \ldots)$. (In other words $U = D(x_1, x_2, \ldots)$.) Show that there is a closed subscheme $Y \subset U$ such that $Y \cap D(x_m)$ is

$$Y \cap D(x_m) = \operatorname{Spec} k[x_1, x_2, \ldots, x_m^{-1}]/(x_1^m, x_2^m, \ldots, x_{m-1}^m, x_{m+1}, x_{m+2}, \ldots).$$

(ii) Show that the smallest closed subscheme of X containing Y is X itself.

(iii) Conclude that Y is not an open subscheme of a closed subscheme of X.

This is a surprisingly important fact.

Exercise 12.11. Show that if X is a noetherian scheme then every locally closed subscheme of X is an open subscheme of a closed subscheme of X.

12.6 Relative schemes

We frequently want to distinguish between coefficients in a ring and variables. For example, if you define the coordinate ring of a curve over $\mathbf{C}$ as $\mathbf{C}[x, y]/(f(x, y))$, don't want to think of automorphisms of $\mathbf{C}$ as automorphisms of the curve. We accomplish this algebraically by introducing the category of $\mathbf{C}$-*algebras*.

Definition 12.12. Let A be a commutative ring. An A-*algebra* is a pair (B, φ) where B is a commutative ring and $\varphi : A \to B$ is a homomorphism of commutative rings. A homomorphism of A-algebras $(B, \varphi) \to (C, \psi)$ is a homomorphism of commutative rings $f : B \to C$ such that $f \circ \varphi = \psi$.

In other words, homomorphisms of A-algebras are homomorphisms of commutative rings that hold the coefficient ring constant. If we translate this geometrically, we obtain the notion of a relative scheme:

Definition 12.13. Let A be a commutative ring. An A-*scheme* is a pair (X, π) where $\pi : X \to \operatorname{Spec} A$ is a morphism of schemes. A morphism of A-schemes $(X, \pi) \to (Y, \tau)$ is a morphism of schemes $f : X \to Y$ such that $\tau \circ f = \pi$.

Should be easy. Do it if it's not obvious.

Exercise 12.14. Verify that A-algebras and affine A-schemes are contravariantly equivalent categories. (You'll have to define an affine A-scheme. There are two obvious definitions, both equivalent.)

We can generalize this definition and think about schemes that are constructed using coefficients coming from the structure sheaf of another scheme:

Definition 12.15. Let S be a scheme. An S-*scheme* is a pair (X, π) where $\pi : X \to S$ is a morphism of schemes. A morphism of S-schemes $(X, \pi) \to (Y, \tau)$ is a morphism of schemes $f : X \to Y$ such that $\tau \circ f = \pi$.

Usually when we are working with S-schemes we refer to an S-scheme (X, π) as X and sometimes don't even introduce a letter for π. This shouldn't be a source of confusion, since π is usually clear from context.

12.7 Examples

Exercise 12.16. Show that the disjoint union of two schemes is a scheme in a natural way (you will have to specify the structure sheaf yourself). Show that your construction has the universal property of a coproduct: $\mathrm{Hom}(X \amalg Y, Z) = \mathrm{Hom}(X, Z) \times \mathrm{Hom}(Y, Z)$ for all Z.

Recommended exercise!

Exercise 12.17. Let X be a scheme (in fact, X can be a locally ringed space). Construct a bijection between maps $X \to \mathbf{P}^1$ and the set of quadruples (V_0, V_1, x_0, x_1) such that

(i) $V_i \subset X$ are open subsets of X with $V_0 \cup V_1 = X$,

(ii) $x_i \in \Gamma(V_i, \mathcal{O}_X)$, and

(iii) $x_0\big|_{V_0\cap V_1} x_1\big|_{V_0\cap V_1} = 1$.

Solution. Suppose that $f : X \to \mathbf{P}^1$ is a morphism of schemes. Set $V_i = f^{-1}U_i$. Note $U_0 = \operatorname{Spec} \mathbf{Z}[t]$ and $U_1 = \operatorname{Spec} \mathbf{Z}[t^{-1}]$. Then set $x_0 = f^*t \in \Gamma(V_0, \mathcal{O}_X)$ and $x_1 = f^*t^{-1} \in \Gamma(V_1, \mathcal{O}_X)$. We have

$$\begin{aligned} x_0\big|_{V_0\cap V_1} x_1\big|_{V_0\cap V_1} &= f^*(t\big|_{U_0\cap U_1} t^{-1}\big|_{U_0\cap U_1}) \\ &= f^*1 = 1. \end{aligned}$$

Going the other way, $x_0 \in \Gamma(V_0, \mathcal{O}_X)$ gives a map $V_0 \to U_0$ and $x_1 \in \Gamma(V_1, \mathcal{O}_X)$ gives $V_1 \to U_1$. We get two maps $V_0 \cap V_1 \to U_0 \cap U_1 = \operatorname{Spec} \mathbf{Z}[t, t^{-1}]$ by restriction. These correspond to maps

$$\mathbf{Z}[t, t^{-1}] \to \Gamma(V_0 \cap V_1, \mathcal{O}_X).$$

One sends t to $x_0\big|_{V_0\cap V_1}$ and the other sends t^{-1} to $x_1\big|_{V_0\cap V_1}$. But $x_0\big|_{V_0\cap V_1}^{-1} = x_1\big|_{V_0\cap V_1}$, so these maps agree. We therefore get a map of topological spaces $f : X \to \mathbf{P}^1$ and we get maps of sheaves

$$\mathcal{O}_{\mathbf{P}^1}\big|_{U_i} \to f_*\mathcal{O}_X\big|_{U_i}.$$

These maps agree on $U_0 \cap U_1$, so they glue to a morphism of sheaves on $\mathbf{P}^1$. We can check that it is a morphism of locally ringed spaces by working locally in $\mathbf{P}^1$, where it reduces to the fact that $V_i \to U_i$ are morphisms of locally ringed spaces. ☐

Chapter 5

Representable functors

13 The functor of points

c:functor-of-points

Reading 13.1. [Mum99, §II.6], [Vak14, §§6.6.1–6.6.2, 9.1.6–9.1.7]

13.1 The problem with the product

The world would be unjust if we could not say that

$$\mathbf{A}^1 \times \mathbf{A}^1 = \mathbf{A}^2.$$

Exercise 13.2. (i) Show that

$$|\mathbf{A}^1| \times |\mathbf{A}^1| \neq |\mathbf{A}^2|,$$

where $|X|$ denotes the underlying topological space of a scheme X. (Hint: Find a point of $\mathbf{A}^2$ that does not correspond to an ordered pair of points. Feel free to work over a field, or even an algebraically closed field, where the important phenomenon will already be visible.)

(ii) Show that $\mathbf{A}^2$ has the correct universal property of a product in the category of schemes.[1] (Hint: A map from a scheme X to $\operatorname{Spec} A$ is a homomorphism of commutative rings $A \to \Gamma(X, \mathcal{O}_X)$. Use the universal property of a polynomial ring or a tensor product.)

This tells us that the universal property is a better way of identify products than by looking at the underlying set of points.

13.2 About underlying sets

Reading 13.3. [Mum99, pp. 112–113]

Many mathematical objects of interest have underlying sets. Algebraic objects like rings and groups are defined by adding an algebraic structure to an underlying set. A topological space is an underlying set and a collection of subsets of that set. A manifold is a topological

[1]This means that a map $X \to \mathbf{A}^2$ corresponds to a pair of maps $X \to \mathbf{A}^1$.

space with additional structure, and its underlying set is the underlying set of the underlying topological space.

In each of these examples, passage to the underlying set defines a functor, sometimes called a *forgetful functor*:

$$F : \mathscr{C} \to \mathbf{Sets}$$

In fact, all of these functors are *representable*. This means that there is some object $X \in \mathscr{C}$ such that $F \simeq h^X$, where $h^X(Y) = \mathrm{Hom}_{\mathscr{C}}(X, Y)$.

Important general knowledge, but not particularly important for this class.

Exercise 13.4. Find objects representing the forgetful functors for groups, rings, topological spaces. (Hint: Free object with one generator.)

All of the forgetful functors described above are *faithful*. This means that the map

$$\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathbf{Sets}}(FX, FY) \qquad (*)$$

eqn:9

is injective. In other words, you can tell if two morphisms in $\mathscr{C}$ are the same by looking at how they behave on the underlying sets.

Not every category $\mathscr{C}$ has an 'underlying set' functor to the category of sets, but (essentially) every category does have a faithful functor to the category of sets:

Think about this, but don't write it up. This exercise will be generalized by the Yoneda lemma later.

Exercise 13.5. Let $\mathscr{C}$ be a category. Prove that the functor

$$F(Y) = \prod_{X \in \mathscr{C}} \mathrm{Hom}(Y, X)$$

is a faithful functor from $\mathscr{C}$ to **Sets**. (If you are the kind of person who likes to worry about set-theoretic issues, assume that $\mathscr{C}$ is a small category so that the product exists. If you are the kind of person that likes to worry about set-theoretic issues *and* work with big categories, find out what a presentable category is and assume that $\mathscr{C}$ is one.)

Of course, the forgetful functors we have encountered are not full. To be *full* means that the map $(*)$ is a surjection. However, the functors above can be promoted to fully faithful functors by recording extra structure:

This exercise won't serve any further purpose in this course, so you shouldn't do it. It is important in the construction of cohomology theories for algebraic structures, though.

Exercise 13.6. In the following problems, you will need to figure out what 'compatible' means.

(i) Show that compatible functions

$$\mathrm{Hom}_{\mathbf{ComRing}}(\mathbf{Z}[x], A) \to \mathrm{Hom}_{\mathbf{ComRing}}(\mathbf{Z}[x], B)$$
$$\mathrm{Hom}_{\mathbf{ComRing}}(\mathbf{Z}[x, y], A) \to \mathrm{Hom}_{\mathbf{ComRing}}(\mathbf{Z}[x, y], B)$$

are induced by a unique homomorphism of commutative rings $A \to B$. (Hint: Use the map $\mathbf{Z}[x] \to \mathbf{Z}[x, y]$ sending x to $x + y$ and the map sending x to xy.)

(ii) Let F_n denote the free group on n generators. Show that compatible functions

$$\mathrm{Hom}_{\mathbf{Grp}}(F_1, A) \to \mathrm{Hom}_{\mathbf{Grp}}(F_1, B)$$
$$\mathrm{Hom}_{\mathbf{Grp}}(F_2, A) \to \mathrm{Hom}_{\mathbf{Grp}}(F_2, B)$$

are induced by a unique homomorphism of groups $A \to B$.

It is harder to come up with a small collection of objects and arrows that give a fully faithful embedding of topological spaces into a category that is similarly set-theoretic, and harder still to do it for schemes. However, if we allow ourselves an *enormous* amount of data, as we shall in the next section, we will see that *every* category has a fully faithful functor to a category that is essentially set-theoretic (meaning its objects are sets with structural morphisms between them). In other words, morphisms in any category can be constructed set-theoretically. This can be quite a coup for categories like schemes, where the definition of the category is very elaborate.

13.3 Yoneda's lemma

Definition 13.7. If C is a category, a *presheaf* on C is a contravariant functor from C to **Sets**. If $X \in C$ then we write h_X for the functor $h_X(Y) = \mathrm{Hom}_C(Y, X)$. If F is a presheaf on C and $F \simeq h_X$ then we say F is *representable* by X.

ex:yoneda

Exercise 13.8 (Yoneda's Lemma)**.** (i) Let C be a category. Show that $X \mapsto h_X$ is a covariant functor from C to $\widehat{C}$.

Solution. For any map $f : X \to Y$, we get a map

$$h_X(Z) = \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Y) = h_Y(Z)$$

by composition with f and this is obviously compatible with composition. The naturality in Z is the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Hom}(Z,X) & \longrightarrow & \mathrm{Hom}(Z,Y) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(W,X) & \longrightarrow & \mathrm{Hom}(W,Y) \end{array}$$

for any map $g : W \to Z$. $\square$

(ii) Show that $X \mapsto h_X$ is fully faithful. (Show in other words that $\mathrm{Hom}_C(X, Y) = \mathrm{Hom}_{\widehat{C}}(h_X, h_Y)$ via the natural map.)

Solution. We need to give an inverse to the map constructed in the first part. If we have $\varphi : h_X \to h_Y$, then $\varphi(\mathrm{id}_X) \in h_Y(X) = \mathrm{Hom}(X, Y)$.

If $f : X \to Y$ is a map in C then the induced map $\varphi : h_X \to h_Y$ sends $g \in h_X(Z)$ to $f \circ g \in h_Y(Z)$. So $\varphi(\mathrm{id}_X) = f \circ \mathrm{id}_X = f$, as desired.

Conversely, if $\varphi : h_X \to h_Y$ is natural, let $f = \varphi(\mathrm{id}_X)$. Let $g : Z \to X$ be any map. We have a commutative diagram by naturality of φ:

$$\begin{array}{ccc} h_X(X) & \longrightarrow & h_X(Z) \\ \downarrow & & \downarrow \\ h_Y(X) & \longrightarrow & h_Y(Z) \end{array}$$

Then $\mathrm{id}_X \in h_X(X)$ maps to $g \in h_X(Z)$ maps to $\varphi(g) \in h_X(Z)$. On the other hand, id_X maps to $f \in h_Y(X)$ maps to $f \circ g \in h_Y(Z)$. Thus $\varphi(g) = f \circ g$, as desired. $\square$

(iii) Show that for any $F \in \widehat{C}$, there is a unique natural bijection $\mathrm{Hom}_{\widehat{C}}(h_X, F) \simeq F(X)$ under which $\varphi : h_X \to F$ corresponds to $\varphi(\mathrm{id}_X) \in F(X)$.

Yoneda's lemma tells us that we can think of a scheme in terms of the contravariant functor it represents. Remarkably this can often be a lot easier than thinking about the scheme as a ringed space.

13.4 Initial and final objects

sec:initial

Let $F : \mathscr{C}^\circ \to \mathbf{Sets}$ be a functor. Out of this, we can build a category $\mathscr{C}/F$ whose objects are pairs (X, ξ) where $X \in \mathscr{C}$ and $\xi \in F(X)$. A morphism $(X, \xi) \to (Y, \eta)$ is a map $f : X \to Y$ in $\mathscr{C}$ such that $f^*\eta = \xi$.

Exercise 13.9. Show that F is representable by (X, ξ) if and only if (X, ξ) a final object of $\mathscr{C}/F$.

Now suppose that $F : \mathscr{C} \to \mathbf{Sets}$ is a functor and let $F/\mathscr{C}$ be the category of pairs (X, ξ) where $X \in \mathscr{C}$ and $\xi \in F(X)$. A morphism $(X, \xi) \to (Y, \eta)$ is a map $f : X \to Y$ such that $f_*\xi = \eta$.

Exercise 13.10. Show that F is representable by (X, ξ) if and only (X, ξ) an initial object of $F/\mathscr{C}$

13.5 The adjoint functor theorem

sec:aft

The adjoint functor theorem is a very powerful tool for showing that a functor is representable.

ex:functor-colimit

Exercise 13.11. Suppose that $F : \mathscr{C} \to \mathscr{D}$ is a functor and that $\{X_i\}$ is a diagram in $\mathscr{C}$. Construct a canonical map:

$$F(\varprojlim_i X) \to \varprojlim_i F(X)$$

Give an example of a functor F and a diagram X where this map is not an isomorphism.

When the morphism constructed in the exercise is an isomorphism for all small (indexed by a set) diagrams X, we say that F *preserves limits*.

Theorem 13.12. *Suppose that $F : \mathscr{C} \to$ **Sets** is a functor such that*

(i) $\mathscr{C}$ admits all small limits;

(ii) F preserves limits; and

(iii) $F/\mathscr{C}$ has an essentially small coinitial subcategory.

Then F is representable.

The condition that $F/\mathscr{C}$ have a small coinitial subcategory means here that there is a small subcategory $\mathscr{C}_0 \subset \mathscr{C}$ such that, for every $(x, \xi) \in F/\mathscr{C}$, there is a $(y, \eta) \in \mathscr{C}_0$ and a morphism $(y, \eta) \to (x, \xi)$.

Proof. Since $F/\mathscr{C}$ has a small coinitial subcategory $\mathscr{C}_0$ and $\mathscr{C}$ admits all small limits, there is a limit $e = \varprojlim_{(x,\xi)\in\mathscr{C}_0} 1$ in $\mathscr{C}$. Since F preserves small limits, we have an bijection:

$$F(e) \xrightarrow{\sim} \varprojlim_{(x,\xi)\in\mathscr{C}_0} F(x)$$

Now, there is a canonical element of the limit, namely the tuple $\eta(x,\xi) = \xi$ so this gives a canonical element $\epsilon \in F(e)$. We will show that (e,ϵ) represents F.

This means that we have to construct a unique morphism $(e,\epsilon) \to (x,\xi)$ for every $(x,\xi) \in F/\mathscr{C}$. But $\mathscr{C}_0$ is coinitial, so if $(x,\xi) \in F/\mathscr{C}$ there is some $(y,\eta) \in \mathscr{C}_0$ and a map $(y,\eta) \to (x,\xi)$. By definition of the limit, we have a projection $(e,\epsilon) \to (y,\eta)$ and composition gives us a map $(e,\epsilon) \to (x,\xi)$.

We still have to prove that the map $(e,\epsilon) \to (x,\xi)$ is unique. Suppose we had two maps $a, b : e \rightrightarrows x$ and let $u : f \to e$ be the equalizer of the two maps $e \rightrightarrows x$ in $\mathscr{C}$ (which exists since $\mathscr{C}$ has small limits). Then, since F preserves small limits, there is a unique $\phi \in F(f)$ such that $u_*(\phi) = \epsilon$. Now, choose $(y,\eta) \in \mathscr{C}_0$ and a map $(y,\eta) \to (f,\phi)$ (since $\mathscr{C}_0$ is coinitial). But now the two compositions

$$y \to f \to e \rightrightarrows x$$

coincide, by definition of the equalizer. Let $v : y \to e$ denote the composition $y \to f \to e$, above. I claim that $v_*\epsilon = \epsilon$. Once we show this, we will be done, since we have a unique map $w : e \to y$ with $w_*\epsilon = \eta$. Then vw is the identity on e and the two compositions

$$e \to y \to f \to e \rightrightarrows x$$

agree.

We check that $v_*\epsilon = \epsilon$. It is sufficient to check that for every $z \in \mathscr{C}_0$ and every projection map $t : e \to z$, that $tv = t$. To check this, we need to show that $t_*\epsilon = t_*v_*\epsilon$. But we have

$$t_*v_*\epsilon = t_*\eta = v_*\epsilon$$

since $(y,\eta) \to (z,\zeta)$ is a morphism in $\mathscr{C}_0$. □

The existence of a small coinitial subcategory looks intimidating, but it is usually very easy to verify. For example, if the objects of $\mathscr{C}$ have some kind of underlying sets and you can find a coinitial subcategory of $F/\mathscr{C}_0$ where the underling sets have some kind of upper bound, then you have a coinitial subcategory. This frequently works in algebraic examples, like the following exercise:

Exercise 13.13. Let **Grp** be the category of groups. Define $F : \mathbf{Grp} \to \mathbf{Sets}$ by $F(G) = G^n$ where n is any integer ≥ 0 (in fact, any cardinal would be fine). Prove that F is representable. This constructs the free group on n generators.

Solution. Let α be an infinite cardinal that is $\geq n$. Let $\mathscr{C}_0$ consist of all groups of cardinality $\leq \alpha$. Then $\mathscr{C}_0$ is essentially small, since the group structures on a group of cardinality $\leq \alpha$ are bounded by $\alpha^{\alpha\times\alpha}$. Furthermore,

$$F(\varprojlim G_i) = (\varprojlim G_i)^n = \varprojlim(G_i^n)$$

so F preserves limits. Therefore F is representable, by the adjoint functor theorem. □

14 Localization of ringed spaces

Reading 14.1. [MO, §I.7], [Gil11]

14.1 The spectrum of a sheaf of rings

In this section **LRS** will be the category of locally ringed spaces and **RS** will be the category of ringed spaces.

Let X be a topological space with a sheaf of rings $\mathscr{A}$. We construct a new topological space, $\operatorname{Spec} \mathscr{A}$. The points of $\operatorname{Spec} \mathscr{A}$ are pairs $(x, \mathfrak{p})$ where $x \in X$ and $\mathfrak{p}$ is a prime ideal of the stalk $\mathscr{A}_x$.

If $(x, \mathfrak{p}) \in \operatorname{Spec} \mathscr{A}$, let $\mathbf{k}(x, \mathfrak{p})$ be the field of fractions of $\mathscr{A}_x/\mathfrak{p}$. For any open $U \subset X$ containing x, let $\mathrm{ev}_{(x,\mathfrak{p})}$ be the composition of homomorphisms:

$$\mathrm{ev}_{(x,\mathfrak{p})} : \mathscr{A}(U) \to \mathscr{A}_x \to \mathscr{A}_x/\mathfrak{p} = \mathbf{k}(x)$$

We write $f(x, \mathfrak{p}) = \mathrm{ev}_{(x,\mathfrak{p})}(f)$.

If $U \subset X$ is open, and $f \in \mathscr{A}(U)$, define $D_U(f)$ to be the set of all $(x, \mathfrak{p}) \in \operatorname{Spec} \mathscr{A}$ such that $x \in U$ and $f(x, \mathfrak{p}) \neq 0$. We give $\operatorname{Spec} \mathscr{A}$ the coarsest topology such the sets $D_U(f)$ are all open.

Exercise 14.2. Show that the map $\pi : \operatorname{Spec} \mathscr{A} \to X$ sending $(x, \mathfrak{p})$ to x is continuous.

Now we give $\operatorname{Spec} \mathscr{A}$ a sheaf of rings. Essentially, we want to imitate the construction of the spectrum over a point, and define $\mathcal{O}(D_U(f)) = \mathscr{A}(U)[f^{-1}]$. The only trouble is that this might not be well-defined, because $D_U(f)$ could coincide with $D_V(g)$ for $U \neq V$ or $f \neq g$. It is possible to get around this with a bit of categorical trickery, but we will instead build the espace étalé of the sheaf directly and take its sheaf of sections.

For each $(x, \mathfrak{p}) \in \operatorname{Spec} \mathscr{A}$, let $\mathscr{A}_{x,\mathfrak{p}}$ be the local ring of $\mathscr{A}_x$ at the prime $\mathfrak{p} \subset \mathscr{A}$.

Let $\mathscr{O}$ be the topological space defined as follows:

- the underlying set of $\mathscr{O}$ is $\coprod_{(x,\mathfrak{p}) \in \operatorname{Spec}} \mathscr{A}_{x,\mathfrak{p}}$;
- for each open $U \subset X$, each $f \in \mathscr{A}(U)$, and each $g \in \mathscr{A}(U)[f^{-1}]$, the image of the map
$$g : D_U(f) \to \prod_{(x,\mathfrak{p}) \in D_U(f)} \mathscr{A}_{x,\mathfrak{p}}$$
is open.

To be clear, this map sends a point $(x, \mathfrak{p}) \in D_U(f)$ to the tuple whose $(x, \mathfrak{p})$-component is the image of g under the homomorphism

$$\mathrm{ev}_{x,\mathfrak{p}} : \mathscr{A}(U)[f^{-1}] \to \mathscr{A}_{x,\mathfrak{p}}.$$

There is a projection $\mathscr{O} \to X$ sending $\mathscr{A}_{x,\mathfrak{p}}$ to x.

Exercise 14.3. Show that the projection $\mathscr{O} \to X$ is a local homeomorphism.

We define $\mathcal{O}_{\operatorname{Spec} \mathscr{A}}$ to be the sheaf of sections of $\mathscr{O}$ on $\operatorname{Spec} \mathscr{A}$.

Exercise 14.4. Construct a morphism of ringed spaces

$$(\operatorname{Spec}\mathscr{A}, \mathcal{O}_{\operatorname{Spec}\mathscr{A}}) \to (X, \mathscr{A})$$

whose underlying morphism of sets is $\pi(x, \mathfrak{p}) = x$.

Exercise 14.5. Suppose that S is a locally ringed space and $\varphi : S \to (X, \mathscr{A})$ is a morphism of ringed spaces. Show that there is a unique factorization of φ through a morphism $\psi : S \to \operatorname{Spec}\mathscr{A}$ (i.e., such that $\pi\varphi = \psi$) such that ψ is a morphism of locally ringed spaces.

Solution. First we show the factorization is unique if it exists. Let ψ be a factorization that is a morphism of locally ringed spaces. Then we have a commutative diagram:

$$\begin{tikzcd}
\mathscr{A}(U) \arrow[r] \arrow[d] & \mathscr{A}_x \arrow[r] & \mathscr{A}_{\psi(s)} \arrow[r] \arrow[d] & \mathbf{k}(\psi(s)) \arrow[d] \\
\mathcal{O}_S(\varphi^{-1}U) \arrow[rr] & & \mathcal{O}_{S,s} \arrow[r] & \mathbf{k}(s)
\end{tikzcd} \tag{14.1}$$

eqn:28

This means that $\psi(s)$ is the pair $(x, \mathfrak{p})$ where $\mathfrak{p}$ is the preimage of $\mathfrak{m}_{S,s}$ under the homomorphism $\mathscr{A}_x \to \mathcal{O}_{S,s}$. The latter map is determined by φ, so the map of sets underlying ψ is determined by φ. Furthermore, the map of sheaves $\psi^{-1}\mathcal{O}_{\operatorname{Spec}\mathscr{A}} \to \mathcal{O}_S$ is determined by the maps on stalks, which the diagram above shows are determined by φ. Hence ψ is uniquely determined by φ.

Now we must construct ψ from φ. The above diagram shows us that we should define $\psi(s)$ to be the preimage of the maximal ideal $\mathfrak{m}_{S,s}$ under the homomorphism:

$$\mathscr{A}_x \to \mathcal{O}_{S,s}$$

This gives $\pi\psi = \varphi$ and we need only construct the map of sheaves

$$\mathcal{O}_{\operatorname{Spec}\mathscr{A}} \to \psi_*\mathcal{O}_S.$$

By definition of the espace étalé, we can regard a section of $\mathcal{O}_{\operatorname{Spec}\mathscr{A}}(U)$ as a tuple $g \in \prod_{(x,\mathfrak{p})\in U} \mathscr{A}_{x,\mathfrak{p}}$. If $\psi(s) = (x, \mathfrak{p})$, we have a map

$$\mathscr{A}_{x,\mathfrak{p}} \to \mathcal{O}_{S,s}$$

where so we get a map

$$\prod_{(x,\mathfrak{p})\in U} \mathscr{A}_{x,\mathfrak{p}} \to \prod_{s\in\psi^{-1}U} \mathcal{O}_{S,s}.$$

We want to show that the image of g under this map is contained in the image of $\mathcal{O}_S(\psi^{-1}U)$. But this is a local question, since $\mathcal{O}_S$ is a sheaf, so, by definition of $\mathcal{O}_{\operatorname{Spec}\mathscr{A}}$, it is sufficient to assume that $U = D_V(f)$ for some open $U \subset X$, some $f \in \mathscr{A}(V)$, and that g is in the image of

$$\mathscr{A}(V)[f^{-1}] \to \prod_{(x,\mathfrak{p})\in D_V(f)} \mathscr{A}_{x,\mathfrak{p}}.$$

But now we have a commutative diagram

$$\begin{tikzcd}
\mathscr{A}(V)[f^{-1}] \arrow[r] \arrow[d] & \mathcal{O}_S(\psi^{-1}D_V(f)) \arrow[d] \\
\mathcal{A}_{x,\mathfrak{p}} \arrow[r] & \mathcal{O}_{S,s}
\end{tikzcd}$$

for each $s \in \psi^{-1} D_V(f)$ with $\psi(s) = (x, \mathfrak{p})$. Taking products in the bottom row, we find that the image of g is contained in $\mathcal{O}_S(\psi^{-1} D_V(f))$, exactly as required.

Finally, we can check that ψ is a morphism of locally ringed spaces. This comes down to the fact that if $g \in \mathcal{O}_{\operatorname{Spec} \mathscr{A}}(U)$ then $g(\psi(s)) = 0$ if and only if $\psi^* g(s) = 0$, which is the commutativity of diagram (14.1), again. □

14.2 The maximal locally ringed subspace

Exercise 14.6. Let X be a ringed space and let $X' \subset X$ be the set of all $x \in X$ such that $\mathcal{O}_{X,x}$ is a localy ring. Give X' the induced topology, let $i : X' \to X$ be the inclusion, and define $\mathcal{O}_{X'} = i^{-1}\mathcal{O}_X$. Show that X' is the largest locally ringed subspace of X.

Exercise 14.7. Let X and Y be locally ringed spaces and let $u : X \to Y$ be a morphism of *ringed spaces*. Show that there is a largest subspace $X' \subset X$ such that $u\big|_{X'}$ is a morphism of locally ringed spaces.

Solution. Let X' be the set of points $x \in X$ such that $\mathcal{O}_{Y,u(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism of local rings. □

Exercise 14.8. Suppose that $u : X \to Y$ is a ringed space morphism between locally ringed spaces and $v : Z \to X$ is a locally ringed space morphism such that $uv : Z \to Y$ is a locally ringed space morphism. Show that u factors through the maximal subspace $X' \subset X$ on which $u\big|_{X'}$ is a morphism of locally ringed spaces.

Solution. By assumption, $\mathcal{O}_{Y,y} \xrightarrow{f} \mathcal{O}_{X,x} \xrightarrow{g} \mathcal{O}_{Z,z}$ is a local homomorphism. So $gf(\mathfrak{m}_y) \subset \mathfrak{m}_z$. But $g^{-1}\mathfrak{m}_z = \mathfrak{m}_x$, since g is a local homomorphism, so this means $f(\mathfrak{m}_y) \subset \mathfrak{m}_x$, which means that u is a local homomorphism at $x \in X$. Therefore $x \in X'$, as required. □

14.3 Fiber products

Exercise 14.9. Let $X \to Z$ and $Y \to Z$ be morphisms of locally ringed spaces. Show that there is a fiber product $X \times_Z^{\mathbf{RS}} Y$ in the category of *ringed spaces*. (Hint: take the fiber product of the topological spaces with the sheaf of rings $p^{-1}\mathcal{O}_X \otimes_{r^{-1}\mathcal{O}_Z} q^{-1}\mathcal{O}_Y$.)

Exercise 14.10. Let W be the maximal subspace of $\ell(X \times_Z^{\mathbf{RS}} Y)$ such that the two projections $W \to X$ and $W \to Y$ are local morphisms of locally ringed spaces. Show that W is the fiber product $X \times_Z Y$ in the category of locally ringed spaces.

14.4 The relative spectrum

Exercise 14.11. Suppose that X a locally ringed space and $\mathscr{A}$ is a sheaf of $\mathcal{O}_X$-algebras. For each locally ringed space S, let $F(S)$ be the set of pairs (u, φ) where $u : S \to X$ is a morphism of locally ringed spaces and $\varphi : \mathscr{A} \to u_*\mathcal{O}_S$ is an $\mathscr{A}$-algebra homomorphism. Show that F is representable by a locally ringed space $\operatorname{Spec}_X \mathscr{A}$. (Hint: take the maximal subspace of $\operatorname{Spec} \mathscr{A}$ such that the projection to X is a local morphism of locally ringed spaces.)

Exercise 14.12. Prove that if X is a scheme and $\mathscr{A}$ is quasicoherent then $\operatorname{Spec}_X \mathscr{A}$ is a scheme.

15 Presheaves representable by schemes

Reading 15.1. [Vak14, §§9.1.6–9.1.7],

Recall that the Yoneda lemma gave us a fully faithful functor

$$\mathbf{Sch} \to \mathbf{Sch}^\wedge$$

where $\mathbf{Sch}^\wedge$ is the category of presheaves on $\mathbf{Sch}$. In this section, we want to characterize the image of this functor. In other words, we want to be able to determine which presheaves on $\mathbf{Sch}$ are representable by schemes. This will give us a new way to construct schemes that will often be easier than constructing a ringed space. In fact, this will give us an entirely new way to think about what a scheme is.

thm:presheaf-scheme

Theorem 15.2. *A presheaf on the category of schemes is representable by a scheme if and only if*

(i) it is a sheaf in the Zariski topology, and

(ii) it has an open cover by presheaves that are representable by affine schemes.

If A is a commutative ring then $h^A : \mathbf{ComRing} \to \mathbf{Sets}$ sending B to $\mathrm{Hom}_{\mathbf{ComRing}}(A, B)$. If X is a scheme then $h_X : \mathbf{Sch}^\circ \to \mathbf{Sets}$ is the functor sending Y to $\mathrm{Hom}_{\mathbf{Sch}}(Y, X)$. Abusively, we think of h^A and $h_{\mathrm{Spec}\, A}$ as being the same object. In reality, h^A is a functor defined on $\mathbf{ComRing} = \mathbf{Aff}^\circ \subsetneq \mathbf{Sch}^\circ$ and $h_{\mathrm{Spec}\, A}$ is defined on all of $\mathbf{Sch}$. More generally, when X is a scheme, we sometimes think of h_X as a covariant functor defined on $\mathbf{ComRing}$ and we abbreviate $h_X(\mathrm{Spec}\, A)$ to $h_X(A)$. We will see below that the composition of the Yoneda embedding with restriction

$$\mathbf{Sch} \to \mathbf{Sch}^\wedge \to \mathbf{Aff}^\wedge$$

is fully faithful, so this abuse of notation does not cause any trouble. Later on, we will even permit ourselves to write $X(A)$ in place of $h_X(A)$.

15.1 Zariski sheaves

Let Y be a scheme. Note that $\mathbf{Open}(Y)$ can be regarded as a subcategory of $\mathbf{Sch}$. If F is a presheaf on $\mathbf{Sch}$ then we can restrict it to $\mathbf{Open}(Y)$ and get a presheaf on Y.

Definition 15.3. A presheaf F on $\mathbf{Sch}$ is said to be a *Zariski sheaf* if, for any scheme Y, the presheaf $F\big|_{\mathbf{Open}(Y)}$ is a sheaf on Y.

The following lemma says that h_X is a Zariski sheaf for any scheme X:

Lemma 15.4. *Suppose X and Y are schemes. Define a presheaf F on X by $F(U) = \mathrm{Hom}(U, Y)$. Then F is a sheaf. (Hint: Use Exercise 4.10 and Exercise 4.13. It may be helpful to think of a map of ringed spaces as a continuous map $f : X \to Y$ and a morphism of sheaves of rings $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. Observe that if $U \subset X$ is open then $f^{-1}\mathcal{O}_Y\big|_U = (f\big|_U)^{-1}\mathcal{O}_Y$.)*

Proof. First we prove **SH1**. Suppose that U is an open subset of X and we have a two maps $f, g : U \to Y$ in $F(U)$. Let $\{U_i\}$ cover U and assume that $f\big|_{U_i} = g\big|_{U_i}$. Since continuous functions form a sheaf (Exercise 4.10) this means that f and g have the same underlying map of topological spaces.

To prove that $f = g$, we need to show that the two maps $f^*, g^* : \mathcal{O}_Y \to f_*\mathcal{O}_U = g_*\mathcal{O}_U$ coincide. Regard these as maps $f^*, g^* : f^{-1}\mathcal{O}_Y \to \mathcal{O}_U$. Then for each i, the maps $f^{-1}\mathcal{O}_Y\big|_{U_i} \to \mathcal{O}_U\big|_{U_i} = \mathcal{O}_{U_i}$ coincide. Since $\mathcal{O}_U$ is a sheaf, this implies that f^* and g^* are the same morphism of sheaves and therefore that f and g are the same morphism of schemes.

Next we prove **SH2**. Suppose we have $f_i : U_i \to Y$ for each U_i in a cover of U with $f_i\big|_{U_i \cap U_j} = f_j\big|_{U_i \cap U_j}$ for all i and j. These glue to a continuous map $f : U \to Y$. We now have maps

$$f^{-1}\mathcal{O}_Y\big|_{U_i} = f_i^{-1}\mathcal{O}_Y \to \mathcal{O}_{U_i} = \mathcal{O}_U\big|_{U_i}$$

that agree when restricted to $U_i \cap U_j$. Since morphisms between two sheaves form a sheaf these glue to a map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. We have to check that this is a morphism of schemes, i.e., that it has the right local form. But each map $U_i \to Y$ is locally a morphism of affine schemes and the U_i cover U, so $U \to Y$ is locally a morphism of affine schemes. $\square$

riski-sheafification

Exercise 15.5. Suppose that F is a presheaf on **Sch**. Show that there is a universal map $F \to F^{\mathrm{sh}}$ where F^{sh} is a Zariski sheaf. This is called the *sheafification* of F. (Hint: Sheafify $F\big|_{\mathbf{Open}(X)}$ for each $X \in$ **Sch**.)

Solution. For each $X \in$ **Sch**, define $F'(X) = \Gamma(X, F\big|^{\mathrm{sh}}_{\mathbf{Open}(X)})$. We show first that F' is a presheaf. If $f : Y \to X$ is a morphism of schemes, we get $F\big|_{\mathbf{Open}(X)} \to f_*F\big|_{\mathbf{Open}(Y)}$, since F is a presheaf on **Sch**. We also have $F\big|_{\mathbf{Open}(Y)} \to F\big|^{\mathrm{sh}}_{\mathbf{Open}(Y)}$, which gives

$$F\big|_{\mathbf{Open}(X)} \to f_*(F\big|_{\mathbf{Open}(Y)}) \to f_*(F\big|^{\mathrm{sh}}_{\mathbf{Open}(Y)})$$

and the universal property of $F\big|^{\mathrm{sh}}_{\mathbf{Open}(X)}$ induces a map $F\big|^{\mathrm{sh}}_{\mathbf{Open}(X)} \to f_*F\big|^{\mathrm{sh}}_{\mathbf{Open}(Y)}$. This gives in particular

$$F'(X) = F\big|^{\mathrm{sh}}_{\mathbf{Open}(X)}(X) \to f_*(F\big|^{\mathrm{sh}}_{\mathbf{Open}(Y)})(X) = F\big|^{\mathrm{sh}}_{\mathbf{Open}\,Y}(Y) = F'(Y).$$

A similar argument shows that these restriction maps are stable under composition, so F' is a presheaf.

To verify that F' is a sheaf, we note that for any presheaf G on **Open**(X), we have $G^{\mathrm{sh}}(U) = (G\big|_U)^{\mathrm{sh}}(U)$. Indeed, the construction of the espace étalé is compatible with pullback. This implies that $F'\big|_{\mathbf{Open}(X)} = F\big|^{\mathrm{sh}}_{\mathbf{Open}(X)}$ and in particular shows that $F'\big|_{\mathbf{Open}(X)}$ is a sheaf for each $X \in$ **Sch**. $\square$

15.2 Open subfunctors

Should be simple

Exercise 15.6. If $\varphi : F \to G$ is a natural transformation between presheaves and $G' \subset G$ is a subpresheaf then define $F'(U) = \varphi^{-1}G'(U)$ for all U. Show that F' is a subpresheaf of F. We denote $F' = \varphi^{-1}G'$.

ti-subsheafification

This exercise is a special case of Exercise 15.5, although we'll only have use for this one right now. The first part is not so important to do, so

subsheafification2

The second part might be good practice.

Exercise 15.7. Suppose that F is a sheaf and G_i, $i \in I$ is a family of subsheaves of F. Let $G(X) = \bigcup_{i \in I} G_i(X)$. Say that $\xi \in F(X)$ *lies locally* in G if there is an open cover $X = \bigcup U_j$ such that for each j, the restriction $\xi\big|_{U_j}$ lies in $G(U_j)$.

(i) Show that G is not necessarily a sheaf.

(ii) Show that there is a smallest subsheaf G' of F that contains all of the G_i. (Hint: Let $G'(X)$ be the set of all $\xi \in F(X)$ that lie locally in G.)

(iii) Suppose that $F = h_X$ is representable. Show that $G' = F$ if and only if id_X lies locally in G. (Hint: h_X is the only subpresheaf of itself containing id_X.)

The sheaf constructed in part (ii) is called the *sheaf theoretic union* or the *sheaf union* of the G_i.

Definition 15.8. Suppose that F is a presheaf on the category of schemes and $F' \subset F$ is a subpresheaf. We say that F' is *open* in F if, for any map $\varphi : h_X \to F$, the preimage $\varphi^{-1}F' \subset h_X$ is representable by an open subscheme of X.

A collection of open subfunctors $F'_i \subset F$ is said to *cover* F if F is the sheaf theoretic union of the F'_i.

ex:zariski-cover

This might be a good exercise to do. It will force you to unpack some of the definitions. You don't necessarily have to prove all of the equivalences.

Exercise 15.9. Let F be a presheaf on **Sch** and $F'_i \subset F$ open subpresheaves. Prove that the following conditions are equivalent:

ex:zariski-cover:1

(i) F is the sheaf union of the F'_i.

ex:zariski-cover:2

(ii) h_X is the sheaf union of the $\varphi^{-1}F'_i$ for all $\varphi \in F(X) = \mathrm{Hom}(h_X, F)$.

ex:zariski-cover:4

(iii) For any scheme X and any $\varphi \in F(X) = \mathrm{Hom}(h_X, F)$ let $U_i \subset X$ be open subschemes such that $\varphi^{-1}F'_i = h_{U_i}$. Then $X = \bigcup U_i$.

ex:zariski-cover:3

(iv) $F(k) = \bigcup_i F'_i(k)$ for all fields k.

Solution. (i) $\Longrightarrow$ (ii). We have $f \in h_X(Y)$. Then $\varphi(f) \in F(Y)$. This lies locally in $\bigcup F'_i(Y)$ because F is the sheaf union of the F'_i. Thus f lies locally in $\varphi^{-1}F'_i(Y)$.

(ii) $\Longrightarrow$ (i). Suppose $\varphi \in F(X)$. Then φ lies locally in F'_i if and only if id_X lies locally in $\varphi^{-1}F'_i$. But id_X lies locally in $\varphi^{-1}F'_i$ if and only if h_X is the sheaf union of the $\varphi^{-1}F'_i$.

(ii) $\Longrightarrow$ (iii). Condition (ii) implies that id_X lies locally in $\bigcup h_{U_i}(X)$. That is, there is an open cover by V_j such that each $\mathrm{id}_X\big|_{V_j} \in h_{U_i}(V_j)$ for some j. In particular, $V_j \subset U_i$, so the U_i cover X.

(iii) $\Longrightarrow$ (iv). Apply (iii) when $X = \operatorname{Spec} k$ and k is a field. Then $\bigcup U_i = \operatorname{Spec} k$, which means that at least one $U_i = \operatorname{Spec} k$, since $\operatorname{Spec} k$ is a point. If $\varphi \in F(k) = \mathrm{Hom}(\operatorname{Spec} k, F)$ then $\mathrm{id}_{\operatorname{Spec} k} \in \bigcup_i \varphi^{-1}F'_i(k)$ so $\varphi = \varphi(\mathrm{id}_{\operatorname{Spec} k}) \in \bigcup_i F'_i(k)$. Thus $\bigcup_i F'_i(k) = F(k)$.

(iii) $\Longrightarrow$ (ii). Suppose that $X = \bigcup U_i$ with $h_{U_i} = \varphi^{-1}F'_i$. Then $\mathrm{id}_X\big|_{U_i} \in h_{U_i}(U_i) = \varphi^{-1}F'_i(U_i)$ so id_X lies locally in $\bigcup \varphi^{-1}F'_i$.

(iv) $\Longrightarrow$ (iii). Pick $\xi \in X$. Then we get $\iota \in \operatorname{Spec} \mathbf{k}(\xi) \to X$. Choose i such that $\varphi \circ \iota \in F'_i(k)$. Then $\iota \in \varphi^{-1}F'_i(k) = h_{U_i}(\mathbf{k}(\xi))$ so $\xi \in U_i$. □

Reality check. Shouldn't be difficult.

Exercise 15.10. Show that every scheme has an open cover by subfunctors that are representable by affine schemes.

Lemma 15.11 ([Vak14, Exercise 9.1.I]). *If F is a Zariski sheaf on schemes that has an open cover by affine schemes then F is representable by a scheme.*

Proof. First we assemble the underlying set of the topological space. Let $|F|$ be the set of isomorphism classes of injections of functors $h_{\operatorname{Spec} k} \to F$, with k restricted to be a field. If $F' \subset F$ is an open subfunctor then declare that $|F'| \subset |F|$ is open. We verify this is a topology.

It is sufficient to show that for any map $h_X \to F$, the preimages of the open subfunctors of F in $|h_X| = X$ are closed under unions, intersections, and contain $\varnothing$ and X. But by definition, a subfunctor of F is open if and only if its preimage in h_X is representable by open subschemes, which are in one to one correspondence with the open subsets of X.

This gives the topological space. Now we find the sheaf of rings. Note that the open subfunctors of F are in bijection with the open subsets of $|F|$. Indeed, suppose F' and F''

induce the same open subset $|F'| = |F''|$. Then consider $h_X \to F$. The preimages of F' and F'' induce the same open subset of $X = |h_X|$. So $F' \times_F h_X$ and $F'' \times_F h_X$ are representable by the same open subscheme (because an open subscheme of a scheme is determined by its underlying subset). But this means that $F'(X) = F''(X)$ for any scheme X.[2]

For any open subfunctor $F' \subset F$, define $\mathcal{O}_X(F') = \mathrm{Hom}(F', h_{\mathbf{A}^1})$. This is a presheaf of rings because $\mathbf{A}^1$ is representable by a ring. Furthermore, it is a Zariski sheaf because it is representable by a scheme!

Thus $(X, \mathcal{O}_X)$ is a ringed space. It just remains to find an open cover by affine schemes. But by assumption, F has a cover by open affine subfunctors F'. By definition, $|F'| \subset |F|$ is an open subset, and if we write $(X', \mathcal{O}_{X'})$ for the ringed space associated to F', then restricting $\mathcal{O}_X$ to X' recovers $\mathcal{O}_{X'}$. Now note that for any open affine subfunctor $h_{\operatorname{Spec} A} \subset F'$ we have $\mathcal{O}_{X'}(|\operatorname{Spec} A|) = \mathrm{Hom}(\operatorname{Spec} A, \mathbf{A}^1) = A$, so that $\mathcal{O}_{X'}$ coincides with $\mathcal{O}_{\operatorname{Spec} A}$, as desired.

Finally, we must check that $(X, \mathcal{O}_X)$ represents F. First we get a map $h_X \to F$ using the fact that F is a sheaf. (Whenver $U \subset X$ is an open affine, we get a corresponding open subfunctor $F' \subset F$ and an identification $F' \simeq h_U$. These identifications are compatible with restriction, so by the sheaf condition, we get an element of $F(X)$, hence $h_X \to F$ by Yoneda.) The rest of the proof is basically to observe that since F and h_X agree locally and are both sheaves, they must be isomorphic. □

15.3 The basis of affines

Since every scheme has an open cover by affine schemes, the full subcategory **Aff** ⊂ **Sch** behaves a lot like a basis, at least with respect to Zariski sheaves:

Exercise 15.12. Show that a Zariski sheaf on **Aff** extends in a unique way (up to unique isomorphism) to a Zariski sheaf on **Sch**.

Using this we can get another perspective on what a scheme is.

Definition 15.13. Let A be a commutative ring and let $h^A : \mathbf{ComRng} \to \mathbf{Sets}$ be the functor represented by A. For any subset $J \subset A$, let $h_{D(J)}$ be the subfunctor of h^A defined as follows:

$$h_{D(J)}(B) = \{\varphi : A \to B \mid \varphi(J)B = B\}.$$

A subfunctor of h^A is called *open* if it is isomorphic to $h_{D(J)}$ for some subset $J \subset A$.

Warning: $h_{D(J)}$ usually is not representable by a commutative ring!

Exercise 15.14. Show that if h^A is regarded as a contravariant functor $\mathbf{Aff} \to \mathbf{Sets}$ then $h_{D(J)}$ is represented by the subscheme $D(J) \subset \operatorname{Spec} A$, whence the notation.

Not really important

Exercise 15.15.

(i) Show that the intersection of two open subfunctors is an open subfunctor.

(ii) Show that the union of two open subfunctors is not necessarily an open subfunctor.

Exercise 15.16. Suppose that $F \subset G$ is an inclusion of Zariski sheaves and F has an open cover by subfunctors that are also open subfunctors of G. Prove that F is an open subfunctor of G.

[2] If $g \in F(X)$ then $g \in F'(X)$ if and only if $\mathrm{id}_X \in (F' \times_F h_X)(X) = (F'' \times_F h_X)(X)$ if and only if $g \in F''(X)$.

Definition 15.17. A morphism $F \to G$ of presheaves on **Aff** is said to be an *open embedding* if, for every morphism $\varphi : h^A \to G$, the preimage $\varphi^{-1}F \subset h^A$ is an open subfunctor.

Definition 15.18. A morphism $F \to G$ of presheaves is said to be a *cover* (with respect to schematic points) if $F(k) \to G(k)$ is a bijection for all fields k.

Definition 15.19 (Alternate definition of a scheme)**.** A presheaf F on **Aff** is called a *scheme* if it is a Zariski sheaf and has a cover by open, representable subfunctors.

Exercise 15.20. Show that the two definitions of schemes (via ringed spaces and via presheaves) yield equivalent categories.

15.4 Fiber products

Reading 15.21. [Vak14, §§9.1–9.3]

Suppose that $p : X \to Z$ and $q : Y \to Z$ are morphisms of schemes. Define $F = h_X \times_{h_Z} h_Y$. That is,

$$F(W) = \{(f, g) \in \mathrm{Hom}(W, X) \times \mathrm{Hom}(W, Y) \bigm| pf = qg \in \mathrm{Hom}(W, Z)\}.$$

Exercise 15.22. Prove that F is a Zariski sheaf.

Exercise 15.23. If $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} C$, and $Z = \operatorname{Spec} A$ then $F \simeq h_{\operatorname{Spec}(B \otimes_A C)}$.

Exercise 15.24. Show that F has an open cover by functors representable by affine schemes. (Hint: For any point $\xi \in F(k)$, choose open affine neighborhoods $U \subset X$, $V \subset Y$, and $W \subset Z$ containing the images of ξ, with $p(U) \subset W$ and $q(V) \subset W$. Let $f : F \to h_X$ and $g : F \to h_Y$ denote the projections. Show that $f^{-1}h_U \cap g^{-1}h_V$ is open in F and affine.)

Fibers

Suppose that $p : X \to S$ is a morphism of schemes. The fiber of p over a point $\xi \in S$ is the fiber product $X \times_S \operatorname{Spec} \mathbf{k}(\xi)$.

Equalizers and the diagonal

Important! **Exercise 15.25.** Suppose that $f, g : X \to Y$ are two morphisms of schemes. Show that there is a universal map $h : Z \to X$ such that $fh = gh$. This is called the *equalizer* of f and g and is sometimes denoted $\mathrm{eq}(f, g)$. (Hint: One way to do this is to construct a sheaf and find an open cover by representable functors. Another way is to build the equalizer using fiber products. It's valuable to think about it both ways, but the second is more common in the algebraic geometry literature.)

15.5 Examples

Exercise 15.26. For any scheme (or locally ringed space) X, let $\mathbf{G}_m(X) = \Gamma(X, \mathcal{O}_X)^*$.

(i) Show directly that $\mathbf{G}_m$ is a Zariski sheaf.

(ii) Show that $\mathbf{G}_m$ is in fact representable by an affine scheme.

Exercise 15.27. For any scheme (or locally ringed space) X, let $\mathrm{GL}_n(X) = \mathrm{GL}_n(\Gamma(X, \mathcal{O}_X))$ be the set of $n \times n$ matrices with coefficients in $\Gamma(X, \mathcal{O}_X)$.

(i) Show directly that GL_n is a Zariski sheaf.

(ii) Show that GL_n is in fact representable by an affine scheme.

Exercise 15.28. Let $X = \operatorname{Spec} A$ be an affine scheme and let $I \subset A$ be an ideal. For each locally ringed space Y, let $U(Y)$ be the set of morphisms of locally ringed spaces $f : Y \to X$ such that the ideal of $\mathcal{O}_Y$ generated by $f^{-1}I$ is all of $\mathcal{O}_Y$.

(i) Prove directly that U is a Zariski sheaf.

(ii) Show that U is representable by the open subscheme $D(I)$.

Exercise 15.29. What scheme represents the functor that sends a scheme (or locally ringed space) X to $\Gamma(X, \mathcal{O}_X)$?

16 Vector bundles

Contrary to what we've come to expect, the definition of a vector bundle in algebraic geometry in exactly the same way as in differential geometry or topology. We will give this definition, as well as two others, one aligned philosophically with thinking of schemes as locally ringed spaces, and the other aligned with thinking in terms of the functor of points.

16.1 Transition functions

In differential geometry, a vector bundle over a manifold S is usually defined as a projection $p : E \to S$ along with

(i) a cover $\mathscr{U}$ of S along with *specified isomorphisms* $p^{-1}U \simeq U \times V$ over U for each $U \in \mathscr{U}$, where V is a vector space, possibly depending on U, such that

(ii) if U_1 and U_2 are two open sets in $\mathscr{U}$, the transition function

$$(U_1 \cap U_2) \times V_1 \xrightarrow{\sim} p^{-1}(U_1 \cap U_2) \xrightarrow{\sim} (U_1 \cap U_2) \times V_2$$

is a family of linear isomorphisms.

It is important to unpack the meaning of a 'family of linear isomorphisms'. For manifolds, this means that the map is of the form $(x, y) \mapsto (x, F(x)y)$ where $F : U_1 \cap U_2 \to \mathrm{Hom}_{\mathbf{Vect}}(V_1, V_2)$ is a C^∞ function. In other words, a family of linear maps from $\mathbf{R}^n$ to $\mathbf{R}^m$ over U is a $m \times n$ matrix of C^∞ functions on U whose determinant takes values in $\mathbf{R}^*$.

This definition makes sense when S is a scheme. We just need to say explicitly what we mean by a 'vector space' and a 'family of linear maps'. By a vector space, we will simply mean $\mathbf{A}^r$. A family of linear maps $U \times \mathbf{A}^r \to U \times \mathbf{A}^s$ is a morphism that is given in coordinates on each $\operatorname{Spec} A$ in an affine open cover of U in the form

$$\begin{aligned} A[t_1, \ldots, t_s] &\to A[t_1, \ldots, t_r] \\ t_i &\mapsto t_i M \end{aligned}$$

for some $M \in \mathrm{Mat}_{s \times r}(A)$.

Definition 16.1 (Vector bundle, version 1)**.** Let S be a scheme. A *vector bundle* over S is a projection $p : E \to S$, along with

(i) a cover $\mathscr{U}$ of S by open subschemes and, for each $U \in \mathscr{U}$, an isomorphism $p^{-1}E \simeq U \times V$ over U for each $U \in \mathscr{U}$, where V is some affine space $\mathbf{A}^r$, such that

(ii) if U_1 and U_2 are two open sets in $\mathscr{U}$, the transition function

$$(U_1 \cap U_2) \times V_1 \xrightarrow{\sim} p^{-1}(U_1 \cap U_2) \xrightarrow{\sim} (U_1 \cap U_2) \times V_2$$

is a family of linear maps over $U_1 \cap U_2$.

16.2 Locally free sheaves

Reading 16.2. [Vak14, Section 13.1]

:locally-free-sheaf

Definition 16.3 (Locally free sheaf)**.** Let S be a scheme. A sheaf of $\mathcal{O}_S$-modules is a sheaf $\mathscr{E}$, along with the structure of a $\mathcal{O}_S(U)$-module on $\mathscr{E}(U)$ for each open $U \subset S$, such that the restriction maps are equivariant in the sense illustrated below:

$$\begin{array}{ccc} \mathcal{O}_S(U) & \longrightarrow & \mathcal{O}_S(V) \\ \circlearrowleft & & \circlearrowleft \\ \mathscr{E}(U) & \longrightarrow & \mathscr{E}(V) \end{array}$$

A *locally free sheaf* over S is a sheaf of $\mathcal{O}_S$-modules $\mathscr{E}$ such that $\mathscr{E}$ is locally isomorphic to $\mathcal{O}_S^{\oplus n}$ for some n. If $\mathscr{E}$ is locally isomorphic to $\mathcal{O}_S^{\oplus n}$ then $\mathscr{E}$ is said to be locally free *of rank* n. Locally free sheaves of rank 1 are also called *invertible sheaves.*

In other words, $\mathscr{E}$ is locally free if there is a cover of S by open subsets U such that $\mathscr{E}\big|_U \simeq \mathcal{O}_U^{\oplus n}$ as a sheaf of $\mathcal{O}_U$-modules. Note that the number n does not have to be the same for every open subset in the cover.

Definition 16.4 (Vector bundle, version 2)**.** A *vector bundle* over a locally ringed space S is a locally free sheaf on S.

16.3 Vector space schemes

Exercise 16.5. Show that, for any locally ringed space X, the set $\mathrm{Hom}_{\mathbf{LRS}}(X, \mathbf{A}^1)$ has the structure of a commutative ring, and for any morphism of schemes $X \to Y$, the induced map

$$\mathrm{Hom}_{\mathbf{Sch}}(Y, \mathbf{A}^1) \to \mathrm{Hom}_{\mathbf{Sch}}(X, \mathbf{A}^1)$$

is a ring homomorphism. Interpret this by saying $\mathbf{A}^1$ is a *commutative ring scheme.* (Hint: $\mathrm{Hom}(X, \mathbf{A}^1) = \Gamma(X, \mathcal{O}_X)$.)

Definition 16.6. An *scheme of* $\mathbf{A}^1$*-modules*[3] *over a scheme* S is an S-scheme E and the structure of a $\mathrm{Hom}_{\mathbf{Sch}/S}(T, \mathbf{A}^1)$-module on $\mathrm{Hom}_{\mathbf{Sch}/S}(T, E)$ for every S-scheme T, such that for every morphism of S-schemes $f : U \to T$, the function

$$\mathrm{Hom}_{\mathbf{Sch}/S}(T, E) \to \mathrm{Hom}_{\mathbf{Sch}/S}(U, E)$$

[3] There doesn't seem to be standard terminology for this object.

is a homomorphism, in the sense that for all $x \in \mathrm{Hom}_{\mathbf{Sch}/S}(T, E)$ and all $\lambda \in \mathrm{Hom}(T, \mathbf{A}^1)$, we have

$$f^*(\lambda x) = f^*(\lambda) f^*(x).$$

A *morphism of schemes of* $\mathbf{A}^1$*-modules over* S is a morphism of S-schemes $E \to F$ such that for any S-scheme T, the map

$$E(T) \to F(T)$$

is an $\mathbf{A}^1(T)$-module homomorphism.

This exercise is a special case of the next one. Do you see how?

Exercise 16.7. Show that there is a natural structure of a scheme of $\mathbf{A}^1$-modules over S on $S \times \mathbf{A}^n$ for any n. (Hint: Show that $\mathrm{Hom}_{\mathbf{Sch}/S}(T, S \times \mathbf{A}^n) = \Gamma(T, \mathcal{O}_T^n)$.) We write $\mathbf{A}^n_S$ for this scheme of $\mathbf{A}^1$-modules.

Exercise 16.8. Suppose that E is a $\mathbf{A}^1$-module over S and $T \to S$ is a morphism of schemes. Put an $\mathbf{A}^1$-module structure on $T \times_S E$ in a natural way.

When $T \subset S$ is an open subscheme, we write $E\big|_T$ for the construction from the previous exercise.

Definition 16.9 (Vector bundle, version 3)**.** A *vector bundle* over S is a $\mathbf{A}^1$-module E such that there is an open cover of S by schemes T with $E\big|_T \simeq T \times \mathbf{A}^n$ for some n. A *morphism of vector bundles* is a morphism of schemes of $\mathbf{A}^1$-modules.

16.4 Comparing the definitions

All of the definitions we have given so far are equivalent. It is easiest to see that the definition in terms of local charts is equivalent to the definition in terms of $\mathbf{A}^1$-modules.

Exercise 16.10. Show that a vector bundle in the sense of the first definition can be equipped with a unique $\mathbf{A}^1$-module structure that restricts to the standard $\mathbf{A}^1$-module structure on each chart. Conversely, show that each $\mathbf{A}^1$-vector space scheme has local charts by $\mathbf{A}^1$-modules.

The symmetric algebra

Exercise 16.11. (i) Let A be a commutative ring. Show that the forgetful functor from A-algebras to A-modules has a left adjoint. In other words, suppose that M is an A-module and let $F : A\text{-}\mathbf{Alg} \to \mathbf{Sets}$ be the functor sending an A-algebra B to $\mathrm{Hom}_{A\text{-}\mathbf{Mod}}(M, B)$. You have to show that this functor is representable. We denote this A-algebra $A[M]$ or $\mathrm{Sym}\, M$.

(ii) Now promote the previous part to sheaves. Let $\mathcal{O}_X$ be a sheaf of rings on a topological space X and show that the forgetful functor from sheaves of $\mathcal{O}_X$-algebras to sheaves of $\mathcal{O}_X$-modules has a left adjoint. (Hint: construct the adjoint on presheaves first and then sheafify.)

(iii) Finally, suppose that X is a scheme and that $\mathscr{M}$ is a *quasicoherent* $\mathcal{O}_X$-module. Prove that $\mathcal{O}_X[\mathscr{M}]$ is a quasicoherent $\mathcal{O}_X$-algebra. (In fact, X only needs to be a locally ringed space for this problem.)

Suppose that $\mathscr{E}$ is a sheaf of $\mathcal{O}_S$-modules on a locally ringed space S. For any S-scheme $f : T \to S$, write $\mathscr{E}\big|_T = f^*\mathscr{E}$, where

$$f^*\mathscr{E} = f^{-1}\mathscr{E} \otimes_{f^{-1}\mathcal{O}_S} \mathcal{O}_T.$$

We obtain a functor on S-schemes:

$$E(T) = \mathrm{Hom}_{\mathcal{O}_T\text{-}\mathbf{Mod}}(\mathscr{E}_T, \mathcal{O}_T)$$

Exercise 16.12. Show that the functor E defined above is representable by $\mathrm{Spec}_S \mathrm{Sym}\,\mathscr{E}$.

Exercise 16.13. Show that, for every S-scheme T, the set $E(T)$ is naturally equipped with the structure of a $\Gamma(T, \mathcal{O}_T)$-module, and that this makes $\mathrm{Spec}_S \mathrm{Sym}\,\mathscr{E}$ into a scheme of $\mathbf{A}^1$-modules.

The space of sections

Suppose again that $\mathscr{E}$ is a locally free sheaf on a locally ringed space S. Define a functor on S-schemes:

$$F(T) = \Gamma(T, \mathscr{E}_T)$$

Exercise 16.14. Show that the functor F is representable by a locally ringed space, and that this locally ringed space is a scheme if S is a scheme. (Hint: Apply the construction from the previous section to the dual locally free sheaf.)

17 Group schemes

Definition 17.1. A *group scheme* is a scheme G, equipped with the structure of a group on $G(S)$ for every scheme S, such that $G(S) \to G(T)$ is a group homomorphism whenever $T \to S$ is a morphism of schemes.

Exercise 17.2. Show that, if $\pi : G \to S$ is a group scheme over S, then G is equipped with a multiplication law, $m : G \times_S G \to G$ over S, an inversion map $i : G \to G$ over S, and a identity section $e : S \to S$ satisfying the following properties:

(i) $m \circ (\mathrm{id}_G, e\pi) = m \circ (e\pi, \mathrm{id}_G) = \mathrm{id}_G : G \to G$;

(ii) $m \circ (\mathrm{id}_G, i) = e\pi$ and $m \circ (i, \mathrm{id}_G) = e\pi : G \to G$;

(iii) $m \circ (\mathrm{id}_G \times m) = m \circ (m \times \mathrm{id}_G) : G \times G \times G \to G$.

Exercise 17.3. Define $\mathbf{G}_m(S) = \Gamma(S, \mathcal{O}_S^*)$. Show that $\mathbf{G}_m$ is representable by the scheme $\mathbf{A}^1 \smallsetminus \{0\} = \mathrm{Spec}\,\mathbf{Z}[t, t^{-1}]$. This is called the *multiplicative group.*

Exercise 17.4. Define $\mathbf{G}_a(S) = \Gamma(S, \mathcal{O}_S)$, viewed as a group under addition. Show that $\mathbf{G}_a$ is representable by the scheme $\mathbf{A}^1$. This is called the *additive group.*

Exercise 17.5. Define $\mathbf{GL}_n(S) = \mathrm{GL}\big(n, \Gamma(S, \mathcal{O}_S)\big)$. Prove that $\mathbf{GL}_n$ is representable by an affine scheme.

Exercise 17.6. Let $u : G \to H$ be a homomorphism of group schemes. Show that there is a group scheme $\ker(u)$ such that $\ker u(S) = \ker(G(S) \to H(S))$.

17.1 Affine group schemes and Hopf algebras

Exercise 17.7. (i) Show that the structure of a group scheme on $\operatorname{Spec} A$ induces a homomorphisms of commutative rings:

(a) (comultiplication) $\Delta : A \to A \otimes A$

(b) (antipode) $\iota : A \to A$

(c) (counit) $\epsilon : A \to \mathbf{Z}$

corresponding to multiplication, inversion, and the identity element. (Note that ϵ can also be viewed as a map from A into *every* commutative ring B.)

(ii) Translate the axioms of a group into identities satisfies by these maps. This structure is called a *Hopf algebra*.

Exercise 17.8. Describe the Hopf algebra structure on $\mathbf{Z}[t, t^{-1}]$ corresponding to the group structure on $\mathbf{G}_m$.

Exercise 17.9. Describe the Hopf algebra structure on $\mathbf{Z}[t]$ corresponding to $\mathbf{G}_a$.

Exercise 17.10. Describe the Hopf algebra structure corresponding to $\mathbf{GL}_n$.

17.2 Representations of the multiplicative group

Reading 17.11. [Vak14, §6.6]

Definition 17.12. An action of a group scheme G on a scheme X is a morphism $G \times X \to X$ such that $G(S) \times X(S) \to X(S)$ is an action of $G(S)$ on $X(S)$ for all schemes S.

Exercise 17.13. Show that an action of $\mathbf{G}_m$ on an affine scheme $X = \operatorname{Spec} A$ corresponds to a grading of A by $\mathbf{Z}$.

Solution. On the level of rings, we get a map $\mu^* : A \to A[t, t^{-1}]$. Define A_n to be the set of all $f \in A$ such that $\mu^*(f) = ft^n$. For any $f \in A$, we can write

$$\mu^*(f) = \sum f_n t^n.$$

Because μ is a group action, we have

$$(\mu^* \otimes \mathrm{id}) \circ \mu^*(f) = (\mu^* \otimes \mathrm{id}) \sum f_n t^n = \sum \mu^*(f_n) t^n$$
$$(\mathrm{id} \otimes \mu^*) \circ \mu^*(f) = (\mathrm{id} \otimes \mu^*) \sum f_n t^n = \sum f_n \mu^*(t)^n.$$

Now, $\mu^*(t) = st$ so we deduce that each component f_n is homogeneous of degree n (i.e., $\mu^*(f_n) = f_n t^n$). Furthermore $A = \sum A_n$ and the sum is direct. To see that multiplication respects the grading, suppose that f has degree n and g has degree m. Then

$$\mu^*(fg) = \mu^*(f)\mu^*(g) = ft^n gt^m = fgt^{n+m}$$

so fg has graded degree $n + m$.

Conversely, if A is graded, define $\mu^*(f) = \sum f_n t^n$ where f_n are the graded pieces of f. This immediately yields an action. $\square$

18 Projective space

18.1 The projective line

We will define two functors P^1 and Q^1 and show that they are isomrphic to one another, and that they are represented by the scheme $\mathbf{P}^1$.

For any scheme S, let $P^1(S)$ be the set of triples (L, s, t) where L is a rank 1 vector bundle on S and $s, t : L \to \mathbf{A}^1_S$ are linear maps such that

$$(s, t) : L \to \mathbf{A}^2_S$$

is injective. Injectivity here means that $L(T) \to \mathbf{A}^2_S(T)$ is injective for all S-schemes T.

Let $Q^1(S)$ be the set of isomorphism classes of triples $(\mathscr{L}, x, y)$ where $\mathscr{L}$ is an invertible sheaf on S (Definition 16.3) and $x, y \in \Gamma(S, \mathscr{L})$ are generators of $\mathscr{L}$. This means that if $z \in \Gamma(U, \mathscr{L})$ then there is a cover of U by open subsets V such that $z\big|_V = ax\big|_V + by\big|_V$ for some $a, b \in \Gamma(V, \mathcal{O}_S)$. In other words, $\mathscr{L}$ is the smallest $\mathcal{O}_S$-submodule of itself that contains both x and y. We say that $(\mathscr{L}, x, y)$ is isomorphic to $(\mathscr{L}', x', y')$ if there is an isomorphism of $\mathcal{O}_S$-modules $\varphi : \mathscr{L} \simeq \mathscr{L}'$ such that $\varphi(x) = x'$ and $\varphi(y) = y'$.

Exercise 18.1. Describe restriction maps making P^1 and Q^1 into functors.

ex:p1-isom-q1 **Exercise 18.2.** Prove that P^1 and Q^1 are isomorphic functors. (Hint: show that a map of vector bundles $E \to F$ is injective if and only if the corresponding map of locally free sheaves $\mathscr{F} \to \mathscr{E}$ is surjective.)

Solution. First we prove the claim suggested in the hint. Suppose that $s \in S$. The map $E(s) \to F(s)$ is injective by hypothesis. This map may be identified with the map

$$\mathrm{Hom}(\mathscr{E}_s, \mathbf{k}(s)) \to \mathrm{Hom}(\mathscr{F}_s, \mathbf{k}(s))$$

which is therefore an injective homomorphism of $\mathbf{k}(s)$-vector spaces. Therefore the dual map

$$\mathbf{k}(s) \otimes_{\mathcal{O}_s} \mathscr{F}_s \to \mathbf{k}(s) \otimes_{\mathcal{O}_s} \mathscr{E}_s$$

is a surjection of $\mathbf{k}(s)$-vector spaces. As $\mathscr{E}_s$ is a finitely generated $\mathcal{O}_{S,s}$-module, Nakayama's lemma implies that

$$\mathscr{F}_s \to \mathscr{E}_s$$

is surjective. But now pick an open U containing S where $\mathscr{E}\big|_U \simeq \mathcal{O}_U^{\oplus n}$. The n generators of $\mathcal{O}_U^{\oplus n}$ each lift to $\mathscr{F}$ in some open neighborhood of s, so, by taking the intersection of these neighborhoods, we find an open neighborhood V of s such that $\mathscr{F}\big|_V \to \mathscr{E}\big|_V$ is surjective. This is valid in a neighborhood of every point, so $\mathscr{F} \to \mathscr{E}$ is surjective as required.

Conversely, if $\mathscr{F} \to \mathscr{E}$ is surjective then, for any $u : T \to S$, the map

$$E(T) = \mathrm{Hom}_{\mathcal{O}_S\text{-}\mathbf{Mod}}(\mathscr{E}, u_* \mathcal{O}_T) \to \mathrm{Hom}_{\mathcal{O}_S\text{-}\mathbf{Mod}}(\mathscr{F}, u_* \mathcal{O}_T) = F(T)$$

is injective.

Now, we can identify $Q^1(T)$ with the set of all pairs $(\mathscr{L}, p)$ where $\mathscr{L}$ is an invertible sheaf on T and $p : \mathcal{O}_T^{\oplus 2} \to \mathscr{L}$ is a surjective homomorphism. Then, by the hint, this corresponds to an injection of locally free sheaves $L \to \mathbf{A}^2_T$, as required. $\square$

Exercise 18.3. Prove that P^1 and Q^1 are Zariski sheaves. (Hint: In view of Exercise 18.2, you only have to show one is a Zariski sheaf.)

Exercise 18.4. Let $U \subset Q^1$ be the subfunctor consisting of all triples $(\mathscr{L}, x, y)$ such that x generates $\mathscr{L}$.

(i) Show that the corresponding subfunctor $V \subset P^1$ consists of all triples (L, s, t) such that $s : L \to \mathbf{A}^1$ is an isomorphism.

(ii) Prove that $U \simeq \mathbf{A}^1$. (Hint: On U, multiplication by x gives an isomorphism $\mathcal{O} \to \mathscr{L}$.)

Solution. Since x generates $\mathscr{L}$, the map $\mathcal{O}_U \to \mathscr{L}$ is an isomorphism. Then yx^{-1} is an $\mathcal{O}_U$-module map $\mathcal{O}_U \to \mathcal{O}_U$, hence is multiplication by some global section of $\mathcal{O}_U$. This gives a map $U(S) \to \Gamma(S, \mathcal{O}_S) = \mathbf{A}^1(S)$. The inverse sends $t \in \mathbf{A}^1(S) = \mathrm{Hom}_{\mathcal{O}_S\text{-}\mathbf{Mod}}(\mathcal{O}_S, \mathcal{O}_S)$ to $(\mathcal{O}_S, 1, t) \in U(S)$. □

(iii) Show that U is an open subfunctor of Q^1.

Solution. To prove that U is open in Q^1, we need to show that for any $\varphi : S \to Q^1$, the sufunctor $\varphi^{-1}U \subset S$ is representable by an open subscheme. In fact, since U is a Zariski sheaf, it is sufficient to replace S with an open cover. We can therefore assume $S = \operatorname{Spec} A$. The map $\varphi : \operatorname{Spec} A \to Q^1$ corresponds to an invertible A-module L and A-module generators x and y.

Choose $f_1, \ldots, f_n \in A$, generating A as an ideal, such that $L_{f_i} \simeq A_{f_i}$ for all i. We argue that $\varphi^{-1}U \cap D(f_i) \subset D(f_i)$ is representable by an open subset. Choose an isomorphism $u_i : L_{f_i} \simeq A_{f_i}$. Then $(\varphi^{-1}U \cap D(f_i))(B)$ is the set of homomorphisms $g : A \to B$ such that $g(f_i) \in B^*$ and $Bx = B \otimes_A L$. But $B \otimes_A u_i$ is an isomorphism $B \otimes_A L \simeq B \otimes_A A = B$ so this is the same as $g(f_i) \in B^*$ and $Bg(u_i(x)) = B$. In other words, the condition is that $g(f_i) \in B^*$ and $g(u_i(x)) \in B^*$. Thus

$$\varphi^{-1}U \cap D(f_i) = D_{D(f_i)}(u_i(x)),$$

which is open in $D(f_i)$. □

(iv) Prove that Q^1 is a scheme. (Hint: Let U_0 be the set of triples $(\mathscr{L}, x, y)$ such that x generates $\mathscr{L}$ and let U_1 be the set of triples $(\mathscr{L}, x, y)$ such that y generates $\mathscr{L}$.)

Solution. We have to show U_0 and U_1 cover Q^1. Suppose we have $(\mathscr{L}, x, y) \in Q^1(k)$ for some field k. Then $\mathscr{L} = \widetilde{L}$ for some 1-dimensional k-vector space L. Choose an isomorphism $u : L \simeq k$. Then $u(x)$ and $u(y)$ generate k as a vector space. Thus either $u(x) \neq 0$ or $u(y) \neq 0$. In the former case, we have $\varphi \in U_0(k)$ and in the latter $\varphi \in U_1(k)$. □

Exercise 18.5. Prove that $Q^1 \simeq \mathbf{P}^1$. (Hint: What is the intersection of U_0 and U_1? Suggestion: Use symbols $\mathbf{U}_0$ and $\mathbf{U}_1$ for the standard charts of $\mathbf{P}^1$ from Section 1.1.)

18.2 Projective space

For any scheme T, define $P^n(T)$ to be the set of injections of vector bundles $L \to \mathbf{A}^{n+1}_T$, where L is a line bundle over T.

Define $Q^n(T)$ to be the set of surjections of $\mathcal{O}_T$-modules $\mathcal{O}_T^{n+1} \to \mathscr{L}$ where $\mathscr{L}$ is locally free of rank 1.

Exercise 18.6. Prove that P^n is isomorphic to Q^n for all n. (Hint: What you need to show here is that if $L \to \mathbf{A}_T^{n+1}$ corresponds to $\mathscr{O}_T^{n+1} \to \mathscr{L}$ then the former is a closed embedding if and only if the latter is a surjection. It's enough to prove this locally in T, so you can assume $L = \mathbf{A}_T^1$ and $\mathscr{L} = \mathscr{O}_T$.)

Solution. We have a contravariant equivalence between vector bundles (viewed as $\mathbf{A}^1$-modules) and locally free $\mathscr{O}_S$-modules under which $\mathscr{O}_T^{n+1}$ corresponds to $\mathbf{A}_T^{n+1}$. Therefore a map of vector bundles

$$L \to \mathbf{A}_T^{n+1}$$

in which L has rank 1 corresponds to a morphism of locally free modules

$$\mathscr{O}_T^{n+1} \to \mathscr{L}$$

in which $\mathscr{L}$ has rank 1. What we need to check is that $L \to \mathbf{A}_T^{n+1}$ is a closed embedding if and only if $\mathscr{O}_T^{n+1} \to \mathscr{L}$ is surjective.

We can verify that a morphism of schemes is a closed embedding locally. Therefore we can assume $T = \operatorname{Spec} A$ is affine $L = \mathbf{A}_T^1 = \operatorname{Spec} A[t]$ and $\mathscr{L} = \mathscr{O}_T$. The map $L = \mathbf{A}_T^1 \to \mathbf{A}_T^{n+1}$ corresponds to a linear map A-algebras $\varphi : A[u_0, \dots, u_n] \to A[t]$ sending each u_i to a multiple of t. In other words, it is a vector $(\xi_0, \dots, \xi_n) \in A$. This vector is precisely the matrix of the map $p : \mathscr{O}_T^{n+1} \to \mathscr{O}_T = \mathscr{L}$. Note that φ is surjective if and only if p is and φ is surjective if and only if $L \to \mathbf{A}_T^{n+1}$ is a closed embedding. □

Do one of the following two exercises. They are two perspectives on the same thing:

Exercise 18.7. Show that P^n is a scheme:

(i) Show that P^n is a Zariski sheaf.

(ii) For each $i = 0, \dots, n$, let U_i be the subfunctor of Q^n consisting of those linear closed embeddings $f : L \to \mathbf{A}_T^{n+1}$ such that if $p_i : \mathbf{A}_T^{n+1} \to \mathbf{A}_T^1$ is the i-th projection the map $f \circ i$ is an isomorphism. Show that $U_i \simeq \mathbf{A}^n$.

Solution. If $p_i \circ f$ is an isomorphism then $f \circ (p_i f)^{-1} : \mathbf{A}_T^1 \to \mathbf{A}_T^{n+1}$ can be represented as $(x_0, \dots, x_n) \in \Gamma(T, \mathscr{O}_T)^{n+1}$ with $x_j = p_j f (f p_i)^{-1}(1)$. But then $x_i = 1$ so by omitting x_i we have $(x_0, \dots, \hat{x}_i, \dots, x_n) \in \Gamma(T, \mathscr{O}_T)^n \simeq \operatorname{Hom}(T, \mathbf{A}^n)$. This gives a map $U_i \to \mathbf{A}^n$.

To reverse it, suppose $(x_0, \dots, \hat{x}_i, \dots, x_n) \in \mathbf{A}^n(T)$. Then set $x_i = 1$ define $\mathbf{A}_T^1 \to \mathbf{A}_T^{n+1}$ by $f(\lambda) = (\lambda x_0, \dots, \lambda x_n)$. This is an element of $U_i(T)$. (Note that $p_i f(\lambda) = \lambda$ is obviously invertible.) □

(iii) Show that each U_i is representable by $\mathbf{A}^n$.

(iv) Show that the U_i cover Q^n.

Solution. We have to show that if $\mathscr{O}_T^{n+1} \to \mathscr{L}$ is a schematic point of $\mathbf{P}^n$ then it lies in U_i for some i. But to be a schematic point means that $T = \operatorname{Spec} k$ for a field k. We can therefore identify $\mathscr{O}_T^{n+1} = k^{n+1}$ and $\mathscr{L} \simeq k$ (since all locally free sheaves on a point are free). The map is represented by $(x_0, \dots, x_n) \in k^{n+1}$ and to be surjective, at least one x_i must be nonzero, which means that the point lies in $U_i(T)$. □

Exercise 18.8. Show that Q^n is a scheme:

(i) Show that Q^n is a Zariski sheaf.

(ii) For each $i = 0, \dots, n$, let U_i be the subfunctor of Q^n consisting of those surjections $\mathscr{O}_T^{n+1} \to \mathscr{L}$ such that $\mathscr{O}_T e_i$ surjects onto $\mathscr{L}$. Show that U_i is an open subfunctor of Q^n.

Solution. We get to fix T and the map $\mathscr{O}_T^{n+1} \to \mathscr{L}$. The preimage of U_i is the set V_i consisting of all points $\xi \in T$ where $\mathscr{O}_T e_i \to \mathscr{L}$ is surjective. To prove that V_i is open, it is sufficient to show $V_i \cap W \subset W$ is open for all W in an open cover of T. We can therefore assume that $\mathscr{L} \simeq \mathscr{O}_T$.

Under this isomorphism, the map $\mathscr{O}_T e_i \to \mathscr{L} \simeq \mathscr{O}_T$ corresponds to an element $f \in \mathscr{O}_T$. Then $V_i = D(f)$ is open. □

(iii) Show that each U_i is representable by $\mathbf{A}^n$.

(iv) Show that the U_i cover Q^n.

Solution. We have to show that if $\mathscr{O}_T^{n+1} \to \mathscr{L}$ is a schematic point of $\mathbf{P}^n$ then it lies in U_i for some i. But to be a schematic point means that $T = \operatorname{Spec} k$ for a field k. We can therefore identify $\mathscr{O}_T^{n+1} = k^{n+1}$ and $\mathscr{L} \simeq k$ (since all locally free sheaves on a point are free). The map is represented by $(x_0, \dots, x_n) \in k^{n+1}$ and to be surjective, at least one x_i must be nonzero, which means that the point lies in $U_i(T)$. □

Exercise 18.9. Prove that $\mathbf{P}^n \simeq P^n$ or $\mathbf{P}^n \simeq Q^n$.

18.3 The tautological line bundle

If S is a scheme then a map $S \to \mathbf{P}^n$ corresponds to a linear embedding of a line bundle $L \subset \mathbf{A}_S^{n+1}$ or to a surjection $\mathscr{O}_S^{n+1} \to \mathscr{L}$ onto an invertible sheaf. In particular, the identity map $\mathbf{P}^n \to \mathbf{P}^n$ gives

$$L \subset \mathbf{A}_{\mathbf{P}^n}^{n+1}$$
$$\mathscr{O}_{\mathbf{P}^n}^{n+1} \to \mathscr{L}.$$

The quotient $\mathscr{L}$ is usually denoted $\mathbf{O}_{\mathbf{P}^n}(1)$ and is called the *tautological (quotient) sheaf.* The subbundle L is called the *tautological line (sub)bundle* and is sometimes denoted $\mathscr{O}_{\mathbf{P}^n}(-1)$ by people who are sloppy about the distinction between quasicoherent sheaves and schemes of modules.

Exercise 18.10. Suppose that $f : S \to \mathbf{P}^n$ corresponds to $(\mathscr{L}, \xi_0, \dots, \xi_n)$. Show that $f^* \mathscr{O}_{\mathbf{P}^n}(1) = \mathscr{L}$ in a canonical way.

Exercise 18.11. Show that $\mathscr{O}_{\mathbf{P}^n}(1)$ is not isomorphic to $\mathscr{O}_{\mathbf{P}^n}$. (Hint: Let A be a commutative ring, like $\mathbf{Z}[\sqrt{-5}]$, that is not a principal ideal domain and let I be a nonprincipal ideal with 2 generators. Use these to construct a map $f : \operatorname{Spec} A \to \mathbf{P}^n$ and show that $f^* \mathscr{O}_{\mathbf{P}^n}(1)$ is not isomorphic to $\mathscr{O}_{\operatorname{Spec} A}$.)

18.4 The Grassmannian

Fix a non-negative integer r and regard $\mathbf{A}^r$ as a vector space. That is, remember that we can use functions in $\Gamma(S, \mathscr{O}_S) = \mathrm{Hom}_{\mathbf{Sch}}(S, \mathbf{A}^1)$ to act on $\mathrm{Hom}_{\mathbf{Sch}}(S, \mathbf{A}^r)$. Define a functor

$$G : \mathbf{Sch}^\circ \to \mathbf{Sets}$$

by taking $G(S)$ to be the set of closed vector bundle subschemes $W \subset \mathbf{A}^r \times S$.

Exercise 18.12. Show that G is the disjoint union of open subfunctors $\coprod_{k=0}^r G_k$ where G_k parameterizes closed vector bundle subschemes $W \subset \mathbf{A}^r \times S$ of rank k.

Correction: $\mathscr{O}_S^{\oplus k}$ was supposed to be $\mathscr{O}_S^{\oplus r}$. Thanks to John Willis.

Exercise 18.13. Show that G_k is isomorphic to the functor $Q_k : \mathbf{Sch}^\circ \to \mathbf{Sets}$ where $Q_k(S)$ is the set of isomorphism classes of surjections $\mathscr{O}_S^{\oplus r} \to \mathscr{W}$, with $\mathscr{W}$ being a locally free sheaf of $\mathscr{O}_S$-modules of rank k.

Exercise 18.14. Show that the functors Q_k are representable by schemes. (Hint: Use the fact that you can glue vector bundles and homomorphisms of vector bundles to prove that Q_k is a Zariski sheaf. To get an open cover observe that at each point of Q_k there is some k-element subset $I \subset \{1, \ldots, r\}$ such that $\mathscr{O}_S^{\oplus I} \to \mathscr{W}$ is surjective. Let $U_I \subset G_k$ be the subfunctor parameterizing surjections $\mathscr{O}_S^{\oplus r} \to \mathscr{W}$ such that $\mathscr{O}_S^{\oplus I} \to \mathscr{W}$ is surjective. Show that U_I is an open subfunctor of G_k and that U_I is representable by $\mathbf{A}^{k \times (r-k)}$.)

The scheme representing G_k is denoted $\mathbf{Grass}(k, r)$ and called the *Grassmannian*.

Exercise 18.15. Define a functor on S-schemes parameterizing closed linear subschemes of a vector bundle V over S. Show that this is representable by an S-scheme. (Hint: After defining the functor and showing it is a sheaf, reduce to the case considered above by passing to an cover of S by open subsets U such that $V\big|_U \simeq U \times \mathbf{A}^r$.)

19 The homogeneous spectrum and projective schemes

Reading 19.1. [Vak14, §4.5], [Har77, §I.2; §II.2, pp. 76–77]

19.1 Some geometric intuition

The exercises in this section are not required (and may not even be well-posed). The idea here is to get an idea of where the construction in the next section comes from. We will see how to make all of these ideas precise later when we talk about algebraic groups.

We limit attention to schemes over $\mathbf{C}$. Recall that $\mathbf{CP}^n$ is the quotient of $\mathbf{C}^{n+1} \smallsetminus \{0\}$ by $\mathbf{C}^*$.

Note that $\mathbf{C}^*$ acts on $\mathbf{C}^{n+1}$ by scaling the coordinates. How does this translate geometrically? If f is a function on $\mathbf{C}^{n+1}$ and $\lambda \in \mathbf{C}^*$ then we get a (right) action of $\mathbf{C}^*$ on f by defining $f.\lambda(x) = f(\lambda x)$. Note that this definition is a bit sloppy because a function on a scheme is not always determined by its values. In this case this turns out to be okay, since functions are determined by their values on $\mathrm{Spec}\, \mathbf{C}[x_0, \ldots, x_n]$, but we will have to wait until later to see how to make this construction make sense more generally.

In other words, $\mathbf{C}^*$ acts on the ring $\mathbf{C}[x_0, \ldots, x_n]$. Now, the actions of $\mathbf{C}^*$ are well-understood. Any time $\mathbf{C}^*$ acts *algebraically* on a complex vector space V, we can decompose that vector space as a direct sum

$$V = \sum_{d \in \mathbf{Z}} V_d$$

where $\lambda \in \mathbf{C}^*$ acts on $v \in V_d$ by $\lambda.v = \lambda^d v$.

Correction! Thanks to Jon Lamar for noticing that the previous version of this exercise was incorrect.

Exercise 19.2. Prove that every *algebraic* representation of $\mathbf{C}^*$ can be written as a direct sum as above. (Hint: Use the fact that commuting diagonalizable endomorphisms of a vector space can be diagonalized simultaneously and that $\mathbf{C}^*$ contains lots of roots of unity. Then show the only linear algebraic actions of $\mathbf{C}^*$ on $\mathbf{C}$ are by $\lambda.x = \lambda^d x$ for some $d \in \mathbf{Z}$.)

Thus there is a grading on $\mathbf{C}[x_0, \dots, x_n]$.

Exercise 19.3. Verify that this is the usual grading by degree.

Thus, at least morally speaking, graded $\mathbf{C}$-algebras correspond to affine schemes over $\mathbf{C}$ with an action of $\mathbf{C}^*$.

We want to take the quotient of $\mathbf{C}^{n+1}$ by this action. However, this has no hope of being a reasonable geometric space because it wants to have a dense point with residue field $\mathbf{C}$ corresponding to the orbit $\{0\}$. We only have a chance of getting something reasonable if we delete the fixed point.

So we hope that $P = (\mathbf{C}^{n+1} \smallsetminus \{0\})/\mathbf{C}^*$ will turn out to be a scheme. Let's try to find a reasonable open cover. The open subsets of P correspond to $\mathbf{C}^*$-invariant open subsets of $\mathbf{C}^{n+1} \smallsetminus \{0\}$.

Exercise 19.4. Show that $D(f) \subset \mathbf{C}^{n+1}$ is $\mathbf{C}^*$-invariant if and only if f is a homogeneous polynomial or zero.

We write $D_+(f) \subset P$ for the open subset corresponding to $D(f) \subset \mathbf{C}^{n+1} \smallsetminus \{0\}$.

Now we figure out what $\mathcal{O}_P(D_+(f))$ should be. A function on $D_+(f)$ should be a function on $D(f)$ that is invariant under the action of $\mathbf{C}^*$. That is, we should have $f.\lambda = f$, which means precisely that f has graded degree zero. (Recall that f has graded degree d if $f.\lambda = \lambda^d f$.) The functions of with this property are exactly the ones of graded degree zero. Thus we get

$$\mathcal{O}_P(D_+(f)) = \mathbf{C}[x_0, \dots, x_n, f^{-1}]_0.$$

How much of this can be generalized? What if we had any affine $\mathbf{C}$-scheme at all with an action of $\mathbf{C}^*$. This corresponds to a graded ring S. We could imitate the above procedure, but we have to delete the locus in $\operatorname{Spec} S$ that is fixed by $\mathbf{C}^*$.

Exercise 19.5. Show that the fixed locus of $\mathbf{C}^*$ acting on $\operatorname{Spec} S$ is $V(S_{\neq 0})$.

Solution. Suppose that $\xi \in \operatorname{Spec} S$ is fixed and f is homogeneous of degree $d \neq 0$. Then $f(\lambda.\xi) = f(\xi)$ and $f.\lambda(\xi) = \lambda^d f(\xi)$. The only way this is possible for all λ is if $f(\xi) = 0$.

Conversely, suppose that $\xi \in V(S_{\neq 0})$. By the (prime) Nullstellensatz, to show that $\lambda.\xi = \xi$ it is equivalent to show that $f(\lambda.\xi) = 0$ if and only if $f(\xi) = 0$ for all $f \in S$. Suppose that $f(\xi) = 0$. We can write $f = g + h$ where $g \in S_0$ and $h \in S_{\neq 0}$. We get $f(\lambda.\xi) = f.\lambda(\xi) = g.\lambda(\xi) + h.\lambda(\xi)$. But $g.\lambda(\xi) = g$ since $\deg g = 0$ and $h.\lambda(\xi) = 0$ because $h.\lambda \in S_{\neq 0}$.

This shows that if $f(\xi) = 0$ then $f(\lambda.\xi) = 0$. For the reverse, replace ξ with $\lambda.\xi$ and λ with λ^{-1}. □

So the first step is to delete $V(S_{\neq 0})$. We then attempt to take a quotient of $D(S_{\neq 0})$ by $\mathbf{C}^*$. Again, the $\mathbf{C}^*$-invariant open subsets are of the form $D(f)$ where f is homogeneous, and these correspond to open subsets $D_+(f) \subset P = D(S_{\neq 0})/\mathbf{C}^*$. We define $\mathcal{O}_P(D_+(f)) = \mathcal{O}_{\operatorname{Spec} S}(D(f))_0$.

19.2 The homogeneous spectrum of a graded ring

Definition 19.6. A *graded ring* is a commutative ring S and a decomposition of the underlying abelian group of S into a direct sum: $S = \sum_{n \in \mathbf{Z}} S_n$ such that $S_n S_m \subset S_{n+m}$ for all $n, m \in \mathbf{Z}$. We write $S_{<0} = \sum_{n<0} S_n$, $S_{>0} = \sum_{n>0} S_n$, and $S_{\neq 0} = \sum_{n \neq 0} S_n$.[4]

An element of S is called *homogeneous* if it is contained in some S_n, in which case it is said to have *degree* n. An ideal of S is called *homogeneous* if it is generated by homogeneous elements. A homomorphism of $\mathbf{Z}$-graded rings $f : S \to T$ is a homomorphism of rings such that $f(S_n) \subset T_n$ for all $n \in \mathbf{Z}$.

The radical ideal generated by $S_{\neq 0}$ is called the *irrelevant ideal* and is denoted S_+.

Exercise 19.7. Show that if S contains a unit of non-zero degree then the irrelevant ideal is S itself.

Solution. Say f is a unit of degree $\neq 0$. Then $f \in S_+$ so $1 = f^{-1} f \in S_+$. □

Exercise 19.8. (i) Show that an ideal $I \subset S$ is homogeneous if and only if $I = \sum (I \cap S_n)$.

Solution. $\sum (I \cap S_n)$ is precisely the submodule of I generated by the homogeneous elements of I. □

(ii) [Vak14, Exercise 4.5.C (a)] Show that $I \subset S$ is a graded ideal if and only if it is the kernel of a homomorphism of graded rings.

Solution. Consider the map $S \to S/I$. If I is graded, define $(S/I)_n = S_n / I_n$. Then

$$(S/I)_n (S/I)_m = S_n S_m / (I_n S_m + S_n I_m) = S_n S_m / (I_{n+m} \cap S_n S_m) \subset S_{n+m} / I_{n+m} = (S/I)_{n+m}$$

so S/I is a graded ring.

Conversely, suppose $f : S \to T$ is a homomorphism of graded rings and let I be the kernel. If $f(\sum_{d \in \mathbf{Z}} a_d) = 0$ (with $\deg a_d = d$) then $\sum f(a_d) = 0$ so $f(a_d) = 0$ for all d so every element of I is a sum of homogeneous elements. □

(iii) [Vak14, Exercise 4.5.C (b)] Show that sums, products, intersections, and radicals of homogeneous ideals are homogeneous ideals.

Solution. $\sum (I \cap S_d) + \sum (J \cap S_d)$ is generated by the $I \cap S_d$ and $J \cap S_d$.

The product of homogeneous elements is homogeneous.

The intersection of I and J is the kernel of $S \to (S/I) \times (S/J)$.

The radical of I is the preimage of the nilradical ideal of S/I. Suppse $a = \sum a_d$ and $a^n = 0$. If d_0 is the smallest value of d for which $a_d \neq 0$ then $a_{d_0}^n = 0$. Thus a_{d_0} is nilpotent. Reduce modulo a_{d_0}. Then a_{d_0+1} is nilpotent modulo a_{d_0} by the same argument. Thus $a_{d_0+1}^k = b a_{d_0}$ for some k, so $a_{d_0+1}^{nk} = 0$. Proceed by induction. □

(iv) [Vak14, Exercise 4.5.C (c)] Show that a homogeneous ideal is prime if and only if $ab \in I$ implies $a \in I$ or $b \in I$ for *homogeneous* elements of S.

[4] In most treatments, S is assumed to be non-negatively graded, i.e., $S_{<0} = 0$.

Solution. Equivalent to S/I being an integral domain. Can assume that $I = 0$. If $ab = 0$ then $a_{d_0} b_{e_0} = 0$. The condition implies $a_{d_0} = 0$ or $b_{e_0} = 0$. Assume the former and induct. □

Definition 19.9 (The homogeneous spectrum)**.** Let S be a graded ring. Define $\operatorname{Proj} S$ to be the set of homogeneous prime ideals in S *that do not contain the irrelevant ideal.* This is called the *homogeneous spectrum* of S. For any homogeneous ideal $J \subset S$ we define $V_+(J) \subset \operatorname{Proj} S$ to be the set of homogeneous primes of S containing J and not containing the irrelevant ideal. We define $D_+(J)$ to be the complement of $V_+(J)$ in $\operatorname{Proj} S$.

Exercise 19.10 (The universal property of an open subset of the homogeneous spectrum)**.**

(i) Suppose $\varphi : S \to T$ is a homomorphism of graded rings such that $\sqrt{\varphi(S_+)T} = T_+$. Show that φ induces a continuous function $\operatorname{Proj} T \to \operatorname{Proj} S$ sending a homogeneous ideal $\mathfrak{p}$ of T to $\varphi^{-1}\mathfrak{p}$.

Correction! Thanks to Jon Lamar for noticing two errors here: the condition $\varphi^{-1}T_+ = S_+$ in the first part and the condition $\sqrt{\varphi(J)T} = T_+$ in the second part were incorrect.

(ii) Suppose that $u : \operatorname{Proj} T \to \operatorname{Proj} S$ is induced from a graded homomorphism $\varphi : S \to T$ as in the first part. Show that u factors through $D_+(J)$ if and only if $\sqrt{\varphi(J)T} \supset T_+$.

Solution. If $\mathfrak{p}$ is a homogeneous prime of T then $\varphi^{-1}\mathfrak{p}$ contains J if and only if $\mathfrak{p} \supset \varphi(J)T$. This holds for all homogeneous primes $\mathfrak{p}$ not containing T_+ if and only if $\sqrt{\varphi(J)T} \supset T_+$. Thus $u(\operatorname{Proj} T) \subset D_+(J)$ if and only if $\sqrt{\varphi(J)T} \supset T_+$. □

Exercise 19.11. Suppose that $S_+ = S$. Show that $\operatorname{Proj} S = \operatorname{Spec} S_0$ as a topological space. (Hint: If this is difficult, use the additional assumption that S contains an invertible element of non-zero degree; this is the only case that we will use. It is possible to reduce the general case to this one. The special case is essentially [Vak14, Exercise 4.5.E] or [Sta15, Tag 00JO].)

Solution. (Adapted from [Sta15, Tag 00JO].)

First we construct a map. If $\mathfrak{P} \subset S$ is a homogeneous prime then $\mathfrak{P} \cap S_0$ is a prime of S_0. We construct an inverse. Suppose that $\mathfrak{p} \in \operatorname{Spec} S_0$. Then we argue $\sqrt{\mathfrak{p}S}$ is prime. Note that $\sqrt{\mathfrak{p}S} \cap S_0 = \sqrt{\mathfrak{p}S \cap S_0} = \sqrt{\mathfrak{p}} = \mathfrak{p}$ since $\mathfrak{p}$ is prime in S_0, so once we show $\sqrt{\mathfrak{p}S}$ is prime it will follow that we have a one-sided inverse to $\operatorname{Proj} S \to \operatorname{Spec} S_0$.

It is equivalent to show that $T = S/\sqrt{\mathfrak{p}S}$ is a domain.

First we reduce to the case where T contains an invertible element of nonzero degree. Suppose $x \in T$ and let J be the annihilator ideal of x. Let K be the annihilator of J. Note that $x \in K$. Note that J is the annihilator of K: we have $JK = 0$ by definition (so $J \subset \operatorname{Ann}(K)$) and if $zK = 0$ then $zx = 0$ so $\operatorname{Ann}(K) \subset \operatorname{Ann}(x) = J$. We want to show that either $K = T$ or $J = T$. If $J = T$ then $x = 0$ and if $K = T$ then $J = 0$ so x is not a zero divisor.

First suppose that K and J are comaximal. Then there is some $y \in J$ and $z \in K$ and $a, b \in T$ such that $ay + bz = 1$. Let $e = ay$ and $f = bz$. Then $e + f = 1$ and $ef = 0$. Thus $e = e(e + f) = e^2$ and $f = f(e + f) = f^2$ are idempotents. They must be elements of $T_0 = S/\mathfrak{p}$, which is a domain because $\mathfrak{p}$ is prime. Thus we have either $e = 0$ or $f = 0$, which implies $f = 1$ or $e = 1$. As $e \in J$ and $f \in K$, this implies $J = T$ or $K = T$.

Now we can assume J and K are not comaximal. We will arrive at a contradiction. Since J and K are not comaximal, there is some maximal ideal $\mathfrak{m}$ of T containing $J + K$. Then $\mathfrak{m}$ does not contain T_+ since $T_+ = T$ so there is some $f \in T \smallsetminus \mathfrak{m}$ *of nonzero degree.* Consider $T[f^{-1}]$. As $JT[f^{-1}]$ and $KT[f^{-1}]$ annihilate each other, the reduction implies we must have $JT[f^{-1}] = 0$ or $KT[f^{-1}] = 0$. Thus either $f \in \sqrt{\operatorname{Ann}(J)} = \sqrt{K} \subset \mathfrak{m}$ or $f \in \sqrt{\operatorname{Ann}(K)} = \sqrt{J} \subset \mathfrak{m}$. This contradicts the choice of f as an element of $T \smallsetminus \mathfrak{m}$.

It remains to prove the statement under the assumption that S contains an invertible element of nonzero degree.

We assume S contains an invertible element f of nonzero degree. Suppose that $ab = 0$ in T. Then $(a^d/f^{\deg a})(b^d/f^{\deg b}) = 0$. But both of these are elements of $T_0 = S_0/\mathfrak{p}$, so at least one must be zero because $\mathfrak{p}$ is prime. Say it is $a^d/f^{\deg a}$. Then a is nilpotent (since f is a unit), so $a = 0$ because T is reduced ($\sqrt{\mathfrak{p}S}$ is a radical ideal).

Thus $\sqrt{\mathfrak{p}S}$ is prime and we have a well-defined map $\operatorname{Spec} S_0 \to \operatorname{Proj} S$. We have $\sqrt{\mathfrak{p}S} \cap S_0 = \mathfrak{p}$ so the composition

$$\operatorname{Spec} S_0 \to \operatorname{Proj} S \to \operatorname{Spec} S_0$$

is the identity. To see that the composition the other way is the identity, consider a homogeneous prime $\mathfrak{P} \subset S$. We want $\mathfrak{P} = \sqrt{(\mathfrak{P} \cap S_0)S}$.

We again reduce to the case where S contains an invertible element of nonzero degree f. By assumption $S_+ = S$ so there are elements f_i of nonzero degree such that we can write $1 = \sum a_i f_i$. Pick $y \in \mathfrak{P}$. By the reduction assumption $y \in \sqrt{(\mathfrak{P} \cap S_0)S S[f_i^{-1}]}$ for all i. Therefore we can choose n such that $f_i^n y \in \sqrt{(\mathfrak{P} \cap S_0)S}$ for all i. Choose a_i such that $\sum a_i f_i^n = 1$. Then $y = \sum a_i f_i^n y \in \sqrt{(\mathfrak{P} \cap S_0)S}$, as desired.

It again sufficient to assume that S contains an invertible element f of non-zero degree d. Let $a \in \mathfrak{P}$. Then $a^d/f^{\deg a} \in \mathfrak{P} \cap S_0$ so $a^d \in (\mathfrak{P} \cap S_0)S$ so $a \in \sqrt{(\mathfrak{P} \cap S_0)S}$, as desired.

Finally, we check the continuity. Suppose $g \in S_0$. Then the preimage of $D(g)$ in $\operatorname{Proj} S$ is $D_+(g)$. If $g \in S$ then the preimage of $D_+(g)$ is the set of all primes $\mathfrak{p} \subset S_0$ such that $g \notin \sqrt{\mathfrak{p}S}$.

Choose homogeneous elements f_i generating $S_+ = S$. We can write $\sum a_i f_i = 1$ for some a_i with $\deg a_i = -\deg f_i$. To show $\operatorname{Spec} S_0 \to \operatorname{Proj} S$ is continuous, it is sufficient to show that the maps

$$D(a_i f_i) \to \operatorname{Spec} S_0 \to \operatorname{Proj} S$$

are continuous. But the latter factor through $D_+(f_i) \subset \operatorname{Proj} S$. Identifying $D_+(f_i)$ with $\operatorname{Proj} S[f_i^{-1}]$ and $D(a_i f_i)$ with an open subset of $\operatorname{Spec} S[f_i^{-1}]_0$, this reduces the problem to the case where S contains an invertible element of nonzero degree.

We can assume there is an invertible element $f \in S$ of nonzero degree. Then $g \in \sqrt{\mathfrak{p}S}$ if and only if $g^{\deg f}/f^{\deg g} \in \sqrt{\mathfrak{p}S} \cap S_0 = \mathfrak{p}$ so that the preimage of $D_+(g)$ is exactly $D(g^{\deg f}/f^{\deg g})$. □

Correction: We only make the definition for f of nonzero degree to make the following exercise easier.

We give $\operatorname{Proj} S$ the sheaf of rings defined on the basis of open sets of the form $D_+(f)$ *where f has nonzero degree* by

$$\mathcal{O}_{\operatorname{Proj} S}(D_+(f)) = S[f^{-1}]_0.$$

Exercise 19.12. (i) Show that the open sets $D_+(f)$, for $f \in S_+$, form a basis for the topology of $\operatorname{Proj} S$.

Solution. For any prime $\mathfrak{p} \in \operatorname{Proj} S$, choose $D_+(g)$ such that $\mathfrak{p} \in D_+(g)$. Then choose $f \in S_+ \smallsetminus \mathfrak{p}$ (which exists by definition of $\operatorname{Proj} S$) and then $\mathfrak{p} \in D_+(fg)$. □

(ii) Verify that $\mathcal{O}_{\operatorname{Proj} S}$ is a presheaf.

Solution. Suppose that $D_+(g) \subset D_+(f)$. Then the map $\operatorname{Proj} S[g^{-1}] \to \operatorname{Proj} S$ has image in $D_+(f)$. Thus f generates $S[g^{-1}]_+$ as a radical ideal. Since g has nonzero degree, $fS[g^{-1}] = S[g^{-1}]_+ = S[g^{-1}]$ so f is a unit in $S[g^{-1}]$ so we get a (uniquely determined) map $S[f^{-1}] \to S[g^{-1}]$ inducing a map

$$\mathcal{O}_{\operatorname{Proj} S}(D_+(f)) = S[f^{-1}]_0 \to S[g^{-1}]_0 = \mathcal{O}_{\operatorname{Proj} S}(D_+(g)).$$

The uniqueness of the map provides the compatibility with $D_+(h) \subset D_+(g) \subset D_+(f)$. □

(iii) Verify that $\mathcal{O}_{\operatorname{Proj} S}$ is a sheaf on the basis $\mathscr{U} = \{D_+(f)\}$. (Hint: Save yourself work and reduce to Exercise 5.4.)

Solution. It is sufficient to assume that S contains an invertible element of nonzero degree, since every element of $\mathscr{U}$ is of that form. We can then identify $\operatorname{Proj} S = \operatorname{Spec} S_0$ and $\operatorname{Proj} D_+(g) = \operatorname{Spec} D(g)$. The exercise therefore reduces to the case of an affine scheme (Exercise 5.4). □

(iv) Extend $\mathcal{O}_{\operatorname{Proj} S}$ to a sheaf on $\operatorname{Proj} S$ in the only possible way. Show that $(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S})$ is a scheme.

Solution. On the open set $D_+(f)$, it coincides with $\operatorname{Spec} S[f^{-1}]_0$. □

20 Quasicoherent sheaves and schemes in modules

20.1 Quasicoherent sheaves

This is important but should feel like repetition of Exercise 5.4. Reuse as much of that exercise as you can.

Exercise 20.1. Let $S = \operatorname{Spec} A$ be an affine scheme. If M is an A-module, define $\widetilde{M}(D(f)) = M_f$ where M_f denotes the $A[f^{-1}]$-module $A[f^{-1}] \otimes_A M$.

(i) Construct restriction morphisms making $\widetilde{M}$ into a presheaf of $\mathcal{O}_S$-modules on the basis of principal open affine subsets of S.

(ii) Show that $\widetilde{M}$ is a sheaf on the basis of principal open affine subsets of S. (Hint: The proof is exactly the same as the proof in Exercise 5.4.)

(iii) Extend $\widetilde{M}$ to a sheaf on $\operatorname{Spec} A$.

`def:quasicoherent`

Definition 20.2 (Quasicoherent sheaf)**.** A sheaf $\mathscr{F}$ of $\mathcal{O}_S$-modules on a scheme S is said to be *quasicoherent* if there is a basis of affines $U = \operatorname{Spec} A$ such that $\mathscr{F}\big|_U \simeq \widetilde{M}$ for some A-module M.

Exercise 20.3. Show that a sheaf of $\mathcal{O}_S$-modules is quasicoherent if and only if it may be presented locally as the cokernel of a homomorphism of free modules (not necessarily of finite rank).

Note the correction! The word quasicoherent was previously missing.

Exercise 20.4. Show that a quasicoherent sheaf of $\mathcal{O}_S$-modules on an affine scheme $\operatorname{Spec} A$ is always of the form $\widetilde{M}$ for some A-module M.

20.2 Morphisms of vector bundles

Charts

If $p : E \to S$ is a vector bundle with charts $p^{-1}U_i \simeq \mathbf{A}^r_{U_i}$ and $q : F \to S$ is a vector bundle with charts *over the same open sets* $q^{-1}U_i \simeq \mathbf{A}^s_{U_i}$ then a morphism of vector bundles $E \to F$ is a morphism of S-schemes such that the induced maps

$$\mathbf{A}^r_{U_i} \simeq p^{-1}U_i \to q^{-1}U_i \simeq \mathbf{A}^s_{U_i}$$

are linear maps.

Not recommended! The point is that this isn't a pleasant thing to do.

Exercise 20.5. Define a morphism of vector bundles E and F whose charts are given on *different* open covers $\{U_i\}$ and $\{V_j\}$.

Locally free sheaves

Definition 20.6. If $\mathscr{F}$ and $\mathscr{G}$ are sheaves of $\mathcal{O}_S$-modules then a homomorphism $\mathscr{F} \to \mathscr{G}$ is a homomorphism of sheaves such that for each open $U \subset S$ the map $\mathscr{F}(U) \to \mathscr{G}(U)$ is a homomorphism of $\mathcal{O}_S(U)$-modules.

Schemes of modules

Definition 20.7. Suppose E and F are schemes of $\mathbf{A}^1$-modules over S. A morphism $E \to F$ is a morphism of S-schemes $\varphi : E \to F$ such that for every S-scheme T, the map $E(T) \to F(T)$ is $\mathbf{A}^1(T)$-linear.

20.3 Pullback of vector bundles and sheaves

Charts

Suppose $p : E \to S$ is a vector bundle with charts $p^{-1}U_i \simeq \mathbf{A}^r_{U_i}$. Let $f : T \to S$ be a morphism of schemes. Then $q : f^{-1}E \to T$ can be given charts $q^{-1}(f^{-1}U_i) \simeq \mathbf{A}^r_{f^{-1}U_i}$.

Exercise 20.8. Verify that the charts for $f^{-1}E$ are compatible and yield a vector bundle on T.

Schemes of modules

Suppose $p : E \to S$ is a scheme of $\mathbf{A}^1$-modules over S and $f : T \to S$ is a morphism of schemes. For any T-scheme $g : U \to T$, define

$$f^{-1}E(U, g) = E(U, fg).$$

Should be a matter of bookkeeping. Probably not worth writing up.

Exercise 20.9. Show that $f^{-1}E$ is naturally equipped with the structure of a sheaf of $\mathbf{A}^1$-modules over T.

Locally free sheaves

-sheaves-of-modules

Exercise 20.10 (Pushforward of sheaves of modules)**.** Suppose that $f : X \to Y$ is a morphism of sheaves and $\mathscr{F}$ is a $\mathcal{O}_X$-module. Show that $f_*\mathscr{F}$ is naturally equipped with the structure of a $\mathcal{O}_Y$-module. Show that this gives a functor

$$f_* : \mathcal{O}_X\text{-}\mathbf{Mod} \to \mathcal{O}_Y\text{-}\mathbf{Mod}$$

called *pushforward* of $\mathcal{O}_X$-modules to Y.

-sheaves-of-modules

Definition 20.11 (Pullback of sheaves of modules)**.** Let $f : X \to Y$ be a morphism of schemes. The *pullback* of an $\mathcal{O}_Y$-module $\mathscr{G}$ is an $\mathcal{O}_X$-module $f^*\mathscr{G}$ with the following universal property: for all $\mathcal{O}_X$-modules $\mathscr{F}$,

$$\mathrm{Hom}_{\mathcal{O}_X\text{-}\mathbf{Mod}}(f^*\mathscr{G}, \mathscr{F}) \simeq \mathrm{Hom}_{\mathcal{O}_Y\text{-}\mathbf{Mod}}(\mathscr{G}, f_*\mathscr{F})$$

naturally in $\mathscr{F}$.

The pullback exists for all sheaves of modules and all morphisms of ringed spaces, but we'll just construct it for quasicoherent sheaves and morphisms of schemes.

Exercise 20.12. (i) Suppose that $f : X \to Y$ is a morphism of *affine* schemes. Construct $f^*\mathscr{F}$ for any quasicoherent sheaf on Y. (Hint: Assume $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $\mathscr{F} = \widetilde{M}$ and take $f^*\mathscr{F} = (B \otimes_A M)^{\sim}$.)

(ii) Suppose that $f : X \to Y$ is an arbitrary morphism of schemes and $\mathscr{F}$ is a sheaf of modules on Y. Suppose that you know $(f\big|_U)^*\mathscr{F}$ exists for all U in an open cover of X. Glue these together to construct $f^*\mathscr{F}$.

(iii) Conclude that $f^*\mathscr{F}$ exists whenever $f : X \to Y$ is a morphism of schemes and $\mathscr{F}$ is quasicoherent.

If you already know about the tensor product of sheaves of modules, the following definition of f^* is more efficient than the one above:

Definition 20.13. Suppose that $\mathscr{F}$ is a sheaf of $\mathscr{O}_Y$-modules on Y and $f : X \to Y$ is a morphism of ringed spaces. Define $f^*\mathscr{F} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathscr{F}$.

Exercise 20.14. Show that $f^*\mathscr{F}$ as defined above satisfies the required universal property.

Should just be a line or two.

Exercise 20.15. Suppose that $\mathscr{F}$ is a locally free sheaf on Y and $f : X \to Y$ is a morphism of schemes. Show that $f^*\mathscr{F}$ is locally free.

Interpretation of sheaf pullback in terms of charts

Fix a map $f : X \to Y$ and a *locally free sheaf* $\mathscr{F}$ on Y. Choose an open cover of Y by U_i and isomorphisms $\alpha_i : \mathscr{F}\big|_{U_i} \simeq \mathcal{O}_{U_i}^{r_i}$. We construct a sheaf of $\mathcal{O}_X$-modules on X by gluing. Take $\mathscr{G}_{f^{-1}U_i} = \mathcal{O}_{f^{-1}U_i}^{r_i}$. On $f^{-1}U_i \cap f^{-1}U_j$, choose the isomorphism

$$\mathscr{G}_{f^{-1}U_i}\big|_{f^{-1}U_i \cap f^{-1}U_j} \simeq \mathcal{O}^{r_i}_{f^{-1}U_i \cap f^{-1}U_j} \to \mathcal{O}^{r_j}_{f^{-1}U_i \cap f^{-1}U_j} \simeq \mathscr{G}_{f^{-1}U_j}\big|_{f^{-1}U_i \cap f^{-1}U_j}$$

to be given by $f^*\varphi_{ij}$, where φ_{ij} is the transition function

$$\varphi_{ij} : \mathcal{O}^{r_i}_{U_i \cap U_j} \xleftarrow{\alpha_i} \mathscr{F}\big|_{U_i}\big|_{U_i \cap U_j} = \mathscr{F}\big|_{U_j}\big|_{U_i \cap U_j} \xrightarrow{\alpha_j} \mathcal{O}^{r_j}_{U_i \cap U_j}.$$

Exercise 20.16. Show that the $\mathscr{G}_{f^{-1}U_i}$ glue together to give $f^*\mathscr{F}$ (via a canonical isomorphism).

Thus the transition functions of $f^*\mathscr{F}$ are pulled back from the transition functions of $\mathscr{F}$. This is one reason it is reasonable to use the notation $f^*\mathscr{F}$ for this construction.

Chapter 6

Some moduli problems

21 Basic examples

21.1 The scheme in modules associated to a quasicoherent sheaf

Reading 21.1. [GD71, §9.4]

Let $\mathscr{F}$ be a quasicoherent sheaf of $\mathcal{O}_S$-modules on S.[1] For each S-scheme T, let

$$F(T) = \operatorname{Hom}_{\mathcal{O}_T\text{-}\mathbf{Mod}}(\mathscr{F}\big|_T, \mathcal{O}_T).$$

We give $F(T)$ the structure of a $\mathbf{A}^1(T)$-module. Suppose that $\lambda \in \mathbf{A}^1(T) = \Gamma(T, \mathcal{O}_T)$ and $x \in F(T)$. Then multiplication by λ gives a morphism $\mathcal{O}_T \to \mathcal{O}_T$ and composition with this homomorphism induces a map $F(T) \to F(T)$. We declare that $\lambda.x$ is the image of x under this map.

We write $F = \mathbf{V}(\mathscr{F})$ for this construction.

ex:O-mod-to-A-mod **Exercise 21.2.** Show that $F = \mathbf{V}(\mathscr{F})$ has the structure of a scheme of modules over S:

(i) Prove that F is a Zariski sheaf on $\mathbf{Sch}/S$.

Solution. Suppose that $f : T \to S$ is in $\mathbf{Sch}/S$ and $T = \bigcup U_i$ and $\xi, \eta \in F(T)$ and $\xi\big|_{U_i} = \eta\big|_{U_i}$ for all i. Then ξ and η are morphisms of sheaves of $\mathcal{O}_T$-modules $f^*\mathscr{F} \to \mathcal{O}_T$. Morphisms of sheaves that agree locally are identical so ξ and η are the same morphism.

Now suppose $f : T \to S$ is in $\mathbf{Sch}/S$, that $T = \bigcup U_i$, and $\xi_i \in F(U_i)$ are such that $\xi_i\big|_{U_i \cap U_j} = \xi_j\big|_{U_i \cap U_j}$ for all i and j. Then the ξ_i determine morphisms of sheaves $f^*\mathscr{F}\big|_{U_i} \to \mathcal{O}_T\big|_{U_i}$, hence glue to a global morphism of sheaves. □

(ii) Prove that F is representable by an affine scheme when $\mathscr{F}$ is quasicoherent and S is affine. (Hint: When $S = \operatorname{Spec} A$ and $F = \widetilde{M}$, represent it by $\operatorname{Spec} \operatorname{Sym} M$.)

(iii) Conclude that F is representable by a scheme over S.

[1] In fact, the definition works for any sheaf of $\mathcal{O}_S$-modules.

Theorem 21.3. *The functor* $\mathbf{V} : \mathbf{QCoh}(S)^\circ \to \mathbf{A}^1\text{-}\mathbf{Mod}/S$ *constructed above is fully faithful. It induces an equivalence between the categories of locally free* $\mathcal{O}_S$*-modules and of vector bundles.*

What we need to show is that the natural map

$$\mathrm{Hom}_{\mathcal{O}_S\text{-}\mathbf{Mod}}(\mathscr{E}, \mathscr{F}) \to \mathrm{Hom}_{\mathbf{A}^1\text{-}\mathbf{Mod}/S}(\mathbf{V}(\mathscr{F}), \mathbf{V}(\mathscr{E})) \tag{$*$}$$

eqn:16

is a bijection for any two quasicoherent sheaves $\mathscr{E}$ and $\mathscr{F}$ on S.

Exercise 21.4. Show that (24.2) may be regarded as the map of global sections between two sheaves on S. Conclude that to prove (24.2) is a bijection it is sufficient to assume S is affine.

The exercise tells us we may assume that $S = \operatorname{Spec} A$. Then $\mathscr{E} = \widetilde{M}$ and $\mathscr{F} = \widetilde{N}$ for two A-modules M and N. The the underlying schemes of $\mathbf{V}(\widetilde{M})$ and $\mathbf{V}(\widetilde{N})$ are $\operatorname{Spec} \operatorname{Sym}_A M$ and $\operatorname{Spec} \operatorname{Sym}_A N$, respectively.

Exercise 21.5. Prove that every $\mathbf{A}^1$-*linear* map $\mathbf{V}(\widetilde{N}) \to \mathbf{V}(\widetilde{M})$ arises from a homomorphism of A-modules $M \to N$.

Solution. A map $\mathbf{V}(\widetilde{N}) \to \mathbf{V}(\widetilde{M})$ over S induces in particular maps

$$\mathrm{Hom}_{A\text{-}\mathbf{Mod}}(N, B) \to \mathrm{Hom}_{A\text{-}\mathbf{Mod}}(M, B)$$

for any A-algebra B. The linearity condition says that this is a homomorphism of B-modules. Apply this in particular when $B = \operatorname{Sym} N$ and we get a map $\varphi : M \to \operatorname{Sym} N$ associated to the inclusion $N \to \operatorname{Sym} N$. We argue that φ factors uniquely through $N \subset \operatorname{Sym} N$.

Suppose that $\varphi(x) = \sum y_k$ with $y_k \in \operatorname{Sym}^k N$. Then the linearity of $\mathbf{V}(\widetilde{M}) \to \mathbf{V}(\widetilde{N})$ implies that the diagram below commutes:

$$\begin{tikzcd}
\operatorname{Sym} M \arrow[r] \arrow[d] & \operatorname{Sym} N \arrow[d] \\
A[t] \otimes_A \operatorname{Sym} M \arrow[r] & A[t] \otimes_A \operatorname{Sym} N
\end{tikzcd}$$

Following $x \in M \subset \operatorname{Sym} M$ both ways around the diagram gives

$$x \mapsto \sum y_k \mapsto \sum t^k \otimes y_k$$
$$x \mapsto t \otimes x \mapsto \sum t \otimes y_k.$$

The only way this can commute is if $y_k = 0$ for $k \neq 0$, which means that $\operatorname{Sym} M \to \operatorname{Sym} N$ is induced from a map $M \to \operatorname{Sym} M$. $\square$

22 Coherent schemes

22.1 The diagonal

Exercise 22.1. The equalizer of a pair of morphisms of schemes is locally closed in the domain.

Solution. Let $f, g : X \to Y$ be a pair of morphisms of schemes and let Z be their equalizer. We have to show that every point of Z has an affine open neighborhood U in X in which $Z \cap U$ is closed. Let $V \subset Y$ be an affine open subset. Then the open sets $f^{-1}V \cap g^{-1}V$ form an open cover of Z. Let U be an open affine neighborhood of $z \in f^{-1}V \cap g^{-1}V$. Then $Z \cap U$ is the equalizer of $U \rightrightarrows V$. Suppose $U = \operatorname{Spec} B$ and $V = \operatorname{Spec} A$. Then this is representable by the coequalizer of $B \rightrightarrows A$, which is a quotient of A. □

Exercise 22.2. Show that the equalizer of a pair of maps $X \rightrightarrows Y$ can be interpreted as the fiber product $X \times_{Y \times Y} \Delta Y$.

22.2 Quasicompact and quasiseparated morphisms

Reading 22.3. [Har77, Exercises 2.13, 3.2], [Vak14, §§3.6.5, 5.1, 10.1.9–12]

Recall that a scheme X is quasicompact if every open subcover of X has a finite subcover.

asicompact-morphism

Definition 22.4. A morphism of schemes $f : X \to Y$ is said to be *quasicompact* if for, for any morphism of schemes $Z \to Y$ with Z quasicompact, the fiber product $Z \times_Y X$ is quasicompact.

It is said to be *quasiseparated* if for every pair of maps $g, h : Z \to X$ with Z quasicompact such that $fg = fh$, the equalizer $W \subset Z$ of g and h in Z is quasicompact.

Morphisms that are both quasicompact and quasiseparated are sometimes called *coherent*.

Exercise 22.5. Show that a morphism of schemes $f : X \to Y$ is quasiseparated if and only if the diagonal map $X \to X \times_Y X$ is quasicompact.

Exercise 22.6. Suppose that $f : X \to Y$ is a morphism of schemes such that for any quasicompact *open subset* $U \subset Y$ the preimage $f^{-1}U$ is also quasicompact. Show that f is quasicompact. (Hint: In the notation of Definition 22.4, reduce to the case where Z and Y are affine.)

Solution. Let $W = X \times_Y Z$. It's sufficient to show that there is a cover of Z by affine opens $U \subset Z$ whose preimages in W is quasicompact. Choose a cover by affine opens U such that $U \to Y$ factors through a cover of Y by open affines V. Then we want to show that $U \times_V X$ is quasicompact. By assumption, X must be quasicompact, so choose a finite cover by affines. Their preimages in W cover it. On the other hand, a fiber product of affines is affine, hence quasicompact. Thus W has a finite cover by affines, hence is quasicompact. □

22.3 Pushforward of quasicoherent sheaves

Exercise 22.7. Show that a sheaf $\mathscr{F}$ of $\mathcal{O}_X$-modules on $X = \operatorname{Spec} A$ is quasicoherent if and only if $\Gamma(D(f), \mathscr{F}) = \Gamma(X, \mathscr{F})_f$ for all $f \in A$.

Solution. Certainly this holds for quasicoherent sheaves. To see the converse, suppose the condition holds and let $M = \Gamma(X, \mathscr{F})$. Then $M_f = \Gamma(D(f), \mathscr{F})$ for all $f \in A$. Therefore $\widetilde{M}$ and $\mathscr{F}$ agree on a basis of open subsets of X, so they must be isomorphic sheaves. □

Exercise 22.8. Show that the kernel and cokernel of a homomorphism of quasicoherent sheaves are quasicoherent sheaves.

Solution. It is sufficient to treat the case of an affine scheme. Suppose that

$$0 \to \mathscr{K} \to \mathscr{F} \to \mathscr{G} \to \mathscr{L} \to 0$$

is exact and $\mathscr{F} = \widetilde{M}$ and $\mathscr{G} = \widetilde{N}$. Let K and L be the cokernel and kernel of $M \to N$, respectively. Then the sequence

$$0 \to K_f \to M_f \to N_f \to L_f \to 0$$

is exact for all $f \in A$. Therefore the kernel of $\mathscr{F}(D(f)) \to \mathscr{G}(D(f))$ is $\mathscr{K}(D(f))$. Thus $\mathscr{K} = \widetilde{K}$.

Furthermore, we see that the presheaf cokernel of $\mathscr{F} \to \mathscr{G}$ is $\widetilde{L}$ by the same reasoning, which gives a map $\widetilde{L} \to \mathscr{L}$ by the universal property of the presheaf cokernel. But $\widetilde{L}$ is a sheaf, so this map is an isomorphism $\mathscr{L} = \widetilde{L}$. □

hm:qcoh-pushforward

Theorem 22.9. *Suppose that $\pi : X \to Y$ is a quasicompact and quasiseparated morphism of schemes and $\mathscr{F}$ is a quasicoherent sheaf on X. Prove that $\pi_*\mathscr{F}$ is a quasicoherent sheaf on Y.*

Proof. We begin with two observations:

(i) Since $\pi_*\mathscr{F}\big|_U = (\pi\big|_{\pi^{-1}U})_*\mathscr{F}\big|_{\pi^{-1}U}$, it is sufficient to assume that $Y = \operatorname{Spec} A$ is affine.

(ii) If $X = \operatorname{Spec} B$ is also affine then $\mathscr{F} = \widetilde{M}$ for some B module M. Then $\pi_*\mathscr{F} = \widetilde{M_A}$ where M_A denotes M, equipped with the A-module structure inherited via the homomorphism $A \to B$.

Choose an open cover of X by affine open subsets U_i. For each indices i and j, choose an open cover $U_i \cap U_j = \bigcup_k U_{ijk}$ with U_{ijk} also affine. Because X is quasicompact and quasiseparated over Y, these collections can all be chosen finite. Abusively, write i, etc. for the inclusions $U_i \to X$. Then we have an exact sequence:

$$0 \to \mathscr{F} \to \prod_i i_*i^*\mathscr{F} \to \prod_{ijk} k_*k^*\mathscr{F}$$

Indeed, if we evaluate this sequence on an open subset $V \subset X$ we get the sequence

$$0 \to \mathscr{F}(V) \to \prod_i \mathscr{F}(V \cap U_i) \to \prod_{ijk} \mathscr{F}(V \cap U_{ijk})$$

and this is exact by the sheaf conditions. By the left exactness of π_*, we now get a sequence

$$0 \to \pi_*\mathscr{F} \to \pi_* \prod_i i_*i^*\mathscr{F} \to \pi_* \prod_{ijk} k_*k^*\mathscr{F}.$$

Now, we can identify $\pi_*i_*i^*\mathscr{F}$ with $(\pi i)_*i^*\mathscr{F}$, which is quasicoherent because U_i is affine and $i^*\mathscr{F}$ is quasicoherent. (Note: π_* commutes with products.) Likewise, $\pi_*k_*k^*\mathscr{F}$ is quasicoherent. Therefore $\pi_*\mathscr{F}$ is the kernel of a homomorphism between quasicoherent $\mathscr{O}_Y$-modules, hence is quasicoherent. □

Exercise 22.10. (i) Show by example that an infinite product of quasicoherent sheaves is not necessarily quasicoherent. (Hint: Use the failure of localization to commute with infinite products.)

Solution. On $\mathbf{A}^1_{\mathbf{C}}$, let $\mathscr{F} = \prod_{i=1}^{\infty} \mathcal{O}$. Then $\mathscr{F}(D(f)) = \prod_{i=1}^{\infty} \mathbf{C}[x, f^{-1}]$ but the element $(f^{-1}, f^{-2}, f^{-3}, \ldots)$ is not contained in $\left(\prod_{i=1}^{\infty} \mathbf{C}[x]\right)_f$. □

(ii) Show by example that an infinite intersection of a quasicoherent subsheaves of a quasicoherent sheaf is not necessarily quasicoherent.

Exercise 22.11. Use the previous exercise to show that both the hypothesis of quasicompactness and quasiseparation are necessary in Theorem 22.9.

Chapter 7

Essential properties of schemes

23 Finite presentation

23.1 Filtered diagrams

def:filtered **Definition 23.1.** A category P is said to be *filtered* if every finite diagram in P has an upper bound.

In practical terms, the definition means the following:

(i) for any pair of objects $x, y \in P$ there is an object $z \in P$ and morphisms $x \to z$ and $y \to z$;

(ii) for any pair of morphisms $x \rightrightarrows y$ in P there is a morphism $y \to z$ in P that coequalizes them.

Note that the second condition holds vacuously for a partially ordered set.[1]

23.2 Remarks on compactness

Exercise 23.2. (i) Suppose that X is a quasicompact topological space and $Y = \bigcup Y_i$ is a filtered union of open subsets. Show that any morphism $X \to Y$ factors through one of the Y_i.

Solution. Let $f : X \to Y$ be a map. Then the $f^{-1}Y_i$ are an open cover of X, hence there is a finite subcover. Choose i to be an upper bound for the indices in this finite collection. □

(ii) Suppose that X is a quasicompact and quasiseparated topological space with a basis of quasicompact open subsets and $Y = \varinjlim Y_i$ is a filtered colimit of a diagram of topological spaces where the transition maps are open embeddings and the Y_i are all étale over Y. Show that any morphism $X \to Y$ factors through one of the Y_i.

[1] The conditions above are usually taken as the definition of a filtered diagram (and the first as the condition of a filtered partially ordered set). However, just as the first condition does not extend trivially, these two conditions do not extend trivially to higher categories. Definition 23.1 does.

Solution. The maps $p_i : Y_i \to Y$ are all local isomorphisms. Their images are therefore an open cover of Y and the $f^{-1}p_i(Y_i)$ cover X. A finite number suffice. Choose an upper bound for these. Then $f(X)$ lies in $p_i(Y_i)$. Choose a finite cover of X by quasicompact open subsets V_j such that $V_j \to Y$ factors through Y_i. On $V_j \cap V_k$ we have two maps $V_j \cap V_k \rightrightarrows Y_i$, which may not agree. However, for each point $x \in V_j \cap V_k$, the two maps $V_j \cap V_k \rightrightarrows Y_{i'}$ agree in a neighborhood of x, for some index $i' \geq i$. But $V_j \cap V_k$ is quasicompact, so we can choose one i' to work for all j and k and all points of $V_j \cap V_k$. Replace i by this i'. Then the two maps $V_j \cap V_k \rightrightarrows Y_i$ agree, hence we can glue to get $X \to Y_i$, as desired. □

23.3 Finite type and finite presentation

Reading 23.3. [GD67, IV.8.14]

Definition 23.4. A morphism of schemes $f : X \to Y$ is said, respectively, to be *locally of finite type* or *locally of finite presentation* if there is an open cover of Y by open affine subsets $V = \operatorname{Spec} A$ such that $f^{-1}V$ is covered by open affines $U = \operatorname{Spec} B$ where B is a finite type or finitely presented A-algebra. The morphism is of *finite type* if it is quasicompact and locally of finite type. It is *finitely presented* if it is locally of finite presentation and quasicompact and quasiseparated.

Exercise 23.5. Suppose that B is an A-algebra and $C = \varinjlim C_i$ is a filtered direct limit of A-algebras. Consider the map

$$\Phi : \varinjlim \operatorname{Hom}_{A\text{-}\mathbf{Alg}}(B, C_i) \to \operatorname{Hom}_{A\text{-}\mathbf{Alg}}(B, C).$$

(i) Prove that Φ is an injection for all $C = \varinjlim C_i$ if B is of finite type over A.

Solution. Choose $A[x_1, \ldots, x_n]$ surjecting onto B. Then we have a commutative diagram

$$\begin{tikzcd}
\varinjlim \operatorname{Hom}_{A\text{-}\mathbf{Alg}}(B, C_i) \arrow[r] \arrow[d] & \operatorname{Hom}_{A\text{-}\mathbf{Alg}}(B, C) \arrow[d] \\
\varinjlim \operatorname{Hom}_{A\text{-}\mathbf{Alg}}(A[x_1, \ldots, x_n]) \arrow[r] \arrow[d, equal] & \operatorname{Hom}_{A\text{-}\mathbf{Alg}}(A[x_1, \ldots, x_n], C) \arrow[d, equal] \\
\varinjlim C_i^n \arrow[r] & C^n
\end{tikzcd}$$

The vertical arrows are injective and the lower horizontal arrow is injective (since filtered colimits commute with finite products). □

(ii) Prove that Φ is a bijection for all filtered unions $C = \bigcup C_i$ if and only if B is of finite type over A.

Solution. We already know Φ is an injection. Say we have a map $B \to C$. Since B is finitely generated, all of a finite set of generators must appear in some C_i, so we get surjectivity as well.

Conversely, recognize $B = \bigcup B_i$, with each B_i finitely generated over A. Then id_B factors through some B_i, so B is finitely generated. □

(iii) Prove that Φ is a bijection for all $C = \varinjlim C_i$ if and only if B is of finite presentation over A.

Solution. We have injectivity from the previous part. To get surjectivity, present B as the quotient of $A[x_1, \ldots, x_n]$ by a finitely generated ideal $J = (f_1, \ldots, f_m)$. Given a map $B \to C$, lift it to a map $A[x_1, \ldots, x_n] \to C_i$ for some i. For each f_j there is some $i(j) > i$ such that $f_j \mapsto 0$ in $C_{i(j)}$. Choose $i' > i(j)$ for all of the finitely many j. Then all f_j map to 0 in $C_{i'}$. We therefore get a factorization of $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m) \to C$ through $C_{i'}$, as desired.

Conversely, recognize $B = \varinjlim B_i$ where each B_i is a subalgebra of finite type. The identity map $B \to B$ must factor through some B_i and $B_i \subset B$ so $B_i = B$. Therefore B is of finite type. Let $x_1, \ldots, x_n$ generate B. Let J be the ideal of B in $A[x_1, \ldots, x_n]$. Let J_i run over the finitely generated submodules of J. Then $B = \varinjlim A[x_1, \ldots, x_n]/J_i$. Therefore the identity map $B \to B$ factors through some $A[x_1, \ldots, x_n]/J_i$. Therefore $J \subset J_i$. Likewise $J_i \subset J$, so $J = J_i$ and $B = A[x_1, \ldots, x_n]/J_i$. □

Should be easy

Exercise 23.6. Show that a morphism of locally noetherian schemes is of locally finite type if and only if it is of locally finite presentation.

lem:lfp

Lemma 23.7. *Let X be an A-scheme and let $C = \varinjlim C_i$ be a colimit of A-algebras. Consider the map*

$$\Phi : \varinjlim X(C_i) \to X(C)$$

(i) *If C is the filtered colimit of the C_i and X is of finite type the Φ is an injection.*

(ii) *If C is the filtered union of the C_i* and C is an integral domain *and X is of finite type then Φ is a bijection.*

(iii) *If C is the filtered colimit of the C_i and X is of finite presentation then Φ is a bijection.*

Proof. Let $f_o, g_o : \operatorname{Spec} C_o \to X$ induce the same map on $\operatorname{Spec} C$. Write $\rho_o : \operatorname{Spec} C \to \operatorname{Spec} C_o$. Choose a finite cover of X by open affine subschemes U_j. Then $\operatorname{Spec} C_o$ is covered by $\rho_o^{-1}(f^{-1}U_j \cap g^{-1}U_j)$. Choose $x_k \in C_o$ such that each $D(x_k)$ is contained in some $f^{-1}U_j \cap g^{-1}U_j$ and the $D(x_k)$ cover $\bigcup (f^{-1}U_j \cap g^{-1}U_j)$. Then the $\rho_o^{-1}(D(x_k))$ cover $\operatorname{Spec} C$, so we can write $\sum a_k x_k = 1$ with coefficients $a_k \in C$. All of these coefficients appear in some C_i, so we can replace o with i and assume that the $f^{-1}U_j \cap g^{-1}U_j$ cover $\operatorname{Spec} C_o$.

Now let $V_k = D(x_k)$ be any affine open subset of $f^{-1}U_j \cap g^{-1}U_j \subset \operatorname{Spec} C_o$. We get a pair of maps $f, g : V_k \to U_j$ corresponding to a pair of ring homomorphisms $\varphi_o, \psi_o : B \to D_o$. Put $D_k = D_o \otimes_{C_o} C_i$ and $D = D_o \otimes_{C_o} C$. Since $\varphi, \psi : D \to B$ agree and D is of finite type, there is some index i such that $\varphi_i = \psi_i$. Replacing o by i, we can assume that the two maps $f, g : V_k \to U_j$ agree.

There are only finitely many V_k in all, so we can repeat the above procedure finitely many times to ensure that $f = g$ on all open subsets in a cover of $\operatorname{Spec} C_o$, which gives the desired injectivity.

Now we prove the surjectivity under the assumption that X is of finite presentation. Assume given a map $f : \operatorname{Spec} C \to X$. Choose a finite open cover of X by affines U_j and a finite open cover of $\operatorname{Spec} C$ by affines $V_k = D(x_k)$ subordinate to the $f^{-1}U_j$. Each x_k lies in some C_i, and since there are only finitely many of them, we can assume they are in the same C_o. Then $D_C(x_k)$ is the preimage of $D_{C_o}(x_k)$.

Fix one value of k. Then $C[x_k^{-1}]$ is the filtered colimit of the $C_i[x_k^{-1}]$. By assumption we have given a map $B_j \to C[x_k^{-1}]$, and since B_j is of finite presentation, this comes from a map $B_j \to C_i[x_k^{-1}]$ for some i. We can choose one i to work for all k, and replace o with this i.

We now have to check that the maps $\operatorname{Spec} C_o[x_k^{-1}] \to U_j \subset X$ agree on intersections. For each j, j' choose an open cover of $U_j \cap U_{j'}$ by open affines W. Cover the preimage $f^{-1}W \subset \operatorname{Spec} C$ by open affines $D(x_k)$. Enlarging o again, we can assume that all x_k comes from C_o. We get two maps $\operatorname{Spec} C_o[x_k^{-1}] \rightrightarrows W = \operatorname{Spec} B$. These maps agree on $\operatorname{Spec} C[x_k^{-1}]$, so by finite presentation, we get agreement on $\operatorname{Spec} C_i[x_k^{-1}]$ if i is sufficiently large. There are only finitely many values of k to contend with, so a single enlargement of o suffices to guarantee that all the maps $\operatorname{Spec} C_o[x_k^{-1}] \rightrightarrows X$ agree. This implies that they glue to a single map $\operatorname{Spec} C_o \to X$ inducing $f : \operatorname{Spec} C \to X$, as desired.

(In the case where C is an integral domain and we have a filtered union, the compatibility is automatic becuase $C_i \subset C$ and the maps to C are compatible.) □

Theorem 23.8. *For any scheme X, the following conditions are equivalent:*

(i) X is locally of finite presentation;

(ii) for any filtered system of commutative rings A_i with $\varinjlim A_i = A$, the map $\varinjlim X(A_i) \to X(A)$.

Proof. We've already seen in Lemma 23.7 that the first condition implies the second. We prove the second implies the first.

Assume that $\varinjlim X(A_i) \to X(A)$ is a bijection. Let $U = \operatorname{Spec} B$ be an open affine of X. We argue that $\varinjlim \operatorname{Hom}(B, A_i) \to \operatorname{Hom}(B, A)$ is a bijection. Indeed, we certainly have $\operatorname{Hom}(B, A) = U(A) \subset X(A)$ and $\varinjlim \operatorname{Hom}(B, A_i) = \varinjlim U(A_i) \subset \varinjlim X(A_i) = X(A)$ so the map is an injection. To show it is surjective, suppose that $f \in U(A)$. Certainly f is induced from some $g \in X(A_i)$. Consider $g^{-1}(U) \subset \operatorname{Spec} A_i$. This is an open subset, so there is some ideal J such that $g^{-1}(U) = D(J)$. The projection $\operatorname{Spec} A \to \operatorname{Spec} A_i$ factors through $D(J)$ so $JA = A$. That is, we can find $x_j \in J$ and $a_j \in A$ such that $\sum a_j x_j = 1$. Enlarging i as necessary, we can assume that $a_j \in A_i$ and $\sum a_j x_j = 1$ holds in A_i. Then $D(J) = A_i$. That is $g \in U(A_i)$, as desired. □

24 Separated and proper morphisms I

Reading 24.1. [Vak14, §§10.1, 10.3, 12.7] [Har77, §II.4]

In this section and the next we will investigate the algebro-geometric analogues of compact and Hausdorff topological spaces. Recall that a topological space X is called *Hausdorff* if, for any pair of points x and y, there are open neighborhoods $x \in U$ and $y \in V$ with $U \cap V = \varnothing$. Equivalently, $U \times V$ is an open neighborhood of (x, y) in $X \times X$ that does not meet the diagonal $X \subset X \times X$. In other words, the diagonal is closed.

Exercise 24.2. Show that the following conditions are equivalent for a topological space X:

(i) X is Hausdorff;

(ii) the diagonal $X \to X \times X$ is a closed embedding;

(iii) for any pair of maps $Z \rightrightarrows X$, their equalizer is closed in Z.

The product of two schemes does not have the product topology, so these conditions are not equivalent for schemes. We know that essentially no scheme is Hausdorff in the literal sense, but the latter two conditions still make sense. We will use these as the definition of a *separated scheme*.

A second interpretation of the Hausdorff condition is that a sequence should have at most one limit. Again, it is hard to make sense of this literally for schemes, but we can reinterpret it in a way that does make sense. Instead of sequences, we look at maps from open curves into X and stipulate that such a map can be completed in at most one way.

A first candidate for such a definition is that any map $\mathbf{A}^1_k \smallsetminus \{0\} \to X$ can be completed in at most one way to $\mathbf{A}^1_k \to X$. Indeed, if X is separated, this must be true. However, there are a few problems owing to the rigidity of algebraic geometry. There are many different kinds of open arcs, of which the above is just one. In order to get a sufficiently large list, we look at *valuation rings*.

24.1 A criterion for closed subsets

pecialization-again

Exercise 24.3 (Repeat of Exercise 7.27)**.** Show that a closed subset of a scheme is closed under specialization but that a subset closed under specialization is not necessarily closed.

The following theorem says that, Exercise 24.3 notwithstanding, being closed under specialization *is* equivalent to being closed in most situations that arise in practice:

m:closure-criterion

Theorem 24.4 ([Sta15, Tags 00HY and 01K9], [GD67, Proposition (II.7.2.1)])**.** *The image of a quasicompact morphism is closed if and only if it is stable under specialization.*

Proof. Suppose $f : Y \to X$ is quasi-compact and its image is stable under specialization. We would like to show it is closed. It is sufficient to pass to a cover of X by open subsets U and show that each $f(Y) \cap U$ is closed in U. We can therefore assume X is affine, say $X = \operatorname{Spec} A$, and that Y is quasi-compact. Choose a surjection $Z \to Y$ with Z affine (we can do this because Y is quasicompact). Replacing Y by Z we can now assume that both $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine.

Let $\mathfrak{p}$ be a point of X (i.e., a prime ideal of $\operatorname{Spec} A$) in the closure of $f(Y)$. We argue that p is the specilization of a point of $f(Y)$.

Since $\mathfrak{p}$ lies in the closure of $f(Y)$, every open neighborhood of $\mathfrak{p}$ meets $f(Y)$. In particular, if $g \in A \smallsetminus \mathfrak{p}$ then $D_X(g)$ is an open neighborhood of $\mathfrak{p}$ so $D_X(g) \cap f(Y) \neq \varnothing$. Thus $D_Y(g) \neq \varnothing$ so the image of $A \smallsetminus p$ in B does not contain any nilpotent elements. Therefore $B_{\mathfrak{p}}$ is not the zero ring so it has at least one prime ideal. Call it $\mathfrak{q}$. Then $f(\mathfrak{q})$ is a prime ideal of $A_{\mathfrak{p}}$ so it specializes to $\mathfrak{p}$ and lies in the image of f. But $f(Y)$ is closed under specialization, so $\mathfrak{p}$ lies in $f(Y)$, as desired. $\square$

Repeat of Exercise 7.12.

Exercise 24.4.1. Let X be a scheme and Z a closed subset. Let $i : Z \to X$ be the inclusion. Define $\mathscr{A}(U)$ to be the quotient of $\mathcal{O}_X(U)$ by the relation $f \sim g$ if $f(p) = g(p)$ for all $p \in U \cap Z$. Set $\mathcal{O}_Z = i^{-1}\mathscr{A}$. Show that $(Z, \mathcal{O}_Z)$ is a reduced scheme. This is called the *reduced scheme structure* on Z.

Solution. (Direct argument, sketch) Suppose that $U \subset X$ is affine, say $U = \operatorname{Spec} B$. Then $\mathscr{A}(U) = B/I(Z \cap U)$, by definition. Moreover, if $f \in B$ then $\mathscr{A}(D(f)) = B_f/I(Z \cap D(f))$ so we get a map $\mathscr{A}(U) \to \mathscr{A}(D(f))$. Thus $\mathscr{A}$ is a presheaf of rings on the basis of affines under principal inclusions. Moreover, we have $Z = V(J)$ for some ideal J, so $I(Z \cap U) = \sqrt{J}$. Likewise, $Z \cap D(f) = V(JB[f^{-1}])$ so $I(Z \cap D(f)) = \sqrt{JB[f^{-1}]}$. Thus $I(Z \cap D(f)) = I(Z \cap U)B_f$, from which it follows that $\mathscr{A}\big|_U$ agrees with the structure sheaf of $\operatorname{Spec} B/I(Z \cap U)$. Therefore $\mathscr{A}$ extends to a sheaf on X and $\mathscr{A} = i_* i^{-1} \mathscr{A}$. (Warning: There is a very subtle point lurking here concerning the extension to a sheaf. Not every inclusion of an open affine in another open affine is a principal inclusion. This doesn't actually cause a problem, but it requires some thought to see why.) Thus Z has an open cover by affine schemes. □

Solution. (Using a sheaf of ideals) Define a sheaf $\mathscr{I} \subset \mathcal{O}_X$ consisting of all $f \in \mathcal{O}_X$ such that $f(z) = 0$ for all schematic points $z \in Z$. Let $\mathscr{A} = \mathcal{O}_X/\mathscr{I}$ and let $\mathcal{O}_Z = i^{-1}\mathscr{A}$. Then $(Z, \mathcal{O}_Z)$ is a ringed space, so we just need to find an open cover by affine schemes. If $U \subset X$ is affine—say $U = \operatorname{Spec} B$—then $Z \cap U = V(J)$ for some ideal $J \subset B$. Then $\mathscr{A} = (B/\sqrt{J})^{\sim}$ so $Z = \operatorname{Spec}(B/\sqrt{J})$. □

n-lifting-criterion

Corollary 24.4.2. *A quasicompact morphism of schemes $f : X \to Y$ is closed if and only if specializations lift along f.*

Proof. Assume first that specializations lift. It is sufficient to prove that the maps $f^{-1}U \to U$ are closed for all U in an open cover of Y. We can therefore assume Y is quasicompact (even affine). Then X is also quasicompact. Let $Z \subset X$ be a closed subset. Then we may give Z a scheme structure (for example, the reduced scheme structure). Since Z is closed in X, specializations lift along $Z \to Y$. Therefore the image of Z is closed under specialization, so by Theorem 24.4, the image of Z is closed.

Conversely, suppose that f is closed and $y \rightsquigarrow y'$ in Y and $y = f(x)$. Let Z be the closure of $\{x\}$ in X. Then $f(Z)$ is a closed subset of Y so $y' \in f(Z)$. Therefore there is some x' in the closure of $\{x\}$ (i.e., some $x \rightsquigarrow x'$) with $f(x') = y'$. □

24.2 Valuation rings

Reading 24.5. [AM69, pp. 65–67], [GD67, §II.7.1]

Definition 24.6. A *valuation ring* is an integral domain A such that for all nonzero x in the field of fractions of A, either $x \in A$ or $x^{-1} \in A$.

Exercise 24.7. (i) Show that $\mathbf{Z}_{(p)}$ is a valuation ring.

(ii) Show that $k[t]_{\mathfrak{p}}$ is a valuation ring when k is a field and $\mathfrak{p}$ is any ideal other than the zero ideal.

(iii) Show that $k[[t]]$ is a valuation ring when k is a field.

(iv) Let k be a field and let $A = \bigcup_{n \to \infty} k[[t^{1/n}]]$ be the ring of *Puiseux series*. Show that A is a valuation ring.

(v) Give an example of a local ring that is not a valuation ring.

These are a few basic facts about valuation rings. They aren't essential but they may help build intuition.

Exercise 24.8.

(i) If A is a valuation ring then the fractional ideals of A are totally ordered under inclusion.

Solution. Let I and J be fractional ideals of A. Assume $x \in I \smallsetminus J$. Choose any $y \in J$. Then $xy^{-1} \in A$ or $x^{-1}y \in A$. But the former cannot happen because $x \notin J$. Therefore $x^{-1}y \in A$ so $y = x^{-1}yx \in AI = I$. Thus $J \subset I$. □

(ii) A valuation ring is a local ring.

Solution. The proper ideals of A form an ascending chain. The union is therefore an ideal. □

(iii) If A is a valuation ring then all finitely generated ideals of A are principal. (Note that this does not mean A is a principal ideal domain!)

Solution. Suppose I has n generators $x_1, \dots, x_n$. Then either $x_n x_{n-1}^{-1}$ or $x_{n-1}x_n^{-1}$ is in A so either x_n or x_{n-1} is redundant. By induction, we can get down to one generator. □

(iv) If A is a valuation ring then the *nonzero* fractional ideals[2] of A form a group under multiplication.

Solution. Every fractional ideal is principal so the group is K^*/A^* where K is the field of fractions. □

(v) [AM69, Chapter 5, Exercise 30] Let K be the field of fractions of a valuation ring A. Let $v(x) = Ax$ for any $x \in K^*$. This gives a homomorphism from K^* into the group of nonzero fractional ideals of K. Show that $v(x+y) \geq \min\{v(x), v(y)\}$ where the nonzero fractional ideals are ordered by inclusion. Thus v is a *valuation*.

ons-valuation-rings

Theorem 24.9. *Let $x \in X(K)$ be a K-point of X and suppose that $x \rightsquigarrow y$.*[3] *Then there is a valuation ring R with field of fractions K and a map $\operatorname{Spec} R \to X$ sending the closed point of $\operatorname{Spec} R$ to y and restricting to x on the generic point.*

Proof. Let A be the quotient of $\mathcal{O}_{X,y}$ by the prime ideal corresponding to x. This is a local ring with closed point y and generic point x. We are given a map $\mathcal{O}_{X,y} \to A \to K$ by $x \in X(K)$. To get a valuation ring, choose R to be maximal among local rings dominating A (which exists by Zorn's lemma). (An ascending union of such local rings is still a local ring dominating A, because an ascending union of fields is a field.)

I claim R is a valuation ring. Indeed, suppose that $t \in K$. We show that either t or t^{-1} is in R. Assume that $t \notin R$. Consider $B = R[t] \subset K$. If $mB \neq B$ then choose a maximal ideal of B containing mB and localize, contradicting the maximality. Thus we must have $mB = B$. But then there is some expression $\sum_{i=0}^n a_i t^i = 1$ with $a_i \in m$. Rewrite this as $\sum_{i=1}^n a_i t^{-n+i} = (1-a_0)t^{-n}$. Note that $1 - a_0$ is a unit, so that t^{-1} is integral over R. Thus $\operatorname{Spec} R[t^{-1}] \to \operatorname{Spec} R$ is surjective, so in particular, $R[t^{-1}]$ is not a field. By the going up theorem, m lifts to a maximal ideal m' of $R[t^{-1}]$ and $R[t^{-1}]_{m'}$ is a local ring dominating R. By the maximality of A, we get $t^{-1} \in R$. □

Thus the inclusion of a valuation ring in its field of fractions is the 'universal specialization'.

[2]A fractional ideal is a *finitely generated* submodule of the field of fractions.

[3]Properly speaking, it is the image of $\operatorname{Spec} K$ under x that specializes to y.

24.3 Separatedness

Definition 24.10 (Separatedness)**.** A morphism of schemes $\pi : X \to Y$ is *separated* if, for any $f, g : Z \to X$ such that $\pi f = \pi g$, the equalizer of f and g is a closed subscheme of Z.

Exercise 24.11. Let k be a field and let $X = \mathbf{A}^1_k \cup_{\mathbf{A}^1_k \smallsetminus \{0\}} \mathbf{A}^1_k$ be the affine line with its origin doubled. Show that $X \to \operatorname{Spec} \mathbf{k}$ is not separated.

Solution. The equalizer of the two inclusions of $\mathbf{A}^1$ is $\mathbf{A}^1 \smallsetminus \{0\}$, which is not closed in $\mathbf{A}^1$. □

Important and easy; correction: 'closed' corrected to 'separated' (thanks to Shawn)

Exercise 24.12. Show that $\pi : X \to Y$ is separated if and only if $\delta = (\mathrm{id}_X, \mathrm{id}_X) : X \to X \times_Y X$ is a closed embedding. This is how separatedness is usually defined.

Exercise 24.13. Prove that a topological space is Hausdorff if and only if its diagonal is a closed embedding.

Important and easy

Exercise 24.14. Prove that every affine scheme is separated.

Solution. This translates into the fact that for any homomorphism of commutative rings $A \to B$, the map $B \otimes_A B \to B$ sending $b_1 \otimes b_2$ to $b_1 b_2$ is a surjection. This is obvious because it has a section $b \mapsto b \otimes 1$. □

closed-locally-closed

Not directly related to this section but useful in the next exercise.

Exercise 24.15. A locally closed embedding of schemes with closed image is closed. (Hint: Use the fact that an embedding can be shown to be a closed embedding on an open cover of the codomain.)

Solution. Let $f : X \to Y$ be a locally closed embedding with closed image. Then there is some open $U \subset Y$ such that f factors through U as a closed embedding. But $V = Y \smallsetminus f(X)$ is an open subset of X such that $f^{-1}V = \varnothing$. Therefore U and V give an open cover of Y such that $f^{-1}U \to U$ and $f^{-1}V \to V$ are closed embeddings. It follows that f is a closed embedding. □

separated-criterion

Exercise 24.16. A morphism $\pi : X \to Y$ is separated if and only if it is quasiseparated and its diagonal is closed under specialization. (Hint: Make use of Exercise 24.15.)

Solution. Suppose π is separated. If $Z \rightrightarrows X$ is a pair of morphisms with Z quasicompact then the equalizer is closed in Z. In particular it is quasicompact, since a closed subscheme of a quasicompact scheme is quasicompact.

Suppose π is quasiseparated and its diagonal is closed under specialization. Separatedness gives that the diagonal is quasicompact, so by Theorem 24.4, the diagonal of π is closed. Therefore the diagonal is a locally closed embedding with closed image. □

This gives an intuitive picture of specialization. A specialization in $X \times_Y X$ of a point x in the diagonal yields a pair of specializations $x \rightsquigarrow x_1'$ and $x \rightsquigarrow x_2'$ in X, with both projecting to the same specialization $y \rightsquigarrow y'$ of Y. If this specialization lifts then $x_1' = x_2'$.

valuative-separated

Theorem 24.17 (Valuative criterion for separatedness)**.** *A quasiseparated morphism of schemes $f : X \to Y$ is separated if and only if whenever R is a valuation ring with field of fractions K, a diagram* (24.1) *admits* at most one *lift.*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{24.1}$$

eqn:15

Proof. Suppose we had two lifts and take the equalizer. This is a closed subscheme of $\operatorname{Spec} R$ containing $\operatorname{Spec} K$. But $\operatorname{Spec} K$ is dense in $\operatorname{Spec} R$ and $\operatorname{Spec} R$ is reduced (R is an integral domain) so the equalizer must be $\operatorname{Spec} R$.

Conversely, suppose that the lifting criterion holds. Then every diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "\delta"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & X \times_Y X \end{tikzcd}$$

admits a (unique) lift. In particular, if we choose $x \in \delta(X)$ and y a specialization of x in $X \times_Y X$ then we can choose a valuation ring $\operatorname{Spec} R \to X \times_Y X$ with its generic point mapping to x and its special point mapping to y. The lifting criterion then guarantees that the specialization lifts. Combined with quasiseparatedness, this means that f is separated (Exercise 24.16). □

24.4 Properness

Definition 24.18. A morphism of schemes $f : X \to Y$ is said to be *universally closed* if, for every Y-scheme Y' the morphism $f' : X' \to Y'$ induced by base change is closed.

Exercise 24.19. Let k be a field. Show that $\mathbf{A}^1_k \to \operatorname{Spec} k$ is closed but not universally closed.

Definition 24.20 (Properness)**.** A morphism of schemes $f : X \to Y$ is *proper* if it is separated, of finite type, and universally closed.

Exercise 24.21. Suppose that $f : X \to Y$ and $g : Y \to Z$ are morphisms of schemes. Show the following:

(i) Show that all closed embeddings are proper.

Solution. The diagonal of an embedding is an isomorphism, so closed embeddings are separated. Closed embeddings are of finite type since a quotient of a commutative ring is always an algebraic of finite type. Surjections are preserved under tensor product, so pullback of a closed embedding is a closed embedding. In particular, closed embeddings are universally closed. □

(ii) If f and g are both proper the gf is proper.

(iii) [Har77, Exercise II.4.8] If g is separated and gf is proper then f is proper.

Solution. We have two maps $X \times_Z Y \to Y$: one is the second projection p_2 and the other is fp_1. We have $gp_2 = gfp_1$ by definition of $X \times_Z Y$. Therefore the equalizer of fp_1 and p_2 is closed in $X \times_Z Y$. On the other hand, the equalizer is simply X (it is the collection of pairs (x, y) such that $f(x) = y$). As $g : Y \to Z$ is separated, $\gamma = (\mathrm{id}_X, f) : X \to X \times_Z Y$ is a closed embedding, hence is proper. Likewise, $p_2 : X \times_Z Y \to Y$ is proper because it is the base change of gf. Therefore $f = p_2 \circ \gamma$ is proper because it is the composition of two proper maps. □

Exercise 24.22. Show that $X \to Y$ is separated if and only if $X \to X \times_Y X$ is proper.

-universally-closed **Theorem 24.23.** *Suppose that $f : X \to Y$ is quasicompact and quasiseparated. Then f is separated and universally closed if and only if every diagram (24.2) admits a* unique *lift.*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{24.2}$$

eqn:16

Proof. This paragraph follows [GD67, Proposition (II.7.3.1)]. Suppose first that f is separated and universally closed. Make a base change via $\operatorname{Spec} R \to Y$. We can assume without loss of generality that $Y = \operatorname{Spec} R$. We can also replace X by the closure of the image of $\operatorname{Spec} K$. Specializations lift via f, and every point of Y is a specialization of the generic point y, so $f^{-1}y$ consists of a single point $x \in X$. Moreover, we have $K = \mathbf{k}(y) \to \mathbf{k}(x) \to K$, which implies that $\mathbf{k}(y) = \mathbf{k}(x) = K$. Now, if $x' \in X$ is any point then x' is a specialization of x. Suppose that x' is such a point and $f(x') = y'$. Then we get $\mathcal{O}_{Y,y'} \subset \mathcal{O}_{X,x'} \subsetneq K$ with $\mathfrak{m}_{x'} \cap \mathcal{O}_{Y,y'} = \mathfrak{m}_{y'}$. That is, $\mathcal{O}_{X,x'}$ dominates $\mathcal{O}_{Y,y'}$. But $\mathcal{O}_{Y,y'}$ is a localization of R, hence is maximal with respect to domination, so $\mathcal{O}_{X,x'} = \mathcal{O}_{Y,y'}$. It follows that $f^* : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is an isomorphism.

Finally, we check that f is a bijection. Suppose that x' and x'' were points of $f^{-1}y'$. Then, as $\mathcal{O}_{X,x'} = \mathcal{O}_{X,x''} = \mathcal{O}_{Y,y}$ we would get two lifts of $\operatorname{Spec} \mathcal{O}_{Y,y} \to Y$ agreeing at the generic point. But by the valuative criterion for separatedness, they must be the same. Now f is a closed bijection, hence a homemorphism, and $f^* : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is an isomorphism. Hence f is an isomorphism.

Now suppose that the lifting criterion holds. By the valuative criterion for separatedness (Theorem 24.17) we know that f is separated. Now we show f is closed. (It will follow that f is universally closed because the lifting criterion is stable under base change.) We use Corollary 24.4.2. Suppose $x \in X$ and $f(x) \rightsquigarrow y$ is a specialization in Y. By Theorem 24.9 there is a valuation ring R with field of fractions K and a map $\operatorname{Spec} R \to Y$ such that the induced map $\operatorname{Spec} K \to Y$ coincides with the composition $\operatorname{Spec} K \xrightarrow{x} X \xrightarrow{f} Y$. The lifting criterion guarantees that there is an $x' \in X$ with $x \rightsquigarrow x'$ and $f(x') = y$, exactly as required. $\square$

Corollary 24.23.1. *Suppose that $f : X \to Y$ is of finite type and is quasiseparated. Then f is proper if and only if f satisfies the right lifting property with respect to* $\operatorname{Spec} K \subset \operatorname{Spec} R$ *for every valuation ring R with field of fractions K.*

24.5 Projective schemes

Definition 24.24. A *projective scheme* is a scheme that can be embedded inside $\mathbf{P}^N$, for some integer N, as a closed subscheme.

m:projective-proper **Theorem 24.25.** *Projective schemes are proper.*

ex:nakayama **Exercise 24.26** (Nakayama's lemma)**.** Let M be a finitely generated module over a local ring A with maximal ideal $\mathfrak{p}$. Show that $\mathfrak{p}M = M$ if and only if $M = 0$.

Solution. We show by induction that M can be generated by 0 elements. If M can be generated by $x_1, \ldots, x_n$ then we can write

$$x_n = \sum_{i=1}^{n} a_i x_i$$

which yields

$$(1 - a_n)x_n = -\sum_{i=1}^{n-1} a_i x_i$$

and $1 - a_n$ is a unit so x_n is redundant. By induction, all the generators are redundant. □

ex:closed-support

Exercise 24.27 (Support of a finitely generated module is closed)**.** Show that the support of a finitely generated module M over an affine scheme A is closed in $\operatorname{Spec} A$. (Hint: Let I be the annihilator ideal of M and show that $M \otimes_A k(p) = 0$ if and only if $p \in D(I)$.)

Solution. Suppose that $p \in D(I)$. Then $Ik(p) = k(p)$ so that $I(M \otimes_A k(p)) = (IM) \otimes_A k(p) = 0$ and $I(M \otimes_A k(p)) = M \otimes_A Ik(p) = M \otimes_A I$.

Conversely, suppose that $M \otimes_A k(p) = 0$. Then $M_p/pM_p = 0$ so $M_p = pM_p$ so $M_p = 0$ by Nakayama's lemma. Then there is some $f \in A \smallsetminus p$ such that $fM = 0$. Then p does not contain I so $p \in D(I)$. □

Elimination theory proof [Vak14, Theorem 7.4.7] of Theorem 24.25. Let S be the homogeneous coordinate ring of a projective scheme Z with $S_0 = k$ a field. Let $\mathfrak{m}$ be the irrelevant ideal of S. Then Z is empty if and only if $\mathfrak{m}$ is nilpotent.

More generally, if S is a homogeneous coordinate ring with $S_0 = A$ and p is a point of $\operatorname{Spec} A$ corresponding to a homomorphism $A \to k$ then $\pi^{-1}p = \varnothing$ if and only if $\mathfrak{m}(S \otimes_A k) \subset S \otimes_A k$ is nilpotent.

Now, set $M_n = \mathfrak{m}^n/\mathfrak{m}^{n+1}$. This is a finitely generated A-module and $\pi(Z) \subset \operatorname{Spec} A$ is

$$\bigcup_{n=1}^{\infty} \{p \in \operatorname{Spec} A \mid M_n \otimes_A k = 0\} = \bigcup_{n=1}^{\infty} D(\operatorname{Ann} M_n)$$

and this is open. □

Valuative criterion proof [GD67, Remarque (II.7.3.9) (ii)] of Theorem 24.25. It is sufficient to show that $\mathbf{P}^N$ is proper since a composition of proper morphisms is proper.

Suppose that R is a valuation ring with field of fractions K and $(x_0, \ldots, x_N) \in \mathbf{P}^N(K)$. Choose an index i such that $v(x_i)$ is minimal. Then $(x_i^{-1}x_0, \ldots, x_i^{-1}x_N) = (x_0, \ldots, x_N)$ in $\mathbf{P}^N(K)$. On the other hand, $v(x_i^{-1}x_j) \geq 0$ for all j so $x_i^{-1}x_j \in R$. That is $(x_i^{-1}x_0, \ldots, x_i^{-1}x_N) \in \mathbf{P}^N(R)$.

To see that this is unique, suppose we had two elements $(y_0, \ldots, y_N)$ and $(z_0, \ldots, z_N)$ in $\mathbf{P}^N(R)$ representing the same element of $\mathbf{P}^N(K)$. Then there is some $\lambda \in K^*$ such that $(y_0, \ldots, y_N) = (\lambda z_0, \ldots, \lambda z_N)$. By definition of $\mathbf{P}^N(R)$, there is at least one index i such that $v(y_i) = 0$. Then if $v(\lambda) < 0$ we have $v(z_i) < 0$, and if $v(\lambda) > 0$ we have $v(z_j) > 0$ for all j. Both are impossible so we have $v(\lambda) = 0$ so $\lambda \in R^*$ and $(y_0, \ldots, y_N)$ and $(z_0, \ldots, z_N)$ represent the same point of $\mathbf{P}^N(R)$. □

Exercise 24.28. Let E be a vector bundle over a scheme Y. Let $\mathbf{P}(E) : (\mathbf{Sch}/Y)^\circ \to \mathbf{Sets}$ be functor sending an Y-scheme X to the set of closed embeddings $L \to E\big|_X$ in which L is a line bundle over X. Verify that $\mathbf{P}(E)$ is representable by a scheme. (Hint: Cover X by open sets U where $E\big|_U$ is trivial and use $\mathbf{P}^N \times U$.)

Definition 24.29 ([GD67, Proposition (II.5.5.1) and Définition (II.5.5.2)])**.** A morphism of schemes $f : X \to Y$ is said to be *projective* if there is a closed embedding $X \to \mathbf{P}(E)$ over Y, for some vector bundle E over Y.

Exercise 24.30. Prove that projective morphisms are proper.

25 Separated and proper morphisms II

Chapter 8

Étale morphisms

26 Separated and proper morphisms III

27 Étale morphisms I

Recall that a morphism of topological spaces $X \to Y$ is said to be *étale* if it is a local homeomorphism. This definition does not work well for schemes, where the Zariski topology is too coarse to detect maps that should be considered local homeomorphisms.

If you have studied differential geometry, this exercise should be essentially immediate. If you have not studied differential geometry, there is no reason to do this exercise.

Exercise 27.1. Show that a morphism of differentiable manifolds $f : X \to Y$ is a local diffeomorphism near a point x if and only if the map $df : T_x X \to T_{f(x)} Y$ is an isomorphism. (Hint: Inverse function theorem.)

Solution. Suppose f is a local diffeomorphism. We can assume without loss of generality that f is a diffeomorphism by replacing X and Y with open neighborhoods of x and $f(x)$, respectively. Then df certainly induces an isomorphism at x.

Conversely, suppose df is an isomorphism at x. We can assume that X and Y are both open subsets of $\mathbf{R}^n$ for some n (necessarily the same since their tangent spaces have the same dimension). Then by the inverse function theorem f possesses an inverse in a neighborhood of $f(x)$. $\square$

Exercise 27.2. (i) Show that the map $\mathbf{C}^* \to \mathbf{C}^*$ sending z to z^n is a local homeomorphism for all nonzero $n \in \mathbf{Z}$.

(ii) Show that the map $\operatorname{Spec} \mathbf{C}[t, t^{-1}] \to \operatorname{Spec} \mathbf{C}[s, s^{-1}]$ sending s to t^n is not a local homeomorphism for any n except ± 1. (Hint: Consider the map on generic points.)

28 Étale morphisms II

Instead of a topological characterization of étale maps, we will use a geometric one. In a sense, a map of topological spaces is a local homeomorphism if its source and target are *locally indistinguishable*. Taking this as our cue, we call a map of schemes étale if its source and target are *infinitesimally indistinguishable*.

Exercise 28.1. Let $i : Z \to Z'$ be a closed embedding. Let I be the kernel of $\mathcal{O}_{Z'} \to i_*\mathcal{O}_Z$ (as a homomorphism of $\mathcal{O}_{Z'}$-modules). Show that I is a quasicoherent sheaf.

Definition 28.2. A morphism of schemes $Z \to Z'$ is said to be an *infinitesimal extension* or a *nilpotent thickening* or a *nilpotent extension* if it is a closed embedding and the sheaf of ideals $I_{Z/Z'}$ is nilpotent.

If $I^2_{Z/Z'} = 0$ then $Z \subset Z'$ is said to be a *square-zero extension* or *square-zero thickening*.

Exercise 28.3. Show that a closed embedding $Z \to Z'$ is an infinitesimal extension if and only if there is a positive integer n and local charts $\operatorname{Spec} A \to \operatorname{Spec} A'$ for $Z \to Z'$ such that, when I is defined to be $\ker(A' \to A)$, we have $I^n = 0$.

Exercise 28.4. Show that every nilpotent thickening can be factored into a sequence of square-zero thickenings. (Hint: Take the closed subschemes defined by $I^n_{Z/Z'}$.)

Exercise 28.5. Show that if $Z \subset Z'$ is an infintesimal thickening then the inclusion of topological spaces $|Z| \subset |Z'|$ is a bijection.

def:etale

Definition 28.6. A morphism of schemes $f : X \to Y$ is said to be *formally étale* if, whenever $Z \subset Z'$ is an infinitesimal thickening, any diagram of solid arrows (28.1) can be completed by a dashed arrow in a unique way.

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Z' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{28.1}$$

eqn:2

If f is also locally of finite presentation then we say f is *étale*.

Exercise 28.7. Show that all open embeddings are étale. In a sense this shows that 'locally indistinguishable' implies 'infintiesimally indistinguishable'. (It is possible to do this directly, but you might find this exercise easier using the results from the next one.)

Correction: "open cover" in the first part changed to "basis of open subsets". Thanks to Paul.

Exercise 28.8.

(i) Show that to prove a diagram (28.1) has a unique lift, it is sufficient produce unique lifts over a basis of open subsets of Z'. (Hint: Use the fact that X and Y are Zariski sheaves.)

(ii) Show that we would have arrived at an equivalent definition of étale morphisms if we had only required liftings with respect to infinitesimal extensions of *affine schemes*.

(iii) Show that we would have arrived at an equivalent definition of étale morphisms if we had only required liftings with respect to square-zero extensions of affine schemes.

Exercise 28.9. (i) Show that the map

$$\operatorname{Spec} \mathbf{C}[t, t^{-1}] \to \operatorname{Spec} \mathbf{C}[s, s^{-1}]$$

sending s to t^n is étale for all $n \neq 0$.

Solution. Let $A' \to A$ be a square-zero extension with ideal J. Suppose that $\tau \in A$ is an A-point of $\operatorname{Spec} \mathbf{C}[t, t^{-1}]$. This means that $\tau \in A^*$. And suppose that $\sigma' \in A'^*$ lifts τ^n. Choose any $\alpha \in A'$ that maps to $\tau \in A$. Then $\alpha^n - \sigma' \in J$. Set $\delta = \frac{1}{n}\alpha^{-(n-1)}(\alpha^n - \sigma') \in J$. Let $\tau' = \alpha - \delta$. We get

$$\tau'^n = \alpha^n - n\alpha^{n-1}\delta = \sigma'$$

so τ' gives the lift. □

(ii) Suppose k is a field of characteristic p. For which values of n is the map

$$\operatorname{Spec} k[t, t^{-1}] \to \operatorname{Spec} k[s, s^{-1}]$$

étale?

Solution. The same argument as above works when p does not divide n. We show the map is not étale when $p|n$. Suppose p^k divides n exactly. Let K be an algebraic closure of $k(x)$, let $Z = \operatorname{Spec} K[\epsilon]/(\epsilon^{p^k})$, and let $Z' = \operatorname{Spec} K[\epsilon]/(\epsilon^{p^{k+1}})$. We have a map $\mathbf{C}[t, t^{-1}] \to K[\epsilon]/(\epsilon^{p^k})$ sending x to $x + \epsilon$. Then

$$s \mapsto (x+\epsilon)^n = \left((x+\epsilon)^{p^k}\right)^{n/p^k} = (x^{p^k} + \epsilon^{p^k})^{n/p^k} = x^n.$$

Therefore we have a lift by $s \mapsto x^n \in K[\epsilon]/(\epsilon^{p^{k+1}})$. But if we try to find a lift of t here, it must have the form $t \mapsto x + \epsilon + \lambda\epsilon^{p^k}$, from which we get

$$t^n \mapsto \left((x + \epsilon + \lambda\epsilon^{p^k})^{p^k}\right)^{n/p^k} = (x^{p^k} + \epsilon^{p^k} + \epsilon^{p^{2k}})^{n/k}$$

Now $k \geq 1$ so $2k \geq k+1$ so $\epsilon^{p^{2k}} = 0$ in $K[\epsilon]/(\epsilon^{p^{k+1}})$. Therefore,

$$t^n \mapsto (x^p + \epsilon^{p^k})^{n/k} = x^n + (n/k)x^{p(n/k-1)}\epsilon^{p^k} + \cdots$$

which does not coincide with x^n. (No choice of λ can make it work.) □

Chapter 9

Smooth morphisms

29 Étale morphisms III

29.1 The module of relative differentials

Definition 29.1. Let A be a commutative ring, let B be a commutative A-algebra, and let J be a B-module. An *A-derivation from B into J* is a function $\delta : B \to J$ such that

Der1 $\delta(A) = 0$ and

Der2 $\delta(xy) = x\delta(y) + y\delta(x)$ for all $x, y \in B$.

The set of A-derivations from B into J is denoted $\mathrm{Der}_A(B, J)$.

Exercise 29.2. Show that $\mathrm{Der}_A(B, J)$ is naturally equipped with the structure of an A-module via $(a\delta)(x) = a\delta(x)$.

Exercise 29.3. Let $B + \epsilon J$ be the commutative ring whose elements are symbols $x + \epsilon y$ with $x \in B$ and $y \in J$ with the addition rules

$$(x + \epsilon y) + (x' + \epsilon y') = (x + x') + \epsilon(y + y')$$
$$(x + \epsilon y)(x' + \epsilon y') = xx' + \epsilon(xy' + x'y).$$

(i) Show that there is a homomorphism $p : B + \epsilon J \to B$ defined by $p(x + \epsilon y) = x$.

(ii) Show that there is a homomorphism $i : B \to B + \epsilon J$ defined by $i(x) = x + \epsilon 0$.

(iii) Suppose that $f : B \to B + \epsilon J$ is an A-algebra homomorphism such that $pf = \mathrm{id}_B$. Show that $f - i$ factors through $\epsilon J \subset B + \epsilon J$ and that regarded as a map $B \to J$ it is a derivation.

(iv) Suppose that $\delta : B \to J$ is a derivation. Show that $\mathrm{id}_B + \epsilon\delta : B \to B + \epsilon J$ is a homomorphism of A-algebras.

(v) Conclude that $\mathrm{Der}_A(B, J) = \mathrm{Hom}_A^B(B, B + \epsilon J)$ (where it's your job to figure out what the notation Hom_A^B means).

ex:omega

Exercise 29.4. Show that there is a universal B-module $\Omega_{B/A}$ and A-derivation $d : B \to \Omega_{B/A}$. (In other words, show that the functor $J \mapsto \mathrm{Der}_A(B, J)$ is representable by a B-module $\Omega_{B/A}$.)

ative-differentials

Definition 29.5. The universal A-derivation $B \to \Omega_{B/A}$ constructed in Exercise 29.4 is called the *module of relative differentials of B over A* or the *module of relative Kähler differentials*.

Exercise 29.6. Compute $\Omega_{B/A}$ when $B = A[x_1, \ldots, x_n]$ is a polynomial ring.

mega-exact-sequence

Exercise 29.7 ([Har77, Proposition II.8.1], [Vak14, Theorem 21.2.9])**.** Suppose $A \to B \to C$ are homomorphisms of commutative rings.

(i) Show that for any C-module J there is a natural exact sequence of C-modules:

$$0 \to \mathrm{Der}_B(C, J) \to \mathrm{Der}_A(C, J) \to \mathrm{Der}_A(B, J)$$

Typo corrected in the exact sequence. Tensor product is over B not over A. Thanks Ryan.

(ii) Deduce an exact sequence

$$C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

(iii) Find an example to show that the sequence can't be completed with a $0 \to C \otimes_A \Omega_{B/A}$ on the left. (Hint: Consider $A = k$ a field, $B = k[x]/(x^2)$, and $C = B/xB \simeq k$.)

Exercise 29.8. Suppose that $B \to C$ is an *epimorphism* of A-algebras.[1]

(i) Show that $\Omega_{C/B} = 0$. (This isn't used in the rest of the exercise.)

(ii) Let I be the kernel of $B \to C$. For any C-module J, construct an exact sequence:

$$0 \to \mathrm{Der}_A(C, J) \to \mathrm{Der}_A(B, J) \to \mathrm{Hom}_{B\text{-}\mathbf{Mod}}(I, J)$$

(iii) Conclude that there is an exact sequence of C-modules:

$$I/I^2 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to 0$$

(Hint: $I/I^2 \simeq C \otimes_B I$. Why?)

(iv) Show by example that the sequence can't be completed by $0 \to I/I^2$ on the left and remain exact. (Hint: Consider $A = k$ a field, $B = k[x]$, and $C = B/x^2B = k[x]/(x^2)$.)

Solution. Follow the hint: $x^3 \in I/I^2$ and $d(x^3) = 3x^2dx = 0$. □

[1] This means that $\mathrm{Hom}_{A\text{-}\mathbf{Alg}}(C, D) \to \mathrm{Hom}_{A\text{-}\mathbf{Alg}}(B, D)$ is injective for any A-algebra D. This includes surjections and localizations.

30 Étale morphisms IV

30.1 Extensions of algebras

Extensions of algebras mean square-zero extensions. Thanks Ryan.

Definition 30.1. Let A be a commutative ring, B a A-algebra, and J a B-module. An A-algebra extension of B by J is a surjective homomorphism *with square-zero kernel* of A-algebras $B' \to B$ and an identification of the kernel of this surjection with J. A morphism from an extension B' to an extension B'' is a homomorphism of A-algebras that induces the identity on J and induces the identity modulo J. In other words, it is a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & J & \longrightarrow & B'' & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

The isomorphism classes of A-algebra extensions of B by J are denoted $\mathrm{Exal}_A(B, J)$.

Exercise 30.2. (i) Show that the automorphism group of $B + \epsilon J$ as an A-algebra extension of B is $\mathrm{Der}_A(B, J)$. Conclude that A-algebra extensions can have nonzero automorphisms.

(ii) Show that every morphism of A-algebra extensions is an isomorphism.

(iii) Construct a bijection between the isomorphisms $B' \simeq B + \epsilon J$ and the A-algebra sections of $B' \to B$.

Exercise 30.3. Let $q : \widetilde{A} \to B$ be a surjection.

(i) Find an identification between $\mathrm{Exal}_{\widetilde{A}}(B, J)$ and

$$\mathrm{Hom}_{\widetilde{A}\textbf{-Alg}}(I_{B/\widetilde{A}}, J) = \mathrm{Hom}_{B\textbf{-Mod}}(B \otimes_{\widetilde{A}} I_{B/\widetilde{A}}, J).$$

Solution. Given $\varphi : \widetilde{A} \to B'$ over $\widetilde{A} \to B$ we must have $\varphi(I_{B/\widetilde{A}}) \subset J$. This gives a map $I_{B/\widetilde{A}} \to J$.

Conversely, given a map $\psi : I_{B/\widetilde{A}} \to J$, we can push out the exact sequence

$$0 \to I_{B/\widetilde{A}} \to \widetilde{A} \to B \to 0$$

to get

$$0 \to J \to B' \to B \to 0.$$

An explicit construction here is to take $\widetilde{A} + \epsilon J$ modulo the ideal generated by all $x - \epsilon\psi(x)$ for $x \in I_{B/\widetilde{A}}$. This shows in particular that B' is a ring. □

(ii) Show that under this identification, the zero element corresponds to $B' = B + \epsilon J$ with the $\widetilde{A}$-algebra structure coming from the homomorphism $q + 0\epsilon$. Show that, up to isomorphism, this is the only $\widetilde{A}$-algebra extension $B' \to B$ that has a section by a $\widetilde{A}$-algebra homomorphism.

Solution. From the explicit construction of B' above, the zero homomorphism corresponds to $\widetilde{A} + \epsilon J / I_{B/\widetilde{A}} + 0\epsilon) = B + \epsilon J$. (Thanks to Matt Grimes for pointing out this simple proof.) □

30.2 An algebraic characterization of étale morphisms

Definition 30.4. Suppose that B is an A-algebra. Let $\widetilde{A} \to B$ be a surjection of A-algebras, where $\widetilde{A}$ is a polynomial ring over A. The *(truncated) cotangent complex* of B over A is the 2-term complex (with respect to this presentation) is the complex

$$B \otimes_{\widetilde{A}} I_{B/\widetilde{A}} \xrightarrow{d} B \otimes_{\widetilde{A}} \Omega_{\widetilde{A}/A}.$$

The map sends $b \otimes f$ to $b \otimes df$. The truncated cotangent complex is denoted $\tau_{\geq -1}\mathbf{L}_{B/A}$.

Exercise 30.5. Show that, up to quasi-isomorphism, $\tau_{\geq -1}\mathbf{L}_{B/A}$ is independent of $\widetilde{A}$.

Theorem 30.6. *A map of affine schemes* $\operatorname{Spec} B \to \operatorname{Spec} A$ *is étale if and only if*

$$d : B \otimes_{\widetilde{A}} I_{B/\widetilde{A}} \to B \otimes_{\widetilde{A}} \Omega_{\widetilde{A}/A}$$

is an isomorphism.

Consider an extension problem in which C' is a square-zero extension of C by the ideal J:

$$\begin{tikzcd} C & B \arrow[l] \arrow[dl, dashed] \\ C' \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd} \tag{30.1}$$

eqn:4

Exercise 30.7. Show that solving the lifting problem (30.1) is equivalent to solving the lifting problem below, in which $B' = C' \times_C B$:

$$\begin{tikzcd} B \arrow[r, equal] & B \arrow[dl, dashed] \\ B' \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd} \tag{30.2}$$

eqn:3

Exercise 30.8. Show that B is formally étale over A if and only if $\operatorname{Der}_A(B, J) = \operatorname{Exal}_A(B, J) = 0$.

Solution. We can identify $\operatorname{Der}_A(B, J)$ with lifts of this diagram:

$$\begin{tikzcd} B \arrow[r, equal] & B \arrow[dl, dashed] \\ B + \epsilon J \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd} \tag{30.3}$$

eqn:19

If B is étale over A there is a unique lift, corresponding to the zero derivation.

If B' is an A-algebra extension of B by J, form the diagram

$$\begin{tikzcd} B \arrow[r, equal] & B \arrow[dl, dashed] \\ B' \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd} \tag{30.4}$$

eqn:18

A lift exists so the extension $B' \to B$ is split. Thus it is the zero element of $\operatorname{Exal}_A(B, J)$.

Conversely, if $\operatorname{Der}_A(B, J) = \operatorname{Exal}_A(B, J) = 0$, consider a lifting problem (30.4). Since $\operatorname{Exal}_A(B, J) = 0$, we know $B' = B + \epsilon J$, so this is really a lifting problem (30.3). But then lifts are in bijection with $\operatorname{Der}_A(B, J)$. $\square$

Two typos corrected here. The target of the map is $\mathrm{Exal}_{\widetilde{A}}(B, J)$ and the ideal is $I_{B/\widetilde{A}}$. Thanks to Ryan for catching them.

Exercise 30.9. Let $\widetilde{A} \to B$ be any surjection. Construct a map

$$\mathrm{Der}_A(\widetilde{A}, J) \to \mathrm{Exal}_{\widetilde{A}}(B, J)$$

and identify it with the map

$$\mathrm{Hom}_{B\text{-}\mathbf{Mod}}(B \otimes_{\widetilde{A}} \Omega_{\widetilde{A}/A}, J) \to \mathrm{Hom}_{B\text{-}\mathbf{Mod}}(B \otimes_{\widetilde{A}} I_{B/\widetilde{A}}, J). \tag{30.5}$$

eqn:6

Exercise 30.10. (i) Suppose that $\widetilde{A} \to B$ is a surjection of A-algebras. Construct a commutative diagram in which the long row is exact and the morphism in the second row is induced by $d : B \otimes_{\widetilde{A}} I_{B/\widetilde{A}} \to B \otimes_{\widetilde{A}} \Omega_{\widetilde{A}/A}$:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Der}_A(B, J) \arrow[r] & \mathrm{Der}_A(\widetilde{A}, J) \arrow[r] \arrow[d, "\wr"'] & \mathrm{Exal}_{\widetilde{A}}(B, J) \arrow[r] \arrow[d, "\wr"] & \mathrm{Exal}_A(B, J) \\
& & \mathrm{Hom}(B \otimes_{\widetilde{A}} \Omega_{\widetilde{A}/A}, J) \arrow[r] & \mathrm{Hom}(B \otimes_{\widetilde{A}} I_{B/\widetilde{A}}, J) &
\end{tikzcd}$$

Solution. We write p for the homomorphism $\widetilde{A} \to B$, which is fixed throughout this discussion.

The map

$$\mathrm{Der}_A(B, J) \to \mathrm{Der}_A(\widetilde{A}, J)$$

is by composition with p. It is injective because p is surjective.

The map $\mathrm{Der}_A(\widetilde{A}, J) \to \mathrm{Exal}_{\widetilde{A}}(B, J)$ sends a derivation δ to the homomrphism $p + \epsilon\delta : \widetilde{A} \to B + \epsilon J$, which makes $B + \epsilon J$ into a $\widetilde{A}$-algebra extension B' of B by J. To see the exactness at this spot, suppose that $B \simeq B + \epsilon J$ as a $\widetilde{A}$-algebra. Then there is a $\widetilde{A}$-algebra section of the projection $B' \to B$. That is $p + \epsilon\delta$ factors through B, so δ factors through a derivation $B \to J$.

The map $\mathrm{Exal}_{\widetilde{A}}(B, J) \to \mathrm{Exal}_A(B, J)$ sends an extension B' to itself, viewed as an A-algebra instead of a $\widetilde{A}$-algebra. If B' lies in the kernel then there is an A-algebra splitting of $B' \to B$. Thus $B' \simeq B + \epsilon J$ as an A-algebra. Choosing such an isomorphism, the $\widetilde{A}$-algebra structure gives an A-algebra map $\widetilde{A} \to B + \epsilon J$, i.e., an A-derivation $\widetilde{A} \to J$. □

(ii) Show that if $\widetilde{A}$ is a *free* A-algebra then the map

$$\mathrm{Exal}_{\widetilde{A}}(B, J) \to \mathrm{Exal}_A(B, J)$$

is surjective.

Solution. Suppose $B' \in \mathrm{Exal}_A(B, J)$. Since $\widetilde{A}$ is free and $\widetilde{A} \to B$ is surjective, we can lift the map $\widetilde{A} \to B$ to a map $\widetilde{A} \to B'$. □

(iii) Prove that

$$d : B \otimes_{\widetilde{A}} \Omega_{\widetilde{A}/A} \to B \otimes_{\widetilde{A}} I_{\widetilde{A}/A}$$

is an isomorphism if and only if B is formally étale over A.

Solution. We have seen that B is formally étale over A if and only if $\mathrm{Der}_A(B, J) = \mathrm{Exal}_A(B, J)$. But by commutativity of the diagram and exactness of the sequence (including the surjectivity on the right), this is the same as d being an isomorphism. □

30.3 A differential characterization of étale morphisms

Exercise 30.11. Let $\widetilde{A} = A[x_1, \ldots, x_n]$ and let $I = (f_1, \ldots, f_m)$.

(i) Show that $B \otimes_{\widetilde{A}} \Omega_{\widetilde{A}/A} = \sum B dx_i$.

(ii) Show that $B \otimes_{\widetilde{A}} I$ is generated by $f_1, \ldots, f_m$.

(iii) Show that the map

$$\sum B f_i \to B \otimes_{\widetilde{A}} I \to B \otimes_{\widetilde{A}} \Omega_{\widetilde{A}/A} = \sum B dx_i$$

is given by the following $n \times m$ matrix:

$$\mathcal{J} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

(iv) Under the assumption $m = n$, conclude that $d : B \otimes_{\widetilde{A}} I \to B \otimes_{\widetilde{A}} \Omega_{\widetilde{A}/A}$ is an isomorphism if and only if $\det \mathcal{J} \in B^*$.

Exercise 30.12. Prove that $\operatorname{Spec} k[t, t^{-1}] \to k[s, s^{-1}]$, given by $s \mapsto t^n$, is étale if and only if the characteristic of k does not divide n. (Hint: Identify $k[t, t^{-1}] = k[s, s^{-1}, t]/(t^n - s)$ and use the differential criterion.)

A Bézout's theorem

sec:bezout

Theorem A.1. *If C and D are algebraic curves in $\mathbf{A}^2_k$ that meet transversally and do not meet at infinity then $|(C \cap D)(k)| = \deg(C)\deg(D)$ for any algebraically closed field k.*

Consider the moduli space of all such polynomials, $\mathbf{A}^N = \operatorname{Spec} A$ where $N = \binom{d+2}{d} + \binom{e+2}{e}$. Let $X \subset \mathbf{A}^N \times \mathbf{A}^2$ be the locus of (f, g, p) such that $f(p) = g(p) = 0$. Let $\pi : X \to \mathbf{A}^N$ be the projection.

Note that $C = V(f)$ and $D = V(g)$ *meet transversally* if and only if the fiber of X over the map $(f, g) : \operatorname{Spec} k \to \mathbf{A}^N$ is étale over $\operatorname{Spec} k$.

ex:trivial-bezout **Exercise A.2.** If

$$f = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_d)$$
$$g = (y - \beta_1)(y - \beta_2) \cdots (y - \beta_e)$$

then $V(f, g)$ consists of de reduced points.

Exercise A.3. There is a non-empty open subset U of $\mathbf{A}^N$ such that $\pi^{-1}U$ is étale over U.

Solution. By definition, X is a closed subset of $Y = \operatorname{Spec} \widetilde{A} = \mathbf{A}^N \times \mathbf{A}^2$, defined by two equations:

$$X = \{(f, g, p) \mid f(p) = g(p) = 0\}.$$

We have $\Omega = \Omega_{\widetilde{A}/A} = \widetilde{A}dx + \widetilde{A}dy \simeq \widetilde{A}^2$. The ideal $I = I_{B/\widetilde{A}}$ is defined by f and g. We therefore obtain a map

$$B^2 \to Bf + Bg = B \otimes_{\widetilde{A}} I \to B \otimes_{\widetilde{A}} \Omega \simeq B^2.$$

The matrix of this map is the Jacobian matrix:

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} & \frac{\partial g}{\partial y} \end{pmatrix}$$

This map is an isomorphism if and only if the Jacobian determinant is invertible. Therefore $D(\det J) \subset \mathbf{A}^N$ is the largest open subset over which this map is an isomorphism. On the other hand, X is étale over $\mathbf{A}^N$ if and only if $\det J$ is an isomorphism by the differential criterion, so this shows there is a largest open subset of $\mathbf{A}^N$ over which X is étale.

The previous exercise shows that this open subset is non-empty. $\square$

Exercise A.4. Show that there is a non-empty open subset of $\mathbf{A}^N$ over which X contains no points at infinity. Show that X is proper over this open subset.

Solution. Let $Y \subset \mathbf{A}^N \times \mathbf{P}^2$ be the projective closure of X and let $q : Y \to \mathbf{A}^N$ be the projection. Note that q is proper because it is projective.[2] Let $Z \subset \mathbf{A}^N \times \mathbf{P}^2$ be the line at infinity (the complement of $\mathbf{A}^N \times \mathbf{A}^2$). Then $Z \cap Y$ is closed in $\mathbf{A}^N \times \mathbf{P}^2$ so it is also proper over $\mathbf{A}^N$. Let $U \subset \mathbf{A}^N$ be the complement of the image of $Z \cap Y$.

Then $q^{-1}U$ is proper over U by stability of properness under base change.[3] On the other hand, $q^{-1}U \cap (Z \cap Y) = U \cap q(Z \cap Y) = \varnothing$ so $q^{-1}U \subset X = Y \smallsetminus Z$. Thus $q^{-1}U = p^{-1}U$ and in particular $p^{-1}U$ is proper over U.

The first exercise shows that this subset is nonempty. Indeed, the intersection of the two curves at infinity is the intersection of $x^d = y^e = 0$ in $\mathbf{P}^1$, which is empty. $\square$

Exercise A.5. Conclude that there is an open subset $U \subset \mathbf{A}^N$ containing the example from Exercise A.2 such that $p^{-1}U$ is proper and étale over U.

Exercise A.6. Show that all geometric fibers of X over U have the same number of points. (Hint: Let k be an algebraically closed field and consider a map $h : \operatorname{Spec} k[[t]] \to U$. Construct a bijection between the closed fiber of $h^{-1}X$ and the set of points of the general fiber with residue field $k((t))$ using the valuative criterion for properness and the formal criterion for étale morphisms.)

31 Smooth and unramified morphisms

Definition 31.1. A morphism of schemes $f : X \to Y$ is said to be *formally unramified* if any infinitesimal lifting problem

$$\begin{tikzcd} S \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ S' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

has *at most one* solution. A morphsim that is formally unramified and locally of finite type is said to be *unramified*.

[2] todo: reference
[3] todo: reference

def:smooth

This definition has been changed! The infinitesimal extension is now required to be affine. This definition is equivalent to the one given earlier, but to prove the equivalence requries cohomology.

Definition 31.2. A morphism of schemes $f : X \to Y$ is said to be *formally smooth* if any infinitesimal lifting problem

$$\begin{tikzcd} S \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ S' \arrow[r] & Y \end{tikzcd}$$

has at least one solution *when S' is affine*. A morphism that is both formally smooth and locally of finite presentation is said to be *smooth*.

Exercise 31.3. Show that formally étale is the conjunction of formally smooth and formally unramified. (Note: This is not completely trivial! You will have to glue some morphisms.)

Exercise 31.4. (i) Suppose that $f : X \to Y$ induces an injection between functors of points. Show that f is unramified.

(ii) Conclude that locally closed embeddings are unramified.

(iii) Give an example of an unramified morphism that is not an injection on functors of points. (Hint: Consider the map $f : \mathbf{A}^1 \to \mathbf{A}^2$ given by $f(x) = (t^2 - 1, (t^2 - 1)t)$. Show that this is a closed embedding away from either of the points $t = \pm 1$.)

Exercise 31.5. (i) Show that $\mathbf{A}^n$ is smooth for all $n \geq 0$.

(ii) Show that the base change of a smooth morphism is smooth.

Deformation theory

Suppose we have a sequence of homomorphisms of commutative rings $A \xrightarrow{f} B \xrightarrow{g} C$. We saw earlier that there is an exact sequence

$$C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

One might be tempted to ask how this sequence can be extended on the left. It turns out that it is easier to consider all C-modules J and the dual sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}(\Omega_{C/B}, J) & \longrightarrow & \mathrm{Hom}(\Omega_{C/A}, J) & \longrightarrow & \mathrm{Hom}(C \otimes_B \Omega_{B/A}, J) \\ & & \| & & \| & & \| \\ 0 & \longrightarrow & \mathrm{Der}_B(C, J) & \longrightarrow & \mathrm{Der}_A(C, J) & \longrightarrow & \mathrm{Der}_A(B, J) \end{array}$$

Exercise 31.6. Show that this sequence can be continued to a 6-term sequence:

$$0 \to \mathrm{Der}_B(C, J) \to \mathrm{Der}_A(C, J) \to \mathrm{Der}_A(B, J) \to \mathrm{Exal}_B(C, J) \to \mathrm{Exal}_A(C, J) \to \mathrm{Exal}_A(B, J)$$

Solution. The map $\mathrm{Der}_B(C, J) \to \mathrm{Der}_A(C, J)$ sends a B-derivation $\delta : C \to J$ to itself, now viewed as an A-derivation.

The map $\mathrm{Der}_A(C, J) \to \mathrm{Der}_A(B, J)$ sends δ to $\delta \circ g$.

The map $\mathrm{Der}_A(B, J) \to \mathrm{Exal}_B(C, J)$ sends a derivation $\delta : B \to J$ to the extension $C + \epsilon J \to J$ with the B-algebra structure coming from $g + \epsilon\delta : B \to C + \epsilon J$.

The map $\mathrm{Exal}_B(C, J) \to \mathrm{Exal}_A(C, J)$ sends a B-algebra extension C' to itself, now viewed as an A-algebra extension.

The map $\mathrm{Exal}_A(C, J) \to \mathrm{Exal}_A(B, J)$ sends a A-algebra extension $C' \to C$ to $g^{-1}C' \to C$.

We have exactness at $\mathrm{Der}_B(C, J)$: If $\delta \in \mathrm{Der}_B(C, J)$ then its image in $\mathrm{Der}_A(C, J)$ is the same map.

We have exactness at $\mathrm{Der}_A(C, J)$: Suppose $\delta \in \mathrm{Der}_B(C, J)$. Then $\delta(g(B)) = 0$ by definition so δ maps to zero in $\mathrm{Der}_A(B, J)$. If $\delta \in \mathrm{Der}_A(C, J)$ maps to zero in $\mathrm{Der}_A(B, J)$ then $\delta(g(B)) = 0$ so δ is a B-derivation.

We have exactness at $\mathrm{Exal}_B(C, J)$. Suppose $\delta \in \mathrm{Der}_A(B, J)$. Then the image of δ in $\mathrm{Exal}_B(C, J)$ is the extension $C + \epsilon J$ with B-algebra structure $g + \epsilon\delta$. Its image in $\mathrm{Exal}_A(B, J)$ is $C + \epsilon J$ with A-algebra structure $(g + \epsilon\delta) \circ f = gf$ since δ is an A-derivation. We therefore get the zero element of $\mathrm{Exal}_A(C, J)$. If $C' \to C$ is a B-algebra extension (fix the map $g' : B \to C'$) by J whose image in $\mathrm{Exal}_A(C, J)$ is zero then choose an A-algebra isomorphism $\varphi : C + \epsilon J \to C'$. Then $\epsilon^{-1}(\varphi^{-1}g' - g) : B \to J$ is a derivation φ witnesses that C' is in the image of $\mathrm{Der}_A(B, J)$.

We have exactness at $\mathrm{Exal}_A(C, J)$. Suppose that C' is a B-algebra extension of C by J (structure map g'). Then $g^{-1}C'$ has a splitting by g'. Suppose that $g^{-1}C'$ has a splitting over B. Then by composition with $g^{-1}C' \to C'$ this gives a B-algebra structure to the extension $C' \to C$. □

Exercise 31.7. (i) Show that f is formally smooth if and only if $\mathrm{Exal}_A(B, J) = 0$ for all J.

Solution. An A-algebra extension B' of B by J is a lifting problem:

$$\begin{tikzcd} B \arrow[r, equal] \arrow[d, leftarrow] & B \arrow[dl, dashed] \\ B' & A \arrow[l] \arrow[u] \end{tikzcd}$$

□

(ii) Show that g is formally unramified if and only if $\mathrm{Der}_B(C, J) = 0$ for all J.

Solution. The difference between two lifts of

$$\begin{tikzcd} C \arrow[r, equal] \arrow[d, leftarrow] & C \arrow[dl, dashed] \\ C' & B \arrow[l] \arrow[u] \end{tikzcd}$$

is a B-derviation $C \to J$. □

(iii) Assume f is formally smooth and g is formally unramified. Show that gf is formally étale if and only if

$$\mathrm{Der}_A(B, J) \to \mathrm{Exal}_B(C, J)$$

is an isomorphism.

Solution. Use the exact sequence:

$$0 \to \mathrm{Der}_A(C, J) \to \mathrm{Der}_A(B, J) \to \mathrm{Exal}_B(C, J) \to \mathrm{Exal}_A(C, J) \to 0$$

We get $\mathrm{Exal}_A(C, J) = \mathrm{Der}_A(C, J) = 0$ if and only if $\mathrm{Der}_A(B, J) \to \mathrm{Exal}_B(C, J)$ is an isomorphism. □

Part II

General properties of schemes

Chapter 10

Dimension

32 Dimension of smooth schemes

32.1 The tangent bundle

Exercise 32.1. Suppose that A is a commutative ring and B is an A-algebra. Show that the natural map

$$B[f^{-1}] \otimes_B \Omega_{B/A} \to \Omega_{B[f^{-1}]/A}$$

is an isomorphism. (Hint: Consider the functors they represent.)

Solution. It is the same to show that for any $B[f^{-1}]$-module J, the map

$$\mathrm{Der}_A(B[f^{-1}], J) \to \mathrm{Der}_A(B, J)$$

is a bijection. Certainly it is injective because $B \to B[f^{-1}]$ is an epimorphism. Suppose $\delta : B \to J$ is an A-derivation. Define $\delta(f^{-n}x) = f^{-n}\delta(x) - nf^{-n-1}x\delta(f)$. To verify this is well-defined note that if $f^k y = 0$ in B then $f^k\delta(y) + kf^{k-1}y\delta(f) = 0$ so $f^{k+1}\delta(y) = 0$, whence $\delta(y) = 0$ in J. $\square$

Exercise 32.2. Suppose $A \to B$ is a homomorphism of commutative rings and let $X \to Y$ be the associated morphism of affine schemes. For each principal open affine $D(f) \subset \operatorname{Spec} B$, define $\Omega_{X/Y}(D(f)) = \Omega_{B[f^{-1}]/A}$. Show that $\Omega_{X/Y}$ is a quasicoherent sheaf on X.

Exercise 32.3. Let $f : X \to Y$ be a morphism of schemes. Construct a quasicoherent sheaf $\Omega_{X/Y}$ on X such that if $U \subset X$ and $V \subset Y$ are open affines with $U \subset f^{-1}V$ we have $\Omega_{X/Y}\big|_U = \Omega_{U/V}$. (Hint: One strategy here is to glue together the constructions from the previous exercise. Another is to construct $d : \mathcal{O}_X \to \Omega_{X/Y}$ as the universal $f^{-1}\mathcal{O}_Y$-derivation. Still another is to take $\Omega_{X/Y} = \Delta^{-1}(\mathscr{I}/\mathscr{I}^2)$ where $\Delta : X \to X \times_Y X$ is the inclusion of the diagonal.)

Exercise 32.4. Let S be a scheme and let $\mathscr{J}$ be a quasicoherent sheaf on S. Define $\mathcal{O}_{S[\mathscr{J}]}(U) = \mathcal{O}_S(U) + \epsilon\mathscr{J}(U)$ for all open $U \subset S$.

(i) Show that $\mathcal{O}_{S[\mathscr{J}]}$ is the structure sheaf of a scheme $S[\mathscr{J}]$ whose underlying topological space is the same as that of S.

(ii) Construct a closed embedding $S \to S[\mathscr{J}]$ and a canonical retraction $S[\mathscr{J}] \to S$.

When $\mathscr{J} = \mathcal{O}_S$ we also write $S[\epsilon]$.

Exercise 32.5. Let $f : X \to Y$ be a morphism of schemes. Define $T_{X/Y}(S)$ to be the set of commutative diagrams

$$\begin{tikzcd} S[\epsilon] \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ S \arrow[r] & Y \end{tikzcd}$$

where $S[\epsilon] \to S$ is the retraction constructed in the last exercise. Show that $T_{X/Y}$ is representable by $\mathbf{V}(\Omega_{X/Y})$.

The scheme $T_{X/Y}$ constructed in the last exercise is known as the relative tangent bundle of X over Y.

32.2 Relative dimension

smooth-tangent-bundle

Theorem 32.6. *Suppose that $f : X \to Y$ is smooth. Then $T_{X/Y}$ is a vector bundle.*

Exercise 32.7. Suppose that B is a formally smooth A-algebra. Show that $\Omega_{B/A}$ is projective as a B-module.

Solution. We need to show $\mathrm{Hom}_{B\text{-}\mathbf{Mod}}(\Omega_{B/A}, M)$ is a right exct functor of M. Consider a surjection $M \to M'$ of B-modules and a map $\Omega_{B/A} \to M'$. This corresponds to an A-derivation $B \to M'$ and therefore to an A-algebra homomorphism $B \to B + \epsilon M'$ lifting the identity. Consider the lifting problem

$$\begin{tikzcd} B + \epsilon M' & B \arrow[l] \arrow[dl, dashed] \\ B + \epsilon M \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

Note that $B + \epsilon M \to B + \epsilon M'$ is a surjection. Therefore a lift exists. Thus the derivation $B \to M'$ lifts to a derivation valued in M and the map $\Omega_{B/A} \to M'$ lifts to M. Thus $\Omega_{B/A}$ is projective. $\square$

Definition 32.8. A sheaf $\mathscr{F}$ of $\mathcal{O}_X$-modules on a scheme X is said to be *locally of finite presentation* if there is a cover of X by open subschemes U such that there is a presentation

$$\mathcal{O}_U^{\oplus n} \to \mathcal{O}_U^{\oplus m} \to \mathscr{F}\big|_U \to 0$$

with both m and n finite.

Exercise 32.9. Suppose that $f : X \to Y$ is locally of finite presentation. Show that $\Omega_{X/Y}$ is locally of finite presentation.

The following exercises will now complete the proof of Theorem 32.6.

Imperative if you haven't done it before. Skip it if you have.

Exercise 32.10 (Nakayama's Lemma)**.** Suppose A is a local ring with residue field k and maximal ideal $\mathfrak{m}$ and M is a finitely generated A-module. Prove that the following conditions are equivalent:

(i) $M = 0$

(ii) $M = \mathfrak{m}M$

(iii) $M/\mathfrak{m}M = 0$

(iv) $M \otimes_A k = 0$

Solution. Let $x_1, \ldots, x_n$ be generators. If $\mathfrak{m}M = M$ then $x_n = \sum a_i x_i$ with $a_i \in \mathfrak{m}$. Then $(1 - a_i)x_n \in \sum_{i=1}^{n-1} Ax_i$ and $1 - a_i$ is a unit because A is local. The remaining implications are easier. □

Exercise 32.11. (i) Prove that a finitely presented A-module M is locally free if and only if $M_\mathfrak{p}$ is free as an $A_\mathfrak{p}$-module for every prime $\mathfrak{p}$ of A.

Solution. If M is locally free then $M_\mathfrak{p}$ is locally free and $A_\mathfrak{p}$ has no nontrivial open covers so $M_\mathfrak{p}$ must be free.

Conversely, if $M_\mathfrak{p}$ is free then choose elements $x_1, \ldots, x_n$ of M that form a basis of $M_\mathfrak{p}$. Recall M is finitely generated, so any of the finitely many generators y_j lies in $\sum A_\mathfrak{p} x_i$. Only finitely many denominators are necessary to write such an expression for y_j, so we can write $y_j \in A[f^{-1}]x_i$ for some $f \in A$. We can make a single f work for all j. Similarly, if z_j is one of the finitely many relations among the x_j in M_f then $z_j = 0$ in $A_\mathfrak{p}$ so $gz_j = 0$ for some $g \notin \mathfrak{p}$. There are only finitely many relations, so we can kill all of them by inverting a single g. Then $D(fg)$ is an open neighborhood of $\mathfrak{p}$ on which M is free. Repeat for all primes $\mathfrak{p}$ to get a cover on which M is free. □

(ii) Prove that a finitely presented A-module M is projective if and only if $M_\mathfrak{p}$ is projective for every prime $\mathfrak{p}$ of A. (Hint: Show that $\mathrm{Hom}_{A_\mathfrak{p}\text{-}\mathbf{Mod}}(M_\mathfrak{p}, N_\mathfrak{p}) = \mathrm{Hom}_{A\text{-}\mathbf{Mod}}(M, N)_\mathfrak{p}$. You will need the finite presentation for this.)

Solution. Choose a presentation of M:

$$A^n \to A^m \to M \to 0.$$

This induces

$$0 \to \mathrm{Hom}(M, N) \to N^m \to N^n.$$

By exactness of localization, we get a diagram of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(M,N)_\mathfrak{p} \arrow[r] \arrow[d] & (N^m)_\mathfrak{p} \arrow[r] \arrow[d] & (N^n)_\mathfrak{p} \arrow[d] \\ 0 \arrow[r] & \mathrm{Hom}(M_\mathfrak{p}, N_\mathfrak{p}) \arrow[r] & (N_\mathfrak{p})^m \arrow[r] & (N_\mathfrak{p})^n \end{tikzcd}$$

By the commutation of localization with finite products the vertical arrows on the right are isomorphisms. Therefore the vertical arrow on the left is too.

To show that $\mathrm{Hom}_{A\text{-}\mathbf{Mod}}(M, N)$ is an exact functor of N for all primes $\mathfrak{p}$. It is sufficient to show that $\mathrm{Hom}_{A\text{-}\mathbf{Mod}}(M, N)_\mathfrak{p} = \mathrm{Hom}_{A\text{-}\mathbf{Mod}}(M_\mathfrak{p}, N_\mathfrak{p})$ is an exact functor of N. But $N_\mathfrak{p}$ is an exact functor of N and $M_\mathfrak{p}$ is projective, so we are done.

Conversely, $M_\mathfrak{p}$ is projective if M is because the inclusion of $A_\mathfrak{p}$-modules in A-modules is an exact right adjoint to localization. □

(iii) Prove that a finitely presented projective module over a local ring is free. (Hint: Choose generators of $M \otimes_A k$ where k is the residue field. Lift these to M and use Nakayama's lemma to conclude that these generate M. Obtain a surjection $A^n \to M$ that induces an isomorphism upon passage to the residue field. Let $N \subset A^n$ be the kernel. Use the fact that M is projective to get an isomorphism $A^n \simeq N \times M$. Conclude that $N \otimes_A k = 0$ and apply Nakayama's lemma again.)

Solution. Suppose M is a finitely presented projective A-module and A is local. Choose a basis for M module $\mathfrak{m}M$. These elements generate M and give a surjection $A^n \to M$ inducing an isomorphism modulo $\mathfrak{m}$. Let N be the kernel. Since M is projective, we can find an identification $A^n \simeq N \times M$. Then $k^n \simeq N/\mathfrak{m}N \times M/\mathfrak{m}M$. But $M/\mathfrak{m}M \simeq k^n$ so $N/\mathfrak{m}N = 0$. But N is finitely generated because M is finitely presented, so this implies $N = 0$ by Nakayama's lemma. $\square$

(iv) Prove that an A-module M of finite presentation is locally free if and only if it is projective.

Solution. We have seen that M is locally free if and only if $M_\mathfrak{p}$ is free for all $\mathfrak{p}$ if and only if $M_\mathfrak{p}$ is projective for all $\mathfrak{p}$ if and only if M is projective. $\square$

Definition 32.12. Suppose that $f : X \to Y$ is a smooth morphism of schemes. If $T_{X/Y}$ has rank n then we say f is *smooth of relative dimension n.*

32.3 The structure of smooth morphisms

Theorem 32.13. *Suppose that $\pi : X \to Y$ is smooth of relative dimension n. Then there is an cover of X by open subsets U such that $U \to Y$ factors as an étale map $U \to \mathbf{A}^n_Y$.*

Proof. Choose U such that $\Omega_{U/Y} \simeq \mathcal{O}_U^{\oplus n}$. Choose $f_1, \ldots, f_n$ such that the df_i form a basis for $\Omega_{U/Y}$. These give a map $U \to \mathbf{A}^n_Y$. We argue that this is étale. Consider a lifting problem in which S' is a square-zero extension of $S = \operatorname{Spec} B$ by J:

$$\begin{tikzcd} S \arrow[r] \arrow[d] & U \arrow[d] \\ S' \arrow[r] \arrow[ur, dashed] & \mathbf{A}^n_Y. \end{tikzcd} \tag{32.1}$$

eqn:22

Note that the lower horizontal arrow is a lift of

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \mathbf{A}^n_Y \arrow[d] \\ S' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{32.2}$$

eqn:20

The lifts of this diagram may be identified (in fact, form a torsor under)

$$\operatorname{Hom}_{B\text{-}\mathbf{Mod}}(B \otimes_{\mathcal{O}_{\mathbf{A}^n_Y}} \Omega_{\mathbf{A}^n_Y/Y}, J).$$

Lifts of the diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d] & U \arrow[d] \\ S' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{32.3}$$

eqn:21

form a torsor under

$$\mathrm{Hom}_{B\text{-}\mathbf{Mod}}(B \otimes_{\mathcal{O}_U} \Omega_{U/Y}, J).$$

But $\mathcal{O}_U \otimes_{\mathcal{O}_{\mathbf{A}^n_Y}} \Omega_{\mathbf{A}^n_Y/Y} \simeq \Omega_{U/Y}$ so lifts of diagram (32.2) are in bijection with lifts of (32.3). In other words, diagram (32.1) has a unique lift. □

33 Dimension I

Reading 33.1. [AM69, Chapter 11], [Vak14, Chapter 11], [GD67, IV.0.16]

We introduce several approaches to the dimension of a commutative ring. The theory works best in the case of a noetherian local ring, and we eventually define the dimension of a non-local noetherian ring to be the maximum of the dimensions of its local rings.

33.1 Chevalley dimension

sec:Chevalley

Definition 33.2. If A is a noetherian local ring, an *ideal of definition* of A is an ideal $\mathfrak{q}$ whose radical is the maximal ideal of A.

The *Chevalley dimension* of A is the minimal number of generators of an ideal of definition of A.

Should be immediate.

Exercise 33.3. Show that the Chevalley dimension of a noetherian local ring A with maximal ideal $\mathfrak{m}$ is the minimal number of elements $f_1, \ldots, f_n$ of A such that $V(f_1, \ldots, f_n) = \{\mathfrak{m}\}$.

33.2 Artin–Rees lemma

This section follows [AM69, Chapter 11].

Definition 33.4. Let A be a commutative ring, $I \subset A$ an ideal, and M an A-module. A decreasing filtration of

$$M = F^0 M \supset F^1 M \supset F^2 M \supset \cdots$$

is called an *I-filtration* if $IF^n M \subset F^{n+1} M$ for all n. It is called a *stable I-filtration* if $IF^n M = F^{n+1} M$ for all $n \gg 0$.

We are really only interested in the filtration $F^n M = I^n M$, but we run into an unfortunate difficulty. If $M' \subset M$ then $I^n M' \neq M' \cap I^n M$. That is, we get a second filtration on M' by setting $F^n M' = M' \cap I^n M$. The Artin–Rees lemma says that when A is noetherian and M is finitely generated, these two filtrations aren't that different.

Theorem 33.5 (Artin–Rees lemma)**.** *If A is noetherian and M is finitely generated, every I-filtration F of M is stable.*

Exercise 33.6. We will prove the Artin–Rees lemma using the *Rees algebra* $B = A[tI] = \sum_{n=0}^{\infty} t^n I^n$ and the modules $N = \sum_{n=0}^{\infty} t^n I^n M$ and $N' = \sum_{n=0}^{\infty} t^n F^n M$.

(i) Prove that the Rees algebra is noetherian if A is noetherian.

(ii) Prove that N' is a B-submodule of N and that N is a finitely generated B-module. Conclude that N' is finitely generated.

(iii) Prove the Artin–Rees lemma. (Hint: Choose n such that all generators of N' have degrees $\leq n$.)

Solution. Choose n as in the hint. Then $IN_m = N_{m+1}$ for all $m \geq n$. That is, $IF^m M = F^{m+1} M$. □

33.3 Hilbert–Samuel dimension

Definition 33.7. Let A be a noetherian local ring and let M be an A-module. If

$$M = F^0 M \supsetneq F^1 M \supsetneq \cdots \supsetneq F^n M = 0.$$

is a maximal filtration of M, the number n is called the *length* of M and is denoted $\operatorname{length}(M)$.

Exercise 33.8. Show that the length of M is the dimension (over the residue field) of the graded module

$$\operatorname{gr}(M) = \sum_{k=0}^{\infty} \mathfrak{m}^k M / \mathfrak{m}^{k+1} M$$

where $\mathfrak{m}$ is the maximal ideal of A. Conclude that the definition of the length does not depend on the choice of filtration F.

Exercise 33.9. Show that the length is additive in short exact sequences: if

$$0 \to M' \to M \to M'' \to 0$$

is exact then $\operatorname{length}(M) = \operatorname{length}(M') + \operatorname{length}(M'')$.

Definition 33.10 (Hilbert–Samuel function)**.** Let A be a noetherian local ring and M an A-module with a descending filtration F. The *Hilbert–Samuel function* associated to F is $h(M, F, n) = \operatorname{length}(M/F^n M)$. When F is the filtration associated to an ideal I, we write $h(M, I, n)$.

The Hilbert function turns out to be a polynomial:

x:hilbert-func-poly **Exercise 33.11.** Let A be a noetherian local ring, $\mathfrak{m}$ its maximal ideal, M a finitely generated A-module, and F a descending $\mathfrak{m}$-filtration on M.

(i) Show that $h(M, F, n) = h(\operatorname{gr} M, n)$ where $\operatorname{gr} M = \sum_{n \geq 0} F^n M / F^{n+1} M$ is the associated graded ring of M, filtered by degree.

Solution. We have

$$\operatorname{length}(M/F^n M) = \sum_{m=0}^{n} \operatorname{length}(F^{m-1} M / F^m M) = \operatorname{length}(\operatorname{gr} M / F^n \operatorname{gr} M).$$

□

(ii) Show that $h(M, F, n)$ agrees with a polynomial for $n \gg 0$.

Solution. By the previous part, we can assume A is graded and M is a graded module. Now proceed by induction on the minimal number of generators for $\mathfrak{m}$. If the number is 0 then $h(M)$ is constant. Otherwise, let x be a generator of $\mathfrak{m}$. Consider the sequence

$$0 \to \mathrm{Ann}_{M(-1)}(x) \to M(-1) \xrightarrow{x} M \to M/xM \to 0$$

where $M(-1)$ is the same module as M with its grading shifted by 1. Then

$$h(M, n) - h(M(-1), n) = h(M/xM, n) - h(\mathrm{Ann}_{M(-1)}(x), n).$$

Since the maximal ideal of A/xA is generated by fewer elements than is $\mathfrak{m}$, the right side of the equality above is a polynomial for $n \gg 0$. On the other hand, the left side is $h(M, n) - h(M, n-1)$, at least when $n > 0$. Thus $h(M, n)$ is a polynomial for $n \gg 0$.[1] □

Definition 33.12. In view of Exercise 33.11, the Hilbert–Samuel function $h(M, F, n)$ agrees with a polynomial for large n. We call this polynomial the *Hilbert–Samuel polynomial* and notate it $P(M, F, n)$.

Exercise 33.13. Show that if $\mathfrak{q}$ and $\mathfrak{p}$ are ideals of definition of A then $P(M, \mathfrak{p})$ and $P(M, \mathfrak{q})$ have the same degree.

Solution. Choose m such that $\mathfrak{q}^m \subset \mathfrak{p}$. Then $\mathfrak{q}^{nm}M \subset \mathfrak{p}^n M$, so there is a surjection $M/\mathfrak{q}^{nm}M \to M/\mathfrak{p}^n M$ so $\mathrm{length}(M/\mathfrak{p}^n M) \leq \mathrm{length}(M/\mathfrak{q}^{nm}M)$. In particular,

$$P(M, \mathfrak{q}, nm) = P(M, \mathfrak{q}^m, n) \geq P(M, \mathfrak{p}, n).$$

But both $P(M, \mathfrak{q})$ and $P(M, \mathfrak{p})$ are polynomials, so this bounds the degree of $P(M, \mathfrak{p})$ by the degree of $P(M, \mathfrak{q})$. The same argument gives the reverse bound. □

Exercise 33.14. Let A be a noetherian local ring, $\mathfrak{q}$ an ideal of definition, F a descending $\mathfrak{q}$-stable filtration on a finitely generated A-module M. Show that $P(M, F)$ and $P(M, \mathfrak{q})$ have the same degree and leading coefficient.

Solution. We have $F^n M \subset \mathfrak{q}^n M$ for all n. Therefore $\lim_{n\to\infty} P(M, F, n)/P(M, \mathfrak{q}, n) \leq 1$. On the other hand, there is some m such that $F^{m+n}M = \mathfrak{q}^n F^m M$ for all $n \geq 0$. Therefore

$$\lim_{n\to\infty} \frac{P(M, F, n)}{P(M, \mathfrak{q}, n)} = \lim_{n\to\infty} \frac{P(M, F, n)}{P(M, F, m+n)} \frac{P(F^m M, F, n)}{P(F^m M, \mathfrak{q}, n)} \frac{P(M, \mathfrak{q}, n)}{P(F^m M, \mathfrak{q}, n)}$$

(Note that $P(F^m M, F, n) = P(M, F, m+n)$.) The first ratio approaches 1 since $P(M, F, m+n)$ and $P(M, F, n)$ have the same leading coefficient and degree. The second ratio is identically 1. Finally $\mathfrak{q}^n F^m M \subset \mathfrak{q}^n M$ so $\mathrm{length}(M/\mathfrak{q}^n F^m M) \geq \mathrm{length}(M/\mathfrak{q}^n M)$. Therefore the last ratio is ≥ 1. The limit is therefore ≥ 1 as well. □

Definition 33.15. The *Hilbert–Samuel dimension* of A is the degree of $P(A, \mathfrak{m})$.

[1]If h is a function of the integers such that $h(n) - h(n-1)$ is a polynomial for $n \gg 0$ then h is a polynomial. Indeed, suppose $\ell(n) = h(n) - h(n-1)$ agrees with a degree d polynomial for all $n \gg 0$. Let $H(n)$ be the unique polynomial that agrees with h at $d+2$ specified values $n_0, n_0+1, \ldots, n_0+d+1$ in the range where ℓ is a polynomial. Let $L(n) = H(n) - H(n-1)$. Then L and ℓ are both polynomials for large n, of the same degree d, agreeing at $d+1$ values. Hence they are equal. Now, $H(n_0) = h(n_0)$ and by induction $H(n+1) = H(n) + L(n+1) = h(n) + \ell(n+1) = h(n+1)$, hence $h(n) = H(n)$ for all $n \gg 0$ is a polynomial.

33.4 Krull dimension

Definition 33.16. The *Krull dimension* of A is the length of the longest chain of nontrivial specializations in $\operatorname{Spec} A$. Equivalently, it is the length of a maximal chain of prime ideals in A.

34 Dimension II

34.1 Equivalence

thm:dimension

Theorem 34.1 (Krull–Chevalley–Samuel [GD67, Théorème (0.16.2.3)], [AM69, Theorem 11.4])**.** *If A is a noetherian local ring, the Krull dimension, the Hilbert–Samuel dimension, and the Chevalley dimension are all the same.*

Let $\delta(A)$ denote the Chevalley dimension, $d(A)$ the degree of the Hilbert–Samuel polynomial, and $\dim(A)$ the Krull dimension.

Not essential, as this follows from the theorem and is not needed to prove it. But it's good practice.

Exercise 34.2. Show that the following statements are all equivalent (without using Theorem 34.1):

(i) $\delta(A) = 0$;

(ii) $d(A) = 0$;

(iii) $\dim(A) = 0$;

(iv) $\mathfrak{m}$ is nilpotent.

Exercise 34.3. Show that $\dim(A) \leq d(A)$.

Solution. Adapted from [GD67, (0.16.2.3.2)].

Choose a maximal chain of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$. Since $d(A/\mathfrak{p}_0) \leq d(A)$ and $\dim A = \dim A/\mathfrak{p}_0$, it is sufficient to prove that $\dim(A/\mathfrak{p}_0) \leq d(A/\mathfrak{p}_0)$. We can therefore assume $\mathfrak{p}_0 = 0$ and A is an integral domain. Pick a nonzero $x \in \mathfrak{p}_1$. Thus $\dim(A/xA) = \dim(A) - 1$: Indeed, we have a chain $\mathfrak{p}_1/xA \subset \cdots \mathfrak{p}_n/xA$ of length $\dim(A) - 1$ and if we could find a longer chain then taking the preimage in A and appending $\mathfrak{p}_0$ would produce a chain of length $> n$ in A. Moreover,

$$P(A/xA, \mathfrak{m}, n) = P(A, \mathfrak{m}, n) - P(A, F, n-1)$$

where $xF^nA = \mathfrak{q}^nA \cap xA$, because of the exact sequence

$$0 \to A \xrightarrow{x} A \to A/xA \to 0.$$

Now, $P(A, \mathfrak{m})$ and $P(A, F)$ both have the same degree and leading coefficient (because F is $\mathfrak{m}$-stable, by Artin–Rees) so $d(A/xA) \leq d(A) - 1$. We can assume by induction that $\dim(A/xA) \leq d(A/xA)$ so

$$\dim(A) \leq \dim(A/xA) + 1 \leq d(A/xA) + 1 \leq d(A)$$

as desired. $\square$

Exercise 34.4. Show that $d(A) \leq \delta(A)$.

Solution. This is a simplified version of [GD67, (0.16.2.3.3)], which deals with a more general situation.

Suppose that $\mathfrak{q}$ is an ideal of definition of A, generated by k elements. Then $\mathfrak{q}^n/\mathfrak{q}^{n+1}$ is generated by $\binom{k+n-1}{n-1}$ elements as a module under $A/\mathfrak{q}^n$. Its length is therefore bounded by $\binom{k+n-1}{n-1}$ length$(A/\mathfrak{q}^n)$, so $P(A, \mathfrak{q})$ is bounded by a polynomial of degree k. □

`ex:prime-avoidance`

Exercise 34.5 ([Vak14, Proposition 11.2.13], [Eis91, Lemma I.3.3])**.** Let $X = \operatorname{Spec} A$ be an affine scheme, let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ points of X, and let I be an ideal of X with $Z = V(I)$. Assume that Z does not contain any of the $\mathfrak{p}_i$. Then there is some $f \in I$ such that $f(\mathfrak{p}_i) \neq 0$ for all i.

Solution. We have $I\mathbf{k}(\mathfrak{p}_i) = \mathbf{k}(\mathfrak{p}_i)$ for all i. Therefore the image of I in $\prod \mathbf{k}(\mathfrak{p}_i)$ is not contained in any maximal ideal. It follows that

$$\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n + I = A.$$

But then there is an expression $g + f = 1$ with $g \in \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$ and $f \in I$. Then $f(\mathfrak{p}_i) = 1$ for all i. □

Exercise 34.6. Prove $\delta(A) \leq \dim(A)$.

Solution. This is by induction on $\dim(A)$. If $\dim(A) = 0$ then $\mathfrak{m}$ is the nilradical. As $\mathfrak{m}$ is finitely generated, this means $\mathfrak{m}$ is nilpotent and $\delta(A) = 0$.

Now proceed by induction on $\dim(A)$. Assume $\dim(A) = n > 0$. Then every prime $\mathfrak{p} \subset A$ such that $\dim(A/\mathfrak{p}) = n$ is a minimal prime. As A is noetherian, $\operatorname{Spec} A$ has finitely many irreducible components, hence A has finitely many minimal primes. There are therefore only finitely many $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ such that $\dim(A/\mathfrak{p}) = n$. As $\dim(A) > 0$, we know by prime avoidance (Exercise 34.5) that $\mathfrak{m} \not\subset \bigcup \mathfrak{p}_i$ so there is at least one $x \in \mathfrak{m}$ not in any of the $\mathfrak{p}_i$. Thus $\dim(A/xA) < \dim A$. Therefore

$$\delta(A) \leq \delta(A/xA) + 1 \leq d(A/xA) + 1 \leq d(A).$$

Indeed, $\delta(A) \leq \delta(A/xA) + 1$ because x may be adjoined to generators for $\mathfrak{m}$ (as a radical ideal) modulo xA to yield generators for $\mathfrak{m}$ (as a radical ideal). The second inequality is by induction on $d(A)$, since $d(A/xA) < d(A)$. The third inequality is again because $d(A/xA) < d(A)$. □

34.2 Codimension

`def:codimension`

Definition 34.7 ([Vak14, §11.1.4])**.** The *codimension* of an *irreducible* closed subset Z of a notherian scheme X is the dimension of the local ring at the generic point of Z.

Exercise 34.8 ([Vak14, Theorems 11.3.3, 11.3.7, §11.5])**.** Prove Krull's Hauptidealsatz: Let A be a noetherian local ring and $f_1, \ldots, f_n \in A$. Show that $\operatorname{codim}_X V(f_1, \ldots, f_n) \leq n$. (Hint: Use Chevalley dimension.)

Solution. Let $\mathfrak{p}$ be a minimal prime containing $(f_1, \ldots, f_n)$. Then $(f_1, \ldots, f_n)A_{\mathfrak{p}}$ is an ideal of definition, so $\dim A_{\mathfrak{p}} \leq n$. □

Exercise 34.9. (i) Prove that for any noetherian local ring A and any prime ideal $\mathfrak{p} \subset A$ we have

$$\dim A/\mathfrak{p} + \dim A_{\mathfrak{p}} \leq \dim A$$

(ii) Give an example of a noetherian local ring A and a prime ideal $\mathfrak{p} \subset A$ such that

$$\dim A/\mathfrak{p} + \dim A_{\mathfrak{p}} \neq \dim A$$

34.3 Examples

Exercise 34.10. Compute $\dim \operatorname{Spec} \mathbf{Z}$.

Exercise 34.11. Let k be a field, let $A = k[x_1, \ldots, x_n]$, and let $\mathfrak{p} \subset A$ be the ideal $(x_1, \ldots, x_n)A$. Compute $\dim A_{\mathfrak{p}}$. (Once we have proved the Nullstellensatz, this will be a calculation of the dimension of $\mathbf{A}^n_k$.)

Exercise 34.12. Compute $\dim \mathbf{A}^n$ at a closed point.

Exercise 34.13. Suppose that X is a smooth scheme over a field k. Prove that $\dim X$ (at any closed point) coincides with the rank of the tangent bundle $T_{X/\operatorname{Spec} k}$.

34.4 Regularity

Definition 34.14. A noetherian local ring A with maximal ideal $\mathfrak{m}$ is said to be *regular* if $\dim \mathfrak{m}/\mathfrak{m}^2 = \dim A$.

Exercise 34.15. Suppose that X is a smooth scheme over a field k. Show that the local ring of X at any point is regular.

Exercise 34.16. Give an example of a regular scheme that is not smooth. (Hint: An inseparable field extension.)

Exercise 34.17. Show that the rank of the tangent bundle of a smooth scheme over a field coincides with the dimension of the scheme.

Chapter 11

Algebraic properties of schemes

35 Finite, quasi-finite, and integral morphisms

Definition 35.1. A morphism of schemes $f : X \to Y$ is said to be *finite* if there is a cover of Y by open affine subschemes $V = \operatorname{Spec} A$ such that $f^{-1}V = \operatorname{Spec} B$ with B finite as an A-module.

Reorganization of definition. Should be easy.

Exercise 35.2. Show that $f : X \to Y$ is finite if and only if it is affine and $f_*\mathcal{O}_X$ is a sheaf of $\mathcal{O}_Y$-modules of finite type.

Trivial.

Exercise 35.3. Show that closed embeddings are finite morphisms.

Definition 35.4. A morphism of commutative rings $A \to B$ is said to be a *integral* if every element of B satisfies a monic polynomial with coefficients in A.[1] A morphism of schemes $f : X \to Y$ is said to be *integral* if there is a cover of Y by open subschemes $V = \operatorname{Spec} A$ such that $f^{-1}V = \operatorname{Spec} B$ where B is an integral extension of A.

Definition 35.5. A morphism of schemes is *quasifinite* if it is of finite type and has finite fibers.

Exercise 35.6. (i) Show that finite morphisms are quasifinite.

(ii) Give an example of a quasifinite morphism that is not finite. (Hint: open embedding.)

Exercise 35.7 (Cayley–Hamilton theorem [Sta15, Tag 00DX])**.** Suppose A is a commutative ring, M is a finitely generated A-module, and f is an endomorphism of M. Then f satisfies an integral polynomial with coefficients in A. If M is free, this polynomial can be taken to be the characteristic polynomial.

(i) Reduce to the case where M is free.

Solution. Choose a surjection $p : A^n \to M$ and lift f to a map $g : A^n \to M$ (i.e., so that $f(px) = pg(x)$ for all $x \in A^n$. Then if $Q(g)x = 0$ for all $x \in A^n$ we have $pQ(g)x = Q(f)px = 0$ for all $x \in A^n$, so $Q(f) = 0$. □

[1] Note that integral morphisms of commutative rings are not necessarily injective. This confused me for a long time.

(ii) Reduce to the case where A is an integral domain.

Solution. Choose a polymomial ring B over $\mathbf{Z}$ and a map $\varphi : B \to A$ containing all of the coefficients of a matrix representative F of f. Then $F = \varphi(G)$ for some matrix G with coefficients in B and if $Q(B) = 0$ then $\varphi(Q(B)) = Q(\varphi(B)) = Q(A) = 0$. □

(iii) Reduce to the case where A is a field and the characteristic polynomial splits into linear factors.

Solution. The characteristic polynomial won't change when applying an injection. Embed A in its field of fractions. Enlarging the field doesn't change the characteristic polynomial either, so we can pass to a splitting field of the characteristic polynomial. □

(iv) Show that M is a finite direct sum of generalized eigenspaces.[2]

Solution. Let $N \subset M$ be the generalized ξ-eigenspace. Then N is the kernel of $(f - \xi\mathrm{id})^r$ for some r. Note that $f - \xi\mathrm{id}$ acts invertibly on M/N, so let h be an inverse on M/N. Note that $(f - \xi\mathrm{id})^r$ descends to a map $g : M/N \to M$. Then $g \circ h^r$ splits the projection $M \to M/N$. Thus $M = M/N \oplus N$ and by induction M/N is a sum of generalized eigenspaces. □

(v) Show the theorem is true when f acts nilpotently on a vector space.

Solution. Assume $f^r = 0$. Then the vector space has a descending filtration $M \supset f(M) \supset \cdots \supset f^r(M) = 0$. Choose a basis compatible with this filtration. Then the matrix of f is upper triangular with zeroes on the diagonal so its characteristic polynomial is X^n with $n \geq r$. □

(vi) Conclude that the theorem is true for all the generalized eigenspaces of M and therefore for M itself.

Solution. If N_ξ is the generalized ξ-eigenspace then the characteristic polynomial of f on N_ξ is $(X - \xi)^r$ and subsituting $f\big|_{N_\xi}$ yields zero. Then if $x \in M$ we write $x = \sum x_\xi$ with $x_\xi \in N_\xi$ and $P_M(f)x = \sum P_{N_\xi}(f\big|_{N_\xi})x_\xi = 0$. □

Exercise 35.8 ([Sta15, Tag 02JJ])**.** Show that a morphism is finite if and only if it is integral and of finite type.

Solution. By the previous exercise, if B is a finite A-module then any element $x \in B$ acts on B as an endomorphism, hence satisfies an integral polynomial. Therefore finite morphisms are integral.

Conversely, an extension $B = A[x]/f(x)$ is clearly finite if f is monic. By induction, a succession of such extensions is also finite. Any finitely generated integral extension may be surjected upon by such an extension, so integral extensions are all finite. □

[2] The generalized ξ-eigenspace of M is the submodule $N \subset M$ containing all $x \in M$ annihilated by some power of $(f - \xi\mathrm{id})$.

Exercise 35.9. Give an example of an integral extension that is not free. (Hint: Normalize a nodal or cuspidal plane curve.)

Solution. Consider $A = \mathbf{C}[x,y]/(y^2 - x^3) \subset \mathbf{C}[y/x] = B$. Write $t = y/x$. Then $y = t^3$ and $x = t^2$. Then t satisfies the monic polynomial $t^2 - x = 0$ but $B \neq A[T]/(T^2 - x)$.

Alternately, $A = \mathbf{C}[x,y]/(y^2 - x^3 + x^2) \subset \mathbf{C}[y/x] = B$ with $xt = y$. Then $t^2 = x - 1$ so the extension is integral. But again, $B \neq A[T]/(T^2 - x + 1)$. □

36 Integral morphisms and dimension

36.1 Lifting inclusions of primes

Exercise 36.1. Suppose $A \subset B$ is an integral *extension.* Show that $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.

This exercise can be used in Exercise 36.3, but so can Exercise **??**, which might be easier.

Exercise 36.2 ([AM69, Proposition 5.7])**.** Let $A \subset B$ be an integral extension of commutative rings. Then A is a field if and only if B is a field.

Solution. Suppose B is a field. Suppose that $x \in A$. Then $x^{-1} \in B$ so x^{-1} satisfies a monic polynomial $f(x^{-1}) = x^{-d} - g(x^{-1})$ with $\deg g = d - 1$. Then $x^{-1} = x^{d-1}g(x^{-1})$ so $x^{-1} \in A$. □

`ex:going-up`

Exercise 36.3 ([Vak14, Theorem 7.2.5])**.** Prove that specializations lift along integral morphisms.

Solution. Let $f : X \to Y$ be an integral morphism and suppose that $f(x') = y' \rightsquigarrow y$. Replace X by the closure of x' (with its reduced structure). This is okay, because specializations lift uniquely along closed embeddings. It is now sufficient to show that $f^{-1}x \neq \varnothing$: any point in $f^{-1}x$ is a specialization of x' by definition. But f is integral, so every fiber is nonempty. □

Exercise 36.4. Prove that integral morphisms are universally closed.

Solution. We can assume the target is affine by passing to an open cover. The source is then affine as well. Any closed subset of the source is representable by an affine scheme, so it is sufficient to show $\operatorname{Spec} B \to \operatorname{Spec} A$ is closed when B is integral over A.

Let A' be the image of A in B. It is sufficient to show that $\operatorname{Spec} B \to \operatorname{Spec} A'$ is closed since $\operatorname{Spec} A'$ is closed in $\operatorname{Spec} A$. But $A' \subset B$ is an integral extension, so it is surjective.

This shows integral morphisms are closed. But the base change of an integral morphism is integral, so integral morphisms are universally closed. □

Exercise 36.5. Show that finite morphisms are proper.

Solution. As finite morphisms are affine, they are quasicompact and separated. They of finite type by definition. They are integral so they are universally closed. □

Exercise 36.6 ([Vak14, Exercise 11.1.E])**.** Suppose $f : X \to Y$ is an integral *extension.* Prove that $\dim X = \dim Y$.

Solution. A chain of specializations in Y lifts to a chain of specializations in X so $\dim X \geq \dim Y$. Conversely, a chain of specializations in X maps to a chain of specializations in Y, *which must have the same length, by uniqueness of lifts*, so $\dim Y \geq \dim X$. □

Exercise 36.7. Let k be a field. Prove that for any maximal ideal $\mathfrak{p}$ of $A = k[x_1, \ldots, x_n]$, we have $\dim A_{\mathfrak{p}} = n$. (Hint: Reduce to the case of an algebraically closed field k and use the Nullstellenstaz.)

Theorem 36.8. *(i) A morphism of schemes is integral if and only if it is both affine and universally closed.*

(ii) A morphism of schemes is finite if and only if it is both affine and proper.

Exercise 36.9 ([ano], [Sta15, Tag 01WM])**.** Prove the theorem using the following steps:

(i) Suppose $\varphi : A \to B$ is injective and the induced map $\operatorname{Spec} B \to \operatorname{Spec} A$ is closed and $f \in A$ is an element such that $\varphi(f) \in B^*$. Show that $f \in A^*$.

Solution. The assumptions that $\operatorname{Spec} B \to \operatorname{Spec} A$ be closed and $\varphi : A \to B$ be injective combine to imply that $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective. (Proof: Let $\operatorname{Spec} C$ be the image of $\operatorname{Spec} B$ in $\operatorname{Spec} A$, with its reduced subscheme structure. Then $A \to B$ factors through a surjection $A \to C$. This must also be injective, hence an isomorphism.) Then for every $\mathfrak{p} \in \operatorname{Spec} A$, choose some $\mathfrak{q} \in \operatorname{Spec} B$ whose image is $\mathfrak{p}$. We have $f(\mathfrak{p}) = \varphi(f)(\mathfrak{q}) \neq 0$ so f does not vanish at any point of $\operatorname{Spec} A$. Therefore f is a unit in A. □

(ii) Suppose $\varphi : A \to B$ is an injection such that the induced map $\operatorname{Spec} B[t] \to \operatorname{Spec} A[t]$ is closed. Then φ is integral.

Solution. Choose $f \in A$ and consider the closed subset $V(tf - 1) \subset \operatorname{Spec} B[t]$. The image of this map in $\operatorname{Spec} A[t]$ is closed by assumption. Let us write $A[f^{-1}]$ for the image of $A[t] \to B[t]/(tf-1) = B[f^{-1}]$. Then $A[f^{-1}] \to B[f^{-1}]$ is injective and induces a closed map on $\operatorname{Spec} B[f^{-1}] \to \operatorname{Spec} A[f^{-1}]$, hence f^{-1} must be a unit in $A[f^{-1}]$. That is $f \in A[f^{-1}]$ also. Thus we can write $f = \sum a_i f^{-i}$ for some $a_i \in A$. In other words, f is integral over A. □

(iii) Complete the proof of the theorem.

Solution. We have seen that an integral morphism is affine (by definition) and universally closed. Conversely, we have just seen above that a universally closed morphism between affine schemes is integral. It follows that a universally closed affine morphism is integral because being integral is a local property.

The second part follows by adding finite type to both sides of the equivalence. □

36.2 Noether normalization

Theorem 36.10 (Noether normalization [Mum99, §I.1], [Vak14, 11.2.4])**.** *Suppose k is a field and B is an integral domain of finite type over k. Then there is a polynomial subring $A \subset B$ such that B is a finite extension of A (as a module).*

Proof. Present B as $k[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. By reordering, we can assume $x_1, \ldots, x_d$ are transcendental over k. We induct on $n - d$. If $d = n$ then $B = k[x_1, \ldots, x_n]$. Otherwise, let $B_0 \subset B$ be the subring generated by $x_1, \ldots, x_{n-1}$. By induction, there is a polynomial subring $A_0 = k[y_1, \dots, y_d] \subset B_0$ such that B_0 is finite over A_0. Then x_n satisfies some polynomial relation $f(x_n) = 0$ with coefficients in $y_1, \dots, y_d$.

Consider $z_i = y_i - y_1^{r_i}$ where the r_i are to be determined later. Let $a(y_1, \dots, y_d, x_n)$ be the monomial of f of highest total in the y_i. Then, viewed as a polynomial in the z_i, it becomes

$$a(y_1, z_2 + y_1^{r_2}, \dots, z_d + y_1^{r_d})$$

and this is monic when viewed as a polynomial in y_1. Moreover, if the r_i are chosen suitably large, the polynomial $f(y_1, z_2, \dots, z_d, x_n)$ will be monic as a polynomial in y_1. Finally, $y_i^{r_i} - y_i - z_i = 0$ for all i, so A_0, and hence B_0, is integral over $A = k[z_2, \dots, z_d, x_n]$ □

Corollary 36.10.1. *Suppose that B is an integral domain of finite type over a field k. Let K be the field of fractions of B. For any maximal ideal $\mathfrak{p} \subset B$, the dimension* $\dim B_{\mathfrak{p}}$ *coincides with* $\operatorname{tr.deg}_k K$.

Proof. We can assume that B is a finite extension of a polynomial ring A. Then specializations lift along $\operatorname{Spec} B \to \operatorname{Spec} A$ so $\dim B_{\mathfrak{p}} = \dim A_{\mathfrak{p}\cap A}$ for all primes $\mathfrak{p}$ of B. Likewise $\operatorname{tr.deg} \operatorname{frac} A = \operatorname{tr.deg} \operatorname{frac} B$ so we can assume $A = B$. But now we know that the dimension of $k[x_1, \dots, x_n]$ at a maximal ideal is n. □

37 Chevalley's theorem

Reading 37.1. [Vak14, §7.4],

hm:generic-freeness

Theorem 37.2. *Let A be a noetherian integral domain, B an A-algebra of finite type, and M is a B-module of finite type. There is a non-zero $f \in A$ such that $A[f^{-1}] \otimes_A M$ is free as a $A[f^{-1}]$-module.*

Proof. We do this by induction on the number of generators of B as an A-algebra. If there are no generators then B is a quotient of A, so a finitely generated B-module can be regarded as a finitely generated A-module. Let K be the field of fractions of A. Choose a map $p : A^n \to M$ that induces an isomorphism $K^n \to K \otimes_A M$. Then let N be the kernel of p and let N' be the cokernel. We get $K \otimes_A N = 0$. Since N and N' are finitely generated (A is noetherian and A^n and M are finitely generated) there is some $f \in A$ such that $fN = fN' = 0$. Then $N \otimes_A A[f^{-1}] = N' \otimes_A A[f^{-1}] = 0$ so $A[f^{-1}] \otimes_A M \simeq A[f^{-1}]^n$.

In the induction step, we assume that the theorem is true when B has one fewer generator. Let t be one of a finite set of generators of B. Consider the maps

$$M/tM \to t^n M/t^{n+1}M$$
$$x \bmod tM \mapsto t^n x \bmod t^{n+1}M$$

with kernels N_n. Then the N_n are increasing submodules of M/tM, which is a finitely generated B/tB-module, so they stabilize because B is noetherian, say to N. Therefore the modules $t^nM/t^{n+1}M$ run through only finitely many distinct isomorphism classes. By induction, we can therefore find a $f \in A$ such that $t^nM/t^{n+1}M$ is a free A-module for all n.

Since $t^nM/t^{n+1}M$ is a free A-module we can choose elements of t^nM whose images in $t^nM/t^{n+1}M$ form a basis. Taking the collection of all of these, for varying n, we obtain a basis for M over A. Indeed, suppose that there is a relation $\sum a_i x_i = 0$ with all $a_i \neq 0$. Let n be the smallest degree among the x_i. Then reduce modulo $t^{n+1}M$ to get a relation among basis elements of $t^nM/t^{n+1}M$. This is necessarily zero, so $a_i = 0$ whenever x_i has minimal degree. □

Definition 37.3 ([GD71, Définition 0.2.3.1, 0.2.3.2, 0.2.3.10])**.** An open subset U of a scheme X is said to be *retrocompact* if the inclusion $U \subset X$ is quasicompact.

Let X be an affine scheme. A subset of X is called *constructible* if it can be constructed using only the retrocompact open subsets of X and a finite process of intersections and passages to complementary subsets.

A subset Z of a scheme X is said to be *locally constructible* if there is an cover of X by affine open subschemes U such that the intersection $Z \cap U$ is a constructible subset of U.

Exercise 37.4. Show that an open subset of an affine scheme is retrocompact if and only if it is the complement of a closed subscheme of finite presentation.

Exercise 37.5. Give an example of an open subset of an affine scheme that is not a retrocompact open subset.

Exercise 37.6. Let $f : X \to Y$ be a morphism of schemes. Show that the pullback of a constructible subset of Y to X is constructible.

Solution. The pullback of a retrocompact open subset is retrocompact, preimage stabilizes intersections and complements. □

Exercise 37.7. Show that a locally constructible subset of an affine scheme is constructible.

Solution. Suppose $Z \cap U_i$ is constructible for all i and $\bigcup U_i = X$. Shrinking the U_i if necessary, we can assume they are affine and in particular retrocompact. Then $Z = \bigcup (Z \cap U_i)$ is a constructible subset of X. □

Exercise 37.8. Show that a subset of a *noetherian* scheme is constructible if and only if it is a finite union of underlying subsets of locally closed subschemes.

Exercise 37.9. Let X be a scheme. For each open $U \subset X$, let $\mathscr{S}(U)$ be the collection of all subsets of U and let $\mathscr{C}(U)$ be the set of all locally constructible subsets.

(i) Show that $\mathscr{S}$ is a sheaf and $\mathscr{C}$ is a subsheaf.

(ii) Show that $\mathscr{C}$ is the smallest subsheaf of $\mathscr{S}$ such that $\mathscr{S}(U)$ includes $D(f)$ and $V(f)$ when $U = \operatorname{Spec} A$ and $f \in A$, and is stable under finite union and finite intersection.

Exercise 37.10. Show that a subset of an affine scheme X is constructible if and only if it is a finite union of sets of the form $U \cap V$ where U is a retrocompact open and V is a closed subset of finite presentation.

Solution. Every locally closed subset is constructible, since closed subschemes of noetherian schemes are always of finite presentation. We have $(U \cap V) \cap (U' \cap V') = (U \cap U') \cap (V \cap V')$. It follows that this class of sets is stable under finite intersection. To prove it is stable under complementation, it is sufficient to show that the complement of $U \cap V$ is also of this form. But the complement of $U \cap V$ is $(X \smallsetminus U) \cup (X \smallsetminus V)$ which is clearly a finite union of the closed subset of finite presentation $X \smallsetminus U$ and the retrocompact open subset $X \smallsetminus V$. □

Theorem 37.11. *Let $f : X \to Y$ be a quasicompact morphism that is locally of finite presentation and $Z \subset X$ a constructible subset. Then $f(Z)$ is a constructible subset of Y.*

Proof. Since constructibility is a local condition, we can assume Y is affine. Since X quasicompact relative to Y, it is quasicompact, so it has a finite cover by affine schemes. We can therefore find a surjection $X' \to X$ with X' affine. Let Z' be the preimage of Z. Then Z' is a constructible subset of X' with the same image in Y as Z. We can therefore assume X is affine.

As Z is constructible, it is a finite union of intersections $U \cap (X \smallsetminus V)$ where U and V are retrocompact opens. It is therefore sufficient to consider $Z = U \cap (X \smallsetminus V)$. Since X is quasicompact, U is quasicompact, in particular has a finite cover by principal open affine subsets. We can therefore assume Z is a closed subset whose complement is quasicompact. Therefore Z can be given the structure of an affine scheme of finite presentation over Y.

We are therefore reduced to the situation where X and Y are affine and $Z = X$. We can find a cartesian diagram

$$\begin{tikzcd} X \arrow[r, "f"'] \arrow[d] & X' \arrow[d, "f'"] \\ Y \arrow[r, "\pi"] & Y' \end{tikzcd}$$

where Y' is of finite type over $\mathbf{Z}$, hence is noetherian. Then $y \in f(X)$ if and only if $\pi(y) \in f'(X')$. That is, $\pi^{-1}f'(X') = f(X)$. We can therefore assume $Y = Y'$ and $X = X'$ and that Y is noetherian.

Now we argue by noetherian induction. Assume that $f(X)$ is constructible when $f(X)$ is contained in a closed subset of Y other than Y itself. By the previous theorem there is an open $U \subset Y$ such that $f^{-1}U$ is free over U (i.e., $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$ where B is a free A-module). If $f^{-1}U = \varnothing$ (it's free of rank 0) then we are done by induction. Otherwise, $f(X)$ contains U and we replace X by $X \smallsetminus f^{-1}U$ (with its reduced subscheme structure) and proceed by induction. □

37.1 A criterion for openness

Exercise 37.12. Show that a subset of an affine scheme X is constructible if and only if it is the image of a morphism of finite presentation.

Solution. Assume $X = \operatorname{Spec} A$. If $Z \subset X$ is constructible then Z is a finite union of subsets of the form $U \cap (X \smallsetminus V)$ where U and V are retrocompact open subsets of X. As U is quasicompact we can assume $U = D(g)$ for some $g \in A$. Write $V = \bigcup D(f_i)$. This union can be taken finite because V is quasicompact. Therefore $X \smallsetminus V = |V(f_1, \dots, f_n)|$ so $Z = D(g) \cap V(f_1, \dots, f_n)$ is the support of $A[g^{-1}]/(f_1, \dots, f_n)$. In particular, it is of finite presentation.

The reverse implication was proved in the last section. □

Exercise 37.13. Show that a locally constructible subset of a scheme X is open if and only if it is stable under generization.

Solution. Let $Z \subset X$ be locally constructible and stable under generization. Then $X \smallsetminus Z$ is locally constructible and stable under specialization. Then it is the image of a morphism of finite presentation and stable under specialization, hence closed, so Z must have been open. □

37.2 Nullstellensatz

Exercise 37.14 ([Vak14, 7.4.3]). Suppose that K is a field extension of k that is finitely generated as a k-algebra. Show that K is finitely generated as a k-module.

Solution. Let ξ be an element of K. Consider the map $k[x] \to K$ sending x to ξ. This corresponds to a map $\operatorname{Spec} K \to \operatorname{Spec} k[x] = \mathbf{A}_k^1$. The image of this map is a constructible subset of $\mathbf{A}_k^1$ consisting of a single point. Therefore it is not the generic point of $\mathbf{A}_k^1$. It is therefore $\operatorname{Spec} k[x]/(f)$ for some nonzero $f \in k[x]$. That is, $f(x) = 0$ in K. Thus ξ satisfies some polynomial equation and K is an algebraic extension of k. $\square$

Chapter 12

Flatness

38 Flatness I

Reading 38.1. [Vak14, Chapter 24], [Har77, §III.9]

Definition 38.2. Let A be a commutative ring. An A-module M is said to be *flat* if $N \otimes_A M$ is an exact functor of N. An A-algebra B is said to be *flat* if it is flat as an A-module.

Definition 38.3. A morphism of schemes $f : X \to Y$ is said to be *flat* if $f^* : \mathbf{QCoh}(Y) \to \mathbf{QCoh}(X)$ is an exact functor. More generally, a quasicoherent sheaf $\mathscr{F}$ on X is said to be *Y-flat* if $\mathscr{F} \otimes_{\mathcal{O}_X} f^*\mathscr{G}$ is an exact functor of $\mathscr{G} \in \mathbf{QCoh}(Y)$.[1]

Exercise 38.4. Show that $f : X \to Y$ is flat if and only if there are open charts by maps $\operatorname{Spec} B \to \operatorname{Spec} A$ where B is a flat A-algebra.

Exercise 38.5 ([Har77, Proposition 9.2], [Vak14, Exercises 24.2.A, 24.2.C, 24.2.D, 24.2.E])**.**

(i) Show that open embeddings are flat.

(ii) Let k be a field. Show that all maps $X \to \operatorname{Spec} k$ are flat.

(iii) Show that the base change of a flat map is flat.

(iv) Show that $\mathbf{A}^n_Y \to Y$ is flat.

(v) Show that a composition of flat maps is flat.

38.1 Openness

Exercise 38.6. (i) Let $f : X \to Y$ be a flat morphism. Show that the image of f is stable under generization.

[1]These definitions can be made more generally for sheaves of $\mathcal{O}_X$-modules and $\mathcal{O}_Y$-modules. The result is equivalent for schemes.

Solution. It is sufficient to assume $Y = \operatorname{Spec} A$ is the spectrum of a valuation ring with closed point y and open point y' where $f^{-1}y \neq \varnothing$. Choose $x \in f^{-1}y$. Replace X with an affine open neighborhood $\operatorname{Spec} B$ of x. Let K be the field of fractions of A. If $f^{-1}y' = \varnothing$ then $K \otimes_A B = 0$ so there is some nonzero $t \in A$ where $tB = 0$. But A is a domain so multiplication by t is an injection on A, and B is flat, so multiplication by t is therefore also an injection on B. In particular, $tB \neq 0$ for all nonzero $t \in A$. □

(ii) Flat morphisms of finite presentation are open.

Solution. The image of a morphism of finite presentation is locally constructible. Flatness implies it is also stable under generization, hence open. □

38.2 Generic flatness

Reading 38.7. [Vak14, §§24.5.8–24.5.13]

hm:generic-flatness **Theorem 38.8.** *Suppose $f : X \to Y$ is a morphism of finite type between noetherian schemes with Y integral. Then there is a dense open subset of Y over which X is flat.*

Proof. In the case where X and Y are both affine, this follows from Theorem 37.2.

We can assume Y is affine, since any open subset of Y is dense. Choose a cover of X by an affine scheme X' that is a local isomorphism in the Zariski topology. Then X is flat over Y if and only if X' is flat over Y and we reduce to the previous case. □

ning-stratification **Exercise 38.9** (Flattening stratification)**.** Under the assumption of the theorem, show that there is a stratification of Y into locally closed subschemes Y_i such that $f^{-1}Y_i$ is flat over Y_i.[2]

Exercise 38.10. Generalize the theorem to a quasicoherent sheaf of finite type on X.

38.3 Fiber dimension

Let $f : X \to Y$ be a morphism of schemes. The *fiber* of f over $y \in Y$ is the scheme $f^{-1}y = y \times_Y X$. We write $\dim_x X = \dim \mathcal{O}_{X,x}$.

Theorem 38.11 ([Har77, Proposition 9.5])**.** *Let $f : X \to Y$ be a flat morphism between locally noetherian schemes. For any $x \in X$ we have*

$$\dim_x X_y + \dim_{f(x)} Y = \dim_x X.$$

Exercise 38.12. (i) Give an example of a non-flat morphism of noetherian schemes where the conclusion of the theorem fails.

Solution. Let $Y = \operatorname{Spec} \mathbf{C}[x, y]$ and let $X = \operatorname{Spec} \mathbf{C}[x, y/x]$. Both are isomorphic to $\mathbf{A}^2_{\mathbf{C}}$, hence have dimension 2, but consider a point p with x-coordinate 0 in X. Then $f(p) = (0, 0) \in Y$ and the fiber over $(0, 0)$ is $\operatorname{Spec} \mathbf{C}[x, y/x]/(x, x(y/x)) = \operatorname{Spec} \mathbf{C}[y/x] \simeq \mathbf{A}^1_{\mathbf{C}}$, which has dimension 1. □

[2] This is weaker than the usual notion. Generally a flattening stratification is also required to be a *universal flattening* [Sta15, Tag 052F].

Exercise 38.13. Prove the theorem:

(i) Show it is sufficient to assume $Y = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$ and both A and B are local rings.

Solution. The localization of a noetherian ring is noetherian. $\square$

(ii) Pick $t \in A$ not contained in any minimal prime. Show that $\dim A/t = \dim A - 1$.

Solution. We certainly have $\dim A/t \leq \dim A - 1$ by interpreting dim as the maximal length of a chain of prime ideals. We also have $\dim A/t \geq \dim A - 1$ by interpreting dim as the minimal number of generators of an ideal of definition. $\square$

(iii) With t as above, show that f^*t is not contained in any minimal prime of B. (Hint: Use the fact that the image of f is stable under generization, hence contains all generic points of $\operatorname{Spec} A$.)

Solution. Suppose $\mathfrak{p}$ is a minimal prime of B. Choose an open neighborhood $\operatorname{Spec} B' \subset \operatorname{Spec} B$ of $\mathfrak{p}$ not containing any other minimal prime. Then $f(\mathfrak{p})$ is a minimal prime of $\operatorname{Spec} A$ because the image of $\operatorname{Spec} B'$ is stable under generization. Thus the vanishing locus of t in $\operatorname{Spec} B$ does not include $\mathfrak{p}$. $\square$

(iv) Conclude that $f^{-1}V(t) \subset X$ has dimension $\dim X - 1$.

(v) Use induction on $\dim Y$ to deduce that $\dim X = \dim Y + \dim f^{-1}y$.

Solution. The only thing left is the base case. But then $y = Y_{\mathrm{red}}$ and $\dim X = \dim f^{-1}(Y_{\mathrm{red}})$ in general. $\square$

Theorem 38.14. *Let $f : X \to Y$ be a morphism of finite type between locally noetherian schemes. Then $\dim X_{f(x)}$ is an upper semicontinuous function of $x \in X$.*

Proof. First of all, fiber dimension is a constructible function on X. Indeed, by Exercise 38.9, we can stratify Y into locally closed subschemes such that fiber dimension is constant over each of the strata. These strata pull back to locally closed subschemes of X on which the fiber dimension is constant.

To show fiber dimension is upper semicontinuous, we therefore only need to show that when $x' \rightsquigarrow x$ is a specialization in X then $\dim_{x'} X_{f(x')} \leq \dim_x X_{f(x)}$. We can therefore replace X with the local ring at x and replace Y with the local ring at $f(x)$. We can also reduce X by the prime ideal corresponding to x' and we can reduce Y by the prime ideal corresponding to $f(x')$. Then $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ where both A and B are noetherian local domains. Let $K = \operatorname{frac} A$ and $L = \operatorname{frac} B$ be the fields of fractions.

Choose a polynomial subring $K[x_1, \ldots, x_d] \subset K \otimes_A B$ such that $K \otimes_A B$ is a finite extension. Let $A' = A[x_1, \ldots, x_d]$. Then every element of B satisfies a polynomial over $K[x_1, \ldots, x_d]$, not necessarily monic. Let $X' = \operatorname{Spec} A'$. Then we have maps

$$X \xrightarrow{g} X' \xrightarrow{h} Y$$

with $f = hg$. The dimensions of the fibers of h are constant. We have $\dim_x X_{f(x)} \geq \dim_x X_{g(x)} + \dim_{g(x)} X'_{f(x)} \geq d$, as desired. $\square$

Exercise 38.15. Let $f : X \to Y$ be a proper morphism of locally noetherian schemes. Show that fiber $\dim X_y$ is an upper semicontinuous function of $y \in Y$.

Solution. Let $Z \subset X$ be the set of points $x \in X$ such that $\dim_x X_{f(x)} \geq n$. Then $f(Z)$ is the set of points $y \in Y$ such that $\dim X_y \geq n$ and this is closed because f is proper. □

Exercise 38.16. Eliminate the noetherian hypotheses in the second theorem.

Solution. Work locally, so assume f is a morphism of finite presentation between affine schemes. Then it is the base change of a morphism of finite type between noetherian schemes. Apply the theorem in that case. □

38.4 Criteria for flatness

Reading 38.17. [Sta15, Tag 00MD], [Vak14, §24.6]

The homological criterion

Exercise 38.18. Let M be an A-module.

(i) Show that $M \otimes_A N$ is a right exact functor of N but is not exact in general.

(ii) Suppose that
$$0 \to N' \to N \to N'' \to 0 \tag{38.1}$$
eqn:23

is an exact sequence. Show that
$$0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$
is exact if *either* M or N is projective.

(iii) Let N be any A-module and choose a surjection $P_0 \to N$ where P is projective. Let P_1 be the kernel. Define T_1^P to be the kernel of $M \otimes_A P_1 \to M \otimes_A P_0$. Show that T_1^P depends on P only up to canonical isomorphism.

(iv) Write $\mathrm{Tor}_1(M, N)$ for the module constructed above. Show that there is an exact sequence
$$\mathrm{Tor}_1(M, N') \to \mathrm{Tor}_1(M, N) \to \mathrm{Tor}_1(M, N'') \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$
associated to any exact sequence (38.1).

(v) Prove that $\mathrm{Tor}_1(M, N) = 0$ if either M or N is projective.

Solution. This is true by definition if N is projective. If M is projective, choose P_0 and P_1 as above, and form the sequence
$$0 = \mathrm{Tor}_1(M, P_0) \to \mathrm{Tor}_1(M, N) \to M \otimes_A P_1 \to M \otimes_A P_0.$$
But M is projective, hence flat, so $M \otimes_A P_1$ injects into $M \otimes_A P_0$. □

(vi) Prove that $\mathrm{Tor}_1(M, N) = \mathrm{Tor}_1(N, M)$.

Solution. Choose $Q_0 \to M$ surjective with Q_0 projective and let Q_1 be the kernel. Choose P_1 and P_0 for N as before. Then we have a commutative diagram:

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
 & & & 0 \arrow[d] & \mathrm{Tor}_1(N,M) \arrow[d] & \\
 & & Q_1 \otimes_A P_1 \arrow[r] \arrow[d] & Q_1 \otimes_A P_0 \arrow[r] \arrow[d] & Q_1 \otimes_A N \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & Q_0 \otimes_A P_1 \arrow[r] \arrow[d] & Q_0 \otimes_A P_0 \arrow[r] \arrow[d] & Q_0 \otimes_A N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Tor}_1(M,N) \arrow[r] & M \otimes_A P_1 \arrow[r] \arrow[d] & M \otimes_A P_0 \arrow[r] \arrow[d] & M \otimes_A N \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & 0 &
\end{tikzcd}$$

Now apply the snake lemma. ☐

(vii) Prove that M is flat if and only if $\mathrm{Tor}_1(M,N) = 0$ for all A-modules N if and only if $\mathrm{Tor}_1(N,M) = 0$ for all A-modules N.

Exercise 38.19. (i) Show that an A-module M is flat if and only if for every injection of A-modules $N' \to N$, the induced map

$$M \otimes_A N' \to M \otimes_A N$$

is injective.

Solution. Tensor product is always right exact. ☐

(ii) Show that in the previous condition, it is sufficient to assume N' and N are finitely generated.

Solution. First we show we can assume N' is finitely generated. Write $N' = \bigcup N_i'$ (filtered union) with the N_i' finitely generated. Then

$$\ker(M \otimes \varinjlim N_i' \to M \otimes_A N) = \varinjlim \ker(M \otimes N_i' \to M \otimes_A N) = 0.$$

Similarly, we can now consider a filtered union $N = \bigcup N_i$ with $N' \subset N_i$ for all i and get

$$\ker(M \otimes N' \to M \otimes \varinjlim N_i) = \varinjlim \ker(M \otimes N' \to M \otimes N_i).$$

We can therefore assume N' and N are finitely generated. ☐

Exercise 38.20. Let M be a finitely generated A-module. Show M is flat if and only if $I \otimes_A M \to IM$ is a bijection for all ideals $I \subset A$.

Solution. Supposing the condition, let N be a finitely generated A-module. Let x be one of the generators. Then we have an exact sequence

$$0 \to I \to A \to Ax \to 0$$
$$0 \to Ax \to N \to N/Ax \to 0$$

By the condition, $\mathrm{Tor}_1(M, Ax) = 0$. By induction on the number of generators, $\mathrm{Tor}_1(M, N/Ax) = 0$. Therefore $\mathrm{Tor}_1(M, N) = 0$ using the long exact sequence:

$$\mathrm{Tor}_1(M, Ax) \to \mathrm{Tor}_1(M, N) \to \mathrm{Tor}_1(M, N/Ax)$$

□

The local criterion

Exercise 38.21. Show that M is flat if and only if $M_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ of A.

Theorem 38.22 ([Vak14, Theorem 24.6.1])**.** *Suppose that $A \to B$ is a local homomorphism of noetherian local rings and M is a finitely generated B-module. Let k be the residue field of A. Then M is A-flat if and only if* $\mathrm{Tor}_1^A(M, k) = 0$.

It is clear that flatness of M implies $\mathrm{Tor}_1^A(M, k) = 0$. We work on the converse. Assume for the rest of the discussion that $\mathrm{Tor}_1^A(M, k) = 0$.

Exercise 38.23. Show that $\mathrm{Tor}_1^A(M, N) = 0$ if $\mathfrak{m}^n N = 0$ for some positive integer n. (Hint: Reduce to the case where $\mathfrak{m}N = 0$ using the long exact sequence, and then observe that $N \simeq k^{\oplus r}$ as an A-module in that case.)

Exercise 38.24. Use the Artin–Rees lemma to prove the following statements about modules over a noetherian local ring B with maximal ideal $\mathfrak{n}$:

(i) If P is a finitely generated B-module and Q is a submodule then $Q \cap \mathfrak{n}^k P \subset \mathfrak{n}^{k-\ell} Q$ for some ℓ and all $k \gg 0$.

Solution. We have $Q \cap \mathfrak{n}^k P = \mathfrak{n}^{k-\ell}(Q \cap \mathfrak{n}^\ell P)$ for some ℓ and all $k \gg 0$. But $Q \cap \mathfrak{n}^\ell \subset Q$ so $Q \cap \mathfrak{n}^k P \subset \mathfrak{n}^{k-\ell} Q$. □

(ii) If P is a finitely generated B-module then $\bigcap \mathfrak{n}^k Q = 0$.

Solution. Let Q be the intersection $\bigcap \mathfrak{n}^k P$. Then consider $\mathfrak{n}^n Q$. By Artin–Rees, we have $Q = Q \cap \mathfrak{n}^k P = \mathfrak{n}(Q \cap \mathfrak{n}^{k-1} P) = \mathfrak{n} Q$ for $k \gg 0$. Therefore $Q = 0$ by Nakayama's lemma. □

Proof of Theorem **??**. This proof is adapted from [Vak14, §24.6.3].

The idea is going to be to approximate an arbitrary finitely generated B-module N by the quotients $N/\mathfrak{m}^n N$. We choose a resolution

$$0 \to Q \to P \to N \to 0$$

where P and Q are both finitely generated, and P is free. Form a commutative diagram with exact rows cand columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Q \cap \mathfrak{m}^n P \arrow[r] \arrow[d] & \mathfrak{m}^n P \arrow[r] \arrow[d] & \mathfrak{m}^n N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Q \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Q/(Q \cap \mathfrak{m}^n P) \arrow[r] \arrow[d] & P/\mathfrak{m}^n P \arrow[r] \arrow[d] & N/\mathfrak{m}^n N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Tensor this diagram with M, taking advantage of the fact that $\mathrm{Tor}_1(M, L) = 0$ whenever $\mathfrak{m}^n L = 0$, and only draw the important part:

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
 & & M \otimes (Q \cap \mathfrak{m}^n P) \arrow[d] & \\
0 \arrow[r] & \mathrm{Tor}_1(M, N) \arrow[r] & M \otimes Q \arrow[r] \arrow[d] & M \otimes P \arrow[d] \\
 & 0 \arrow[r] & M \otimes (Q/(Q \cap \mathfrak{m}^n P)) \arrow[r] & M \otimes (P/\mathfrak{m}^n P)
\end{tikzcd}$$

Note that $\mathrm{Tor}_1(M, N) \subset M \otimes Q$ is contained in the image of $M \otimes (Q \cap \mathfrak{m}^n P)$ for all n. By Artin–Rees, $Q \cap \mathfrak{m}^n P \subset \mathfrak{m}^n Q$ for $n \gg 0$. Therefore $\mathrm{Tor}_1(M, N)$ is contained in the image of $M \otimes \mathfrak{m}^n Q$, which is just $\mathfrak{m}^n(M \otimes Q)$. If $\mathfrak{n}$ denotes the maximal ideal of B then $\mathrm{Tor}_1(M, N)$ is also contained in $\mathfrak{n}^n(M \otimes Q)$. But $M \otimes Q$ is a finitely generated B-module. Therefore, again by Artin–Rees, $\bigcap \mathfrak{n}^n(M \otimes Q) = 0$. $\square$

The slicing criterion

Theorem 38.25 ([Vak14, Theorem 24.6.5])**.** *Suppose $A \to B$ is a local homomorphism of local rings and t is not a zero divisor in A. Then B/tB is flat over A/tA if and only if B is flat over A and t is not a zero divisor in B.*

Exercise 38.26. Prove the theorem:

(i) Suppose B is flat over A. Prove that t is not a zero divisor in B if and only if $\mathrm{Tor}_1^A(B, A/tA) = 0$.

Solution. Tensor the exact sequence

$$0 \to A \xrightarrow{t} A \to A/tA \to 0$$

by B. $\square$

(ii) Suppose B is flat over A. Show that $B \otimes_A A'$ is flat over A'. Conclude that B/tA is flat over A/tA.

(iii) Suppose that t is not a zero divisor in B. Show that $\mathrm{Tor}_1^A(k, B) = \mathrm{Tor}_1^{A/tA}(k, B/tB)$.

Solution. Choose a surjection $P_0 \to B$ with P_0 free and let P_1 be the kernel. We have an exact sequence:

$$0 \to P_1 \to P_0 \to B \to 0$$

Tensor with A/tA. Using that $\mathrm{Tor}_1(B, A/tA) = 0$ gives an exact sequence

$$0 \to P_1/tP_1 \to P_0/tP_0 \to B/tB \to 0$$

with P_0/tP_0 free over A/tA. Therefore we have a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}_1^A(tB, k) \arrow[r] \arrow[d] & tP_1 \otimes_A k \arrow[r] \arrow[d] & tP_0 \otimes_A k \arrow[r] \arrow[d] & tB \otimes_A k \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Tor}_1^A(B, k) \arrow[r] \arrow[d] & P_1 \otimes_A k \arrow[r] \arrow[d] & P_0 \otimes_A k \arrow[r] \arrow[d] & B \otimes_A k \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Tor}_1^{A/tA}(B/tB, k) \arrow[r] \arrow[d] & P_1/tP_1 \otimes_{A/tA} k \arrow[r] \arrow[d] & P_0/tP_0 \otimes_{A/tA} k \arrow[r] \arrow[d] & B/tA \otimes_{A/tA} k \arrow[r] \arrow[d] & 0 \\
 & {} & 0 & 0 & 0 &
\end{tikzcd}$$

But $\mathrm{Tor}_1^A(tB, k) \to \mathrm{Tor}_1^A(B, k)$ is the zero map, since t maps to 0 in k. $\square$

(iv) Prove the theorem.

Solution. If B is flat over A then $\mathrm{Tor}_1^A(B, A/tA) = 0$ so t is not a zero divisor in B, and by a previous part B/tB is flat over A/tA.

Conversely, if t is not a zero divisor in B then $\mathrm{Tor}_1^A(B, k) = \mathrm{Tor}_1^{A/tA}(B/tB, k)$. But if B/tB is flat over A/tA then $\mathrm{Tor}_1^{A/tA}(B/tB, k) = 0$ so B is flat over A. $\square$

The infinitesimal criterion

The equational criterion

`sec:bezout-2`

38.5 Bézout's theorem

In our proof of Bézout's theorem in Section A, we showed that there was an open subset $U \subset \mathbf{A}^N$ such that $p^{-1}U$ is proper over U. By construction, X is affine over $\mathbf{A}^N$ so $p^{-1}U$ is both proper and affine, hence finite over U.

Exercise 38.27. Show that $p^{-1}U$ is flat over U.

Solution. Let $A = \mathbf{Z}[t_1, \ldots, t_N]$ and let $B = A[x,y]/fA[x,y]$. To see that this is flat over A, it is sufficient to replace A by one of its local rings, say with maximal ideal $\mathfrak{m}$. As long as f is injective modulo $\mathfrak{m}$—that is, f is not contained in $\mathfrak{m}$—we know B will be flat at $\mathfrak{m}$. But if f were contained in $\mathfrak{m}$ then the fiber of X over $\mathfrak{m}$ would be $\mathbf{A}^2$, which is not proper. Therefore B is flat over A.

Now consider B/gB. To show this is flat, it is again enough to show that g is not a zero divisor modulo any maximal ideal in A. But if $gh = 0$ in B, with $h \neq 0$, then gh is a multiple of f in $A[x,y]$. Now, $A[x,y]$ is a unique factorization domain, so there must be some irreducible q such that q^n divides f but q^n does not divide h. Then q must divide g.

Then $p^{-1}U = \mathbf{A}^2_U \cap V(f,g)$ contains $V(q)$ which is 1-dimensional, so $p^{-1}V$ is not finite over U. □

Exercise 38.28. Show that a flat module that is of finite presentation is locally free.

Solution. It is sufficient to show that a flat module of finite presentation over a local ring is free. Let A be a local ring, M an A-module of finite presentation, and k the residue field of A. Choose elements $x_1, \ldots, x_n$ of M forming a basis of $M \otimes_A k$. Then by Nakayama's lemma we get a surjection $A^n \to M$. Let N be the kernel. Then since M is flat, the sequence

$$0 \to N \otimes_A k \to k^n \to M \otimes_A k \to 0$$

is exact. But the map $k^n \to M \otimes_A k$ is an isomorphism by assumption, so $N \otimes_A k = 0$. Now, M is of finite presentation, so N is finitely generated. Nakayama's lemma (again) implies $N = 0$. □

Exercise 38.29. Conclude that $\dim_{\mathbf{k}(q)} \mathcal{O}_{p^{-1}(q)}$ is independent of $q \in U$.

Solution. Then $p_* \mathcal{O}_{p^{-1}U}$ is a quasicoherent sheaf that is flat and finitely generated (hence of finite presentation, because we are in a noetherian situation). In particular, it is locally free. □

39 Flatness II

40 Flatness III

Chapter 13

Projective space

41 Group schemes and quotients

41.1 Graded rings and quotients

Let $X = \operatorname{Spec} A$ be an affine scheme with an action of $\mathbf{G}_m$. This corresponds to a grading of A by $\mathbf{Z}$, as we saw in the last section.

def:fixed-locus

Definition 41.1. Let G be an algebraic group acting on a scheme X. The *fixed locus* of X is the functor $X^G \subset X$ consisting of all $x \in X$ such that $g.x = x$ for all $g \in G$. More precisely, $X^G(S)$ is the set of all $x \in X(S)$ such that for all S-schemes T and all $g \in G(T)$ we have $g.x\big|_T = x\big|_T$.

Exercise 41.2. Let $\mathbf{G}_m$ act on an affine scheme $X = \operatorname{Spec} A$. Show that the fixed locus is $V(A_+)$ where A_+ is the ideal generated by elements of nonzero degree.

Solution. Suppose that $f \in A_+$ is a homogeneous element. We argue that f is zero on X^G. Indeed, consider a map $s : S \to X^G$. Then $f(ts)$ is a map $S \times \mathbf{G}_m \to X^G$ coincides with $f(t)$, by definition. That is $f(ts) - f(s) = 0$. On the other hand, $f(ts) = t^n f(s)$. We conclude that $(t^n - 1)f = 0$ as an element of the ring $A[t, t^{-1}]$. But $n \neq 0$ by assumption so $t^n - 1$ is not a zero divisor, hence $f = 0$.

Conversely, we argue that $\mathbf{G}_m$ acts trivially on $\operatorname{Spec} A/A_+$. Indeed, suppose that $s : S \to \operatorname{Spec} A/A_+ \subset X$ is a morphism of schemes. Then for any $f \in A$, we have

$$f(ts) = \sum t^n f_n(s) = \sum f_n(s) = f(s).$$

Thus s is fixed by $\mathbf{G}_m$. $\square$

Definition 41.3. Let X be a scheme with an action of an algebraic group G. If it exists, the initial G-morphism from X to a scheme on which G acts trivially is called the *quotient* of X by G. It is denoted X/G if it exists.

Exercise 41.4. Let $\mathbf{G}_m$ act on $X = \operatorname{Spec} A$. Show that the $D(f)$, as f ranges among homogeneous elements of A, form a basis for the $\mathbf{G}_m$-invariants open subsets of X.

Solution. Suppose U is invariant and $\mathfrak{p} \in U$. Then there is some $g \in A$ such that $g \notin \mathfrak{p}$ and $D(g) \subset U$. If $Z = X \smallsetminus U$ then $g(Z) = 0$. But Z is $\mathbf{G}_m$-invariant, so all of the homogeneous components of g vanish on Z. But at least one of the homogeneous components—say g_n—is not contained in $\mathfrak{p}$. Therefore $\mathfrak{p} \in D(g_n) \subset U$. $\square$

Exercise 41.5. Show that when an algebraic group G acts on $X \times G$ by $g.(x,h) = (x, gh)$, the quotient $(X \times G)/G$ is X.

Solution. Consider a G-invariant $f : X \times G \to Y$. Define $f'(x) = f(x,e)$. Then $f(x,g) = f(g.(x,e)) = f(x,e) = f'(x)$ so f factors uniquely through the projection $X \times G \to G$. $\square$

thm:affine-quotient

Theorem 41.6 ([MFK, Chapter 1, Theorem 1.1])**.** *Suppose that $\mathbf{G}_m$ acts on an affine scheme $X = \operatorname{Spec} A$, corresponding to a grading $A = \sum A_n$. Show that $X/\mathbf{G}_m$ exists and is equal to $\operatorname{Spec} A_0$.*[1]

Proof. The proof is adapted from [MFK, Chapter 1, Theorem 1.1] and [MFK, Chapter 0, §2, Remark (6)].

Let $Y = \operatorname{Spec} A_0$. We certainly have a $\mathbf{G}_m$-invariant map $\pi : X \to Y$ from the inclusion $A_0 \subset A$. First note that if $W \subset \operatorname{Spec} A$ is a $\mathbf{G}_m$-invariant closed subscheme then $\pi(W)$ is closed. Indeed, suppose that y is in the closure of $\pi(W)$. Let $\mathfrak{p}$ be the prime ideal of A_0 corresponding to y. Then $\pi^{-1}\{y\} = V(\mathfrak{p}A)$. We have $(\mathfrak{p}A + I)_0 = \mathfrak{p} + I_0$. Thus $\overline{\pi(W \cap \pi^{-1}\{y\})} = \overline{\pi(W)} \cap \{y\}$. If $y \in \overline{\pi(W)}$ then $W \cap \pi^{-1}\{y\} \neq \varnothing$ so $y \in \pi(W)$.

Furthermore, if W_j are closed subschemes of X then $\pi(\bigcap W_j) = \bigcap \pi(W_j)$ by the same argument: Let I_j be the ideals defining the W_j. Then $\pi(V(\sum I_j)) = V(A_0 \cap \sum I_j) = V(\sum(A_0 \cap I_j)) = \bigcap \pi(V(I_j))$.

Now consider a map $f : X \to Z$. If Z is affine then $X \to Z$ certainly factors through Y, by the universal property of A_0 as the ring of invariants in A. If Z is not affine, choose an open cover $Z = \bigcup Z_i$ with each Z_i affine, and let $W_i \subset Z$ be the complement of Z_i. Then $f^{-1}W_i$ is open in X so $\pi(f^{-1}W_i)$ is a closed subset of Y. Let $U_i \subset Y$ be the complement of $\pi(f^{-1}W_i)$. Then $\bigcap \pi(f^{-1}W_i) = \pi(f^{-1}\bigcap W_i) = \varnothing$, so the U_i are an open cover of Y. Moreover, $\pi(f^{-1}W_i) \cap U_i = \varnothing$ so $\pi^{-1}(U_i) \cap f^{-1}(W_i) = \varnothing$ so $f(\pi^{-1}U_i) \cap W_i = \varnothing$ so $f(\pi^{-1}U_i) \subset Z_i$.

We can therefore choose a basis of open subsets V_j for Y with $V_j = D(f_j)$ such that $\pi^{-1}V_j \subset f^{-1}Z_i$ for some i. Now, $\pi^{-1}V_j = \operatorname{Spec} A[f_j^{-1}]$. As f_j has degree zero, we have $A[f_j^{-1}]_0 = A_0[f_j^{-1}]$. By the affine case of the theorem, mentioned above, the map $\pi^{-1}V_j \to Z_i$ factors through $\operatorname{Spec} A_0[f_j^{-1}] = V_j$ in a unique way. These maps therefore glue to give a factorization of $f : X \to Z$ through Y, as desired. $\square$

Exercise 41.7. Let $X = \operatorname{Spec} A$ be an affine scheme with an action of $\mathbf{G}_m$ corresponding to a grading $A = \sum A_n$. Let $X^\circ \subset X$ be the complement of $X^G \subset X$. Show that $X^\circ/\mathbf{G}_m$ exists and is equal to $\operatorname{Proj} A$:

(i) Show that $\mathbf{G}_m$ acts on X°.

(ii) Show that $D(f)$, for $f \in A$ homogeneous of nonzero degree, form a basis for the $\mathbf{G}_m$-invariant open subsets of X°.

(iii) Show that for each $f \in A_+$, the quotient $D(f)/\mathbf{G}_m$ exists and is equal to $\operatorname{Spec} A[f^{-1}]_0 = \operatorname{Proj} A[f^{-1}]$.

(iv) Construct a map $X^\circ \to \operatorname{Proj} A$ and show that it has the universal property of $X/\mathbf{G}_m$.

[1] This holds more generally for an action of a reductive group scheme.

Solution. We have already constructed the map on a basis of open subsets (using the open inclusions $D_+(f) = \operatorname{Proj} A[f^{-1}] \subset \operatorname{Proj} A$). Local agreement comes from the universality of the construction in Theorem 41.6 and yields the map.

To verify the universal property, imitate the proof of the unviersal property in Theorem 41.6. □

42 Quasicoherent sheaves and graded modules

Reading 42.1. [Har77, §II.5]

Definition 42.2. Let A be a graded ring. A *graded A-module* is an A-module M that is decomposed as a direct sum $M = \sum M_n$ with $A_m M_n \subset M_{m+n}$ for all $m, n \in \mathbf{Z}$.

This is essentially equivalent to Exercise 42.5. You might want to regard this exercise as a hint for or a step in the solution of that one.

Exercise 42.3. Let A be a graded ring, corresponding to a comultiplication map $\mu^* : A \to A[t, t^{-1}]$. Show that to give a grading on an A-module M is the same as to give a map $\mu^* : M \to M[t, t^{-1}]$ such that $\mu^*(fx) = \mu^*(f)\mu^*(x)$ for any $f \in A$ and $x \in M$. Interpret this geometrically as an isomorphism $\mu^* \widetilde{M} \simeq p^* \widetilde{M}$ where $p : \mathbf{G}_m \times \operatorname{Spec} A \to \operatorname{Spec} A$ and $\mu : \mathbf{G}_m \times \operatorname{Spec} A \to \operatorname{Spec} A$ are, respectively, the projection and the action.

Exercise 42.4 (Flat base change for global sections)**.** Consider a cartesian diagram of schemes:

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "p'"'] & Y' \arrow[d, "p"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

Assume that p is coherent (quasicompact and quasiseparated). Show that $g^* p_* \mathscr{F} = p'_* f^* \mathscr{F}$ for any quasicoherent sheaf $\mathscr{F}$ on Y'.

ex:qcoh-to-graded

Exercise 42.5. Let $X = \operatorname{Spec} A$ and let $Y = \operatorname{Proj} A$. Write $\pi : X^\circ \to Y$ for the projection and let $j : X^\circ \to X$ be the inclusion. Suppose that $\mathscr{F}$ is a quasicoherent sheaf on Y. Show that $\pi^* \mathscr{F}$.

(i) Show that $j_* \pi^* \mathscr{F}$ is a quasicoherent sheaf on X. Conclude that $j_* \pi^* \mathscr{F} = \widetilde{M}$ for some A-module M.

(ii) Show that M is naturally equipped with the structure of a graded A-module. (Hint: Pull back via the projection $p : \mathbf{G}_m \times X \to X$ and $\mu : \mathbf{G}_m \times X \to X$ and compare.)

Solution. Consider the modules $\mu^* j_* \pi^* \mathscr{F} = (\mathrm{id} \times j)_* \mu^* \pi^* \mathscr{F} = (\mathrm{id} \times j)_* p^* \pi^* \mathscr{F} = p^* j_* \pi^* \mathscr{F}$. Now, $p^* \widetilde{M} = M[t, t^{-1}]$ with the $A[t, t^{-1}]$-module structure by $t.(t^n x) = t^{n+1} x$. On the other hand, $\mu^* : M \to \mu^* \widetilde{M}$ gives a map $M \to M[t, t^{-1}]$. Define M_n to be the set of all $x \in M$ such that $\mu^*(x) = t^n x$.

In general $\mu^*(x) = \sum t^n x_n$ and because μ is an action, we have $x_n \in M_n$. If $f \in A_m$ and $x \in M_n$ then $\mu^*(fx) = \mu^*(f)\mu^*(x) = t^m f t^n x = t^{m+n} f x$ so $fx \in M_{m+n}$, as desired. □

Exercise 42.6. Suppose that A is a graded ring and M is a graded A-module. Let $X = \operatorname{Spec} A$, $Y = \operatorname{Proj} A$, and let $\pi : X^\circ \to Y$ be the projection. Let $\mathscr{F}$ be the sheaf on X associated to M. Define $\mathscr{G}(U) = \mathscr{F}(\pi^{-1} U)_0$ for all open $U \subset Y$.

(i) Show that $\mathscr{G}$ is a sheaf on Y.

(ii) Suppose that $\mathscr{F} = j_*\pi^*\mathscr{F}'$ for a quasicoherent sheaf on Y. Construct a canonical isomorphism $\mathscr{G} \simeq \mathscr{F}'$.

Exercise 42.7. Let $X = \operatorname{Spec} A$ and let $Y = \operatorname{Proj} A$. Assume that A_+ is generated by elements of degrees 1 and -1.

(i) Show that the category $\mathbf{QCoh}(Y)$ is equivalent to the category of graded quasicoherent sheaves of $\mathcal{O}_{X^\circ}$-modules.

Solution. We have functors in both directions; we just need to check that they are inverse to one another. Consider a graded quasicoherent $\mathcal{O}_{X^\circ}$-module $\mathscr{F}$. The induced sheaf on $Y = \operatorname{Proj} A$ has $\mathscr{G}(D_+(f)) = \mathscr{F}(D(f))_0$. Then

$$\pi^*\mathscr{G}(D(f)) = \mathscr{G}(D(f)) \otimes_{\mathcal{O}_Y(D_+(f))} \mathcal{O}_X(D(f)) = \mathscr{F}(D(f))_0 \otimes_{A[f^{-1}]_0} A[f^{-1}].$$

On the other hand, $\mathscr{F}(D(f)) = \widetilde{M}$ for some graded $A[f^{-1}]$-module M and if f has degree ± 1 (or is divisible by an element of degree ± 1) then $M = \sum_{n \in \mathbf{Z}} f^n M_0$. Thus

$$\mathscr{F}(D(f)) = \sum f^n \mathscr{F}(D(f))_0 = \mathscr{F}(D(f))_0 \otimes_{A[f^{-1}]_0} A[f^{-1}].$$

Now, the open sets associated to homogeneous elements A_+ divisible by an element of degree ± 1 form a class that is stable under intersection and covers X°. We have agreement of $\pi^*\mathscr{G}$ and $\mathscr{F}$, in a compatible way, on these sets, so we have agreement on all of X°.

Conversely, we have

$$\begin{aligned}(\pi^*\mathscr{G})(D_+(f)) = (\pi^*\mathscr{G}(D(f)))_0 = (\mathscr{G}(D_+(f)) \otimes_{A[f^{-1}]_0} A[f^{-1}]) \\ = \sum \mathscr{G}(D_+(f)) \otimes_{A[f^{-1}]_0} A[f^{-1}]_n = \mathscr{G}(D_+(f)).\end{aligned}$$

This is a natural identification for all homogeneous $f \in A_+$, as required. $\square$

(ii) Show that the category $\mathbf{QCoh}(X^\circ)$ is equivalent to the category of objects $\mathscr{F} \in \mathbf{QCoh}(X)$ such that $\mathscr{F} \to j_*j^*\mathscr{F}$ is an isomorphism.

Solution. We have a functor in one direction by j^*. If $\mathscr{F} \in \mathbf{QCoh}(X^\circ)$ then $j^*j_*\mathscr{F} \to \mathscr{F}$ is an isomorphism. Thus $j_*\mathscr{F} \simeq j_*j^*j_*\mathscr{F}$ so that j_* gives a functor in the other direction. This also shows that $j^*j_* \simeq \mathrm{id}$

To go the other way, suppose that $\mathscr{F} \to j_*j^*\mathscr{F}$ is an isomorphism. Then $j_*j^*\mathscr{F} \to j_*j^*j_*j^*\mathscr{F}$. But we know $j^*j_*j^* \simeq \mathrm{id}$ (naturally) so we get

$$j_*j^*\mathscr{F} \to j_*j^*j_*j^*\mathscr{F} \simeq j_*j^*\mathscr{F}$$

as required. $\square$

43 Line bundles and divisors

Reading 43.1. [GD67, IV.21], [Har77, II.6], [Vak14, Chapter 14]

Invertible sheaves are examples of quasicoherent sheaves, so we can use the classification of quasicoherent sheaves on projective space to classify line bundles.

Exercise 43.2. Suppose that $\mathscr{L}$ and $\mathscr{L}'$ are invertible sheaves. Show that $\mathscr{L} \otimes \mathscr{L}'$ and $\underline{\mathrm{Hom}}(\mathscr{L}, \mathscr{L}')$ are invertible sheaves as well.[2] Show that isomorphism classes of invertible sheaves on a scheme X form an abelian group where addition is $\otimes$, difference is $\underline{\mathrm{Hom}}$, and the zero element is $\mathcal{O}_X$.

Exercise 43.3. Let $A = \mathbf{Z}[x_0, \ldots, x_n]$ and $X = \operatorname{Spec} A$ and $Y = \operatorname{Proj} A$. Construct an equivalence of categories between the category of line bundles on Y and the category of graded invertible sheaves on X°.

We will have fully classified invertible sheaves on $\mathbf{P}^n$ when we show that

(1) a sheaf on X° is invertible if and only if $j_* X^\circ$ is invertible, and

(2) all invertible sheaves on $X = \mathbf{A}^{n+1}$ are trivial.

43.1 Cartier divisors

Definition 43.4 (Meromorphic functions)**.** Let X be a scheme. Let $\mathscr{M}_X$ be the sheaf obtained by adjoining inverses to all nondivisors of zero in $\mathcal{O}_X$. This is known as the sheaf of *meromorphic functions* on X. An invertible sheaf on X is called an *invertible fractional ideal* if it can be embedded, as an $\mathcal{O}_X$-module, in $\mathscr{M}_X$.

ular-in-meromorphic **Exercise 43.5.** Show that there is an injection $\mathcal{O}_X \to \mathscr{M}_X^*$.

Definition 43.6 (Cartier divisors)**.** Let $\underline{\mathrm{Div}}_X = \mathscr{M}_X^* / \mathcal{O}_X^*$. This is known as the sheaf of *Cartier divisors* on X. If f is a section of $\mathscr{M}_X$, the associated divisor is denoted (f). Divisors associated to meromorphic functions are called *principal*.

Exercise 43.7. Suppose that $X = \operatorname{Spec} A$ and A is a unique factorization domain. Show that the map

$$\Gamma(X, \mathscr{M}_X^*) \to \Gamma(X, \underline{\mathrm{Div}}_X)$$

is a surjection.

Exercise 43.8. Suppose that X is an integral scheme with generic point η. Show that $\mathscr{M}_X(U) = \mathbf{k}(\eta)$ for all nonempty $U \subset X$.

Exercise 43.9. Let D be a divisor on X (an element of $\Gamma(X, \underline{\mathrm{Div}}_X)$).

Exercise 43.10. Suppose that D and E are Cartier divisors. We say that $D \geq E$ if $D - E = (f)$ for some $f \in \mathcal{O}_X$. Show that this gives $\underline{\mathrm{Div}}_X$ the structure of a sheaf of partially ordered groups. Let $\underline{\mathrm{Div}}_X^+$ be the subsheaf of divisors $D \in \underline{\mathrm{Div}}_X$ such that $D \geq 0$.

Exercise 43.11. Let D be a divisor on X. Let $\mathcal{O}_X(D)$ be the set of $f \in \mathscr{M}_X$ such that $(f) \geq -D$.

[2] The notation $\underline{\mathrm{Hom}}$ refers to the sheaf of homomorphisms.

(i) Show that $\mathcal{O}_X(D)$ is an invertible sheaf on X.

(ii) Show that this gives a map

$$\Gamma(X, \underline{\mathrm{Div}}_X) \to \mathrm{Pic}(X).$$

(iii) Show that the image of this map consists of all equivalence classes of invertible fractional ideals of X.

Exercise 43.12. Suppose that X is an integral scheme. Show that every invertible sheaf is isomorphic to an invertible fractional ideal. Conclude that there is an isomorphism:

$$\Gamma(X, \underline{\mathrm{Div}}_X)/\Gamma(X, \mathscr{M}_X^*) \simeq \mathrm{Pic}(X)$$

43.2 Weil divisors

Let X be a locally noetherian scheme and $x \in X$ a point. We say that x has codimension 1 in X if $\dim \mathcal{O}_{X,x} = 1$.

Definition 43.13 (Weil divisor)**.** Suppose X is noetherian. A *Weil divisor* on X is a formal sum of codimension 1 points of X. The abelian group of Weil divisors is denoted $Z^1(X)$.

Let $D \geq 0$ be a Cartier divisor on X. Then $\mathcal{O}_X(-D) \subset \mathcal{O}_X$ so it is an ideal. It therefore defines a closed subscheme $V(\mathcal{O}_X(-D))$. Furthermore, it defines a Weil divisor: For each codimension 1 point $x \in X$, set

$$c_x(D) = \mathrm{length}\, \mathcal{O}_{X,x}/\mathcal{O}_{X,x}(-D).$$

Exercise 43.14. Show that $c_x(D) = 0$ for all but finitely many points $x \in X$. Conclude that $c(D) = \sum c_x(D)[x]$ is a Weil divisor of X.

Exercise 43.15. Show that $c_x(D+E) = c_x(D) + c_x(E)$. Conclude that c_x extends to homomorphisms defined on $\mathrm{Div}(X) \to \mathbf{Z}$ and c extends to $\mathrm{Div}(X) \to Z^1(X)$.

Exercise 43.16 ([GD67, Théorème (IV.21.6.9)])**.** An element of $Z^1(X)$ is called locally principal if is locally $c([f])$ for some $f \in \mathscr{M}_X$. Show that $\mathrm{Div}(X) \to Z^1(X)$ is injective and its image consists of the locally principal cycles.

Solution. Fix a Weil divisor D. Let $\mathscr{L}$ be the subsheaf of $\mathscr{M}_X$ consisting of all f such that $c([f]) \geq D$. If D is locally principal then $\mathscr{L}$ is an invertible fractional ideal and $c(\mathscr{L}) = D$. □

43.3 The Picard group of projective space

Exercise 43.17. Let A be a graded ring and let $X = \mathrm{Spec}\, A$ and $Y = \mathrm{Proj}\, A$. If M is an A-module, define $M(n)_k = M(n+k)$.

(i) Show that $A(n)$ is an invertible sheaf on $\mathrm{Proj}\, A$ for all $n \in \mathbf{Z}$. This sheaf is denoted $\mathcal{O}_Y(n)$. We write $\mathscr{F}(n) = \mathscr{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(n)$.

(ii) Show that $\Gamma(Y, \mathscr{F}(n)) = M_n$ whenever $\widetilde{M} = j_* \pi^* \mathscr{F}$.

Solution.

$$\Gamma(Y, \mathscr{F}(n)) = \Gamma(X^\circ, \pi^*\mathscr{F}(n))_0 = \Gamma(X^\circ, \pi^*\mathscr{F})_n = \Gamma(X, j_*\pi^*\mathscr{F})_n = \Gamma(X, \widetilde{M})_n = M_n$$

□

le-Pn-An-complement

Exercise 43.18. Let $A = \mathbf{Z}[x_0, \ldots, x_n]$ and let $X = \operatorname{Spec} A = \mathbf{A}^{n+1}$ and $Y = \operatorname{Proj} A = \mathbf{P}^n$. Show that a quasicoherent sheaf $\mathscr{L}$ on Y is invertible if and only if $\pi^*\mathscr{L}$ is invertible if and only if $j_*\pi^*\mathscr{L}$ is invertible.[3]

Solution. First we show $\mathscr{L}$ is invertible if and only if $\pi^*\mathscr{L}$ is invertible. Certainly, if $\mathscr{L}$ is invertible then so is $\pi^*\mathscr{L}$. Conversely, suppose that $\pi^*\mathscr{L}$ is invertible. Then it is sufficient to show $\mathscr{L}\big|_{U_i}$ is invertible for each $i = 0, \ldots, n$. But we have a section $\sigma_i : U_i \to X^\circ$ of $\pi\big|_{\pi^{-1}U_i}$ so that $\mathscr{L}\big|_{U_i} = \sigma_i^*\pi^*\mathscr{L}$, so $\mathscr{L}\big|_{U_i}$ is a line bundle.

Now we show that a quasicoherent sheaf $\mathscr{L}$ on X° is invertible if and only if $j_*\mathscr{L}$ is. We have $\Gamma(U, j_*\mathscr{L}) = \Gamma(U \cap X^\circ, \mathscr{L})$ so $j^*j_*\mathscr{L} = \mathscr{L}$. Thus if $j_*\mathscr{L}$ is invertible, so is $\mathscr{L}$.

Now, suppose that $\mathscr{L}$ is invertible. We can represent $\mathscr{L}$ as $\mathcal{O}_{X^\circ}(D)$ for some divisor D. Let $\bar{D}$ be the closure of D in X. Since X is factorial, $\bar{D}$ is a Cartier divisor. Then $\mathcal{O}_X(\bar{D})$ is an invertible sheaf on X and $j^*\mathcal{O}_X(\bar{D}) = \mathscr{L}$. Now, note that

$$j_*\mathscr{L} = j_*(\mathcal{O}_{X^\circ} \otimes j^*\mathcal{O}_X(\bar{D})) = (j_*\mathcal{O}_{X^\circ}) \otimes \mathcal{O}_X(\bar{D}).$$

Therefore it is sufficient to assume $\mathscr{L} = \mathcal{O}_{X^\circ}$. But in that case, we can easily calculate that $j_*\mathcal{O}_{X^\circ} = \mathcal{O}_X$. □

Exercise 43.19. Prove that $\operatorname{Pic} \mathbf{A}^n = 0$.

Solution. The polynomial ring is a unique factorization domain. □

Exercise 43.20. Prove directly that $\operatorname{Pic}(\mathbf{A}^n \smallsetminus \{0\}) = 0$. This gives another solution to Exercise 43.18.

Solution. Let $\mathscr{L}$ be an invertible sheaf on $X = \mathbf{A}^n \smallsetminus \{0\}$. Cover X by $U_1, \ldots, U_n$ with $U_i = \operatorname{Spec} A_i$ and $A_i = \mathbf{Z}[x_1, \ldots, x_n, x_i^{-1}]$. Each A_i is a unique factorization domain, so $\mathscr{L}\big|_{U_i} \simeq \mathcal{O}_{U_i}$ for all i. Choose these isomorphisms arbitrarily and call them ϕ_i. Then $\phi_{ji} = \phi_i\big|_{U_{ij}} \circ \phi_j^{-1}\big|_{U_{ij}} \in \mathcal{O}_{U_i}^*$. We can use the ϕ_{ji} to recover $\mathscr{L}$.

An isomorphism $v : \mathscr{L} \simeq \mathscr{L}'$ induces maps $\alpha_i = \phi_i' \circ v\big|_{U_i} \circ \phi_i^{-1}$. Likewise $\alpha_j = \phi_j' \circ v\big|_{U_j} \circ \phi_j^{-1}$ so

$${\phi_i'}^{-1}\alpha_i\phi_i = v\big|_{U_{ij}} = {\phi_j'}^{-1}\alpha_j\phi_j.$$

Thus $\phi_{ij}{\phi_{ij}'}^{-1} = \alpha_j\alpha_i^{-1}$.

The trivial line bundle has $\phi_i = 1$ for all i. To prove that $\mathscr{L}$ is isomorphic to the trivial line bundle, we need to show that there is some choice of α_i such that $\alpha_j\alpha_i^{-1} = \phi_{ij}$. All of this shows that we can identify $\operatorname{Pic}(X)$ with the first cohomology of the sequence

$$\prod_i \Gamma(U_i, \mathcal{O}_{U_i}^*) \to \prod_{i<j} \Gamma(U_{ij}, \mathcal{O}_{U_{ij}}^*) \to \prod_{i<j<k} \Gamma(U_{ij}, \mathcal{O}_{U_{ijk}}^*)$$

[3]This calculation also works over a field. With small modification, it even works over an arbitrary base ring replacing $\mathbf{Z}$.

One way to do this is to consider the simplex and interpret the entries above as computing the Čech cohomology of the sheaf of (discontinuous) functions on the vertices valued in $\mathbf{Z}$ and the sheaf of locally constant functions, valued in the units of the base. $\square$

Exercise 43.21. Prove that $\operatorname{Pic} \mathbf{P}^n = \mathbf{Z}$ with $1 \in \mathbf{Z}$ corresponding to $\mathcal{O}(1)$.

Part III

Cohomology

Chapter 14

Sheaf cohomology

44 Divisors

45 Sheaves III

45.1 Injective resolutions

Definition 45.1. Let X be a scheme.[1] A sheaf of $\mathcal{O}_X$-modules $\mathscr{I}$ is said to be *injective* if $\mathrm{Hom}_{\mathcal{O}_X\text{-}\mathbf{Mod}}(\mathscr{F}, \mathscr{I})$ is an exact functor of $\mathscr{F}$.

Should be easy, but it's important to know and understand why this is true

Exercise 45.2. Show that $\mathscr{I}$ is injective if and only if, for every injection of sheaves of $\mathcal{O}_X$-modules $\mathscr{F}' \to \mathscr{F}$, the map $\mathrm{Hom}(\mathscr{F}, \mathscr{I}) \to \mathrm{Hom}(\mathscr{F}', \mathscr{I})$ is surjective.

Theorem 45.3 (Grothendieck [Gro57, Théorème 1.10.1])**.** *Every sheaf of $\mathcal{O}_X$-modules can be embedded in an injective module.*

The proof uses a few facts about the category of $\mathcal{O}_X$-modules:

(i) the category is *abelian*: it has kernels, cokernels, and images that behave as we are accustomed;

(ii) the category has a *set of generators*: every object is a quotient of a direct sum of $\mathcal{O}_U$ for $U \subset X$ open;

(iii) arbitrary (small) colimits exist and filtered colimits are exact.

The proof is known as the 'small object argument'. The idea is that if we have a witness $\mathscr{F}' \subset \mathscr{F}$ to the failure of injectivity of a sheaf $\mathscr{I}$ then we pushout:

$$\begin{tikzcd} \mathscr{F}' \arrow[r] \arrow[d] & \mathscr{F} \\ \mathscr{I} & \end{tikzcd}$$

Iterating this process enough, we get an injective module. The details of the proof will be a series of exercises, following [Gro57, §1.10].

[1] or, really, a ringed space

Exercise 45.4. Show that the category of sheaves of $\mathcal{O}_X$-modules has a generator.[2] (Hint: Take the direct sum of all $\mathcal{O}_U$, with U ranging among open subsets of X.)

Exercise 45.5. Let $\mathscr{G}$ be the generator. Show that $\mathscr{I}$ is injective if and only if, for every subobject $\mathscr{G}' \subset \mathscr{G}$, every morphism $\mathscr{G}' \to \mathscr{I}$ extends to $\mathscr{G} \to \mathscr{I}$.

Solution. Suppose that $\mathscr{F}' \subset \mathscr{F}$ and $\mathscr{F}' \to \mathscr{I}$ are given. Let $\mathscr{F}'' \subset \mathscr{F}$ be the largest submodule to which the map extends. (This exists by the existence and exactness of filtered colimits.) To show $\mathscr{F}'' = \mathscr{F}$, we consider a map $\mathscr{G} \to \mathscr{F}$ and let $\mathscr{G}''$ be the preimage. The map $\mathscr{G}'' \to \mathscr{I}$ extends to $\mathscr{G} \to \mathscr{I}$. Therefore the map $\mathscr{F}'' \to \mathscr{I}$ extends to the image of $\mathscr{G}$ in $\mathscr{F}$. By assumption, this is contained in $\mathscr{F}''$, since $\mathscr{F}''$ is maximal. Thus every map $\mathscr{G} \to \mathscr{F}$ has image in $\mathscr{F}''$, so $\mathscr{F}'' = \mathscr{F}$. $\square$

Exercise 45.6. Fix $\mathscr{F}$. For each successor ordinal $n+1$, let $\mathscr{F}_{n+1}$ be the pushout of the diagram below:

$$\begin{tikzcd} \sum_{\mathscr{G}' \subset \mathscr{G}} \mathscr{G}' \times \mathrm{Hom}(\mathscr{G}', \mathscr{F}_n) \arrow[r] \arrow[d] & \sum_{\mathscr{G}' \subset \mathscr{G}} \mathscr{G} \times \mathrm{Hom}(\mathscr{G}', \mathscr{G}_n) \\ \mathscr{F}_n & \end{tikzcd}$$

When n is a limit ordinal, let $\mathscr{F}_n = \varinjlim_{m<n} \mathscr{F}_m$. Show that $\mathscr{F}_n$ is injective for large n.

Solution. Choose a limit ordinal n larger than the cardinality of the set of subobjects of $\mathscr{G}$. Consider a map $v : \mathscr{G}' \to \mathscr{F}_n$ for some $\mathscr{G}' \subset \mathscr{G}$. Let $\mathscr{G}'_m = v^{-1}\mathscr{F}_m$ for each $m \le n$. Then by the pigeonhole principle, $\mathscr{G}'_m$ is constant for all $m \ge m_0$, where $m_0 < n$. Thus $\mathscr{G}' = v^{-1}\mathscr{F} = v^{-1}\mathscr{F}_{m_0}$. In other words, $v(\mathscr{G}') \subset \mathscr{F}_{m_0}$. But then $\mathscr{G}' \to \mathscr{F}_{m_0}$ extends to $\mathscr{G} \to \mathscr{F}_{m_0+1} \subset \mathscr{F}_n$, as required. $\square$

45.2 Flaccid sheaves

Definition 45.7. A sheaf $\mathscr{I}$ is said to be *flaccid* if

$$\mathscr{I}(U) \to \mathscr{I}(V)$$

is surjective for all open $V \subset U$.

Exercise 45.8. Show that injective sheaves are flaccid.

Exercise 45.9. Suppose that

$$0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$$

is exact and $\mathscr{A}$ is flaccid.

(i) Show that
$$0 \to \Gamma(X, \mathscr{A}) \to \Gamma(X, \mathscr{B}) \to \Gamma(X, \mathscr{C}) \to 0$$
is exact.

[2] A generator is an object A such that an injection $B \to C$ is an isomorphism if and only if the induced map $\mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$ is an isomorphism.

(ii) Show that $\mathscr{B}$ is flaccid if and only if $\mathscr{C}$ is flaccid.

Solution. Suppose $x \in \mathscr{B}(V)$. Let $\bar{x}$ be its image in $\mathscr{C}(V)$. Choose $\bar{y} \in \mathscr{C}(U)$ with image $\bar{x}$. Lift this to $y \in \mathscr{B}(U)$ using the fact that $\mathscr{A}$ is flaccid. Then $\bar{x} - \bar{y} = 0$ in $\mathscr{C}(V)$, so $x - y$ is the image of $z \in \mathscr{A}(V)$. Lift z to $w \in \mathscr{A}(U)$, using again that $\mathscr{A}$ is flaccid. Then $y + w$ restricts to x in $\mathscr{B}(V)$.

Now suppose $x \in \mathscr{C}(V)$ and $\mathscr{B}$ is flaccid. Lift x to $y \in \mathscr{B}(V)$, using that $\mathscr{A}$ is flaccid. Lift y to $z \in \mathscr{B}(U)$ using that $\mathscr{B}$ is flaccid. Then let w be the image of z in $\mathscr{C}(U)$ and we are done. $\square$

45.3 Cohomology as a derived functor

Definition 45.10. Let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules. Choose an embedding $\mathscr{F} \subset \mathscr{I}$ where $\mathscr{I}$ is a flaccid $\mathcal{O}_X$-module. Define

$$\begin{aligned} H^1(X, \mathscr{F}) &= \Gamma(X, \mathscr{I}/\mathscr{F})/\Gamma(X, \mathscr{I}) \\ H^n(X, \mathscr{F}) &= H^{n-1}(X, \mathscr{I}/\mathscr{F}) \qquad \text{for } n \geq 2. \end{aligned}$$

45.4 Torsors

Definition 45.11 (Torsor)**.** Let $\mathscr{G}$ be a sheaf of groups on X. A *$\mathscr{G}$-torsor* is a sheaf of sets $\mathscr{P}$, equipped with an action of $\mathscr{G}$, such that there is a cover of X by open subsets U such that $\mathscr{P}\big|_U$ is isomorphic to $\mathscr{G}\big|_U$ as a sheaf on U with $\mathscr{G}\big|_U$-action.

A $\mathscr{G}$-torsor is said to be *trivial* if it is isomorphic to $\mathscr{G}$ as a sheaf of $\mathscr{G}$-sets. Define $H^1(X, \mathscr{G})$ to be the set of isomorphism classes of $\mathscr{G}$-torsors on X.

Exercise 45.12. Show that a $\mathscr{G}$-torsor $\mathscr{P}$ is trivial if and only if $\Gamma(X, \mathscr{P}) \neq \varnothing$.

Exercise 45.13. Suppose that $\mathscr{G}$ is a flaccid sheaf of groups. Show that every $\mathscr{G}$-torsor is trivial.

Solution. Let $\mathscr{P}$ be a $\mathscr{G}$-torsor. View the open subsets of X as a partially ordered set, ordered by inclusion. Let $U \subset X$ be an open subset that is maximal among those on which $\mathscr{P}$ has a section. We argue that $U = X$. Suppose that $x \in X$. Choose an open neighborhood V of x such that $\mathscr{P}\big|_V$ is trivial. Pick $p \in \mathscr{P}(U)$ and $q \in \mathscr{P}(V)$. Then there is some $g \in \mathscr{G}(V \cap U)$ such that $g.p\big|_{V\cap U} = q\big|_{V\cap U}$. Lift g to $g' \in \mathscr{G}(V)$. Then let $p' = g'.p$. We get $g'\big|_{V\cap U} = q\big|_{V\cap U}$ so we can find $q \in \mathscr{P}(V \cup U)$ lifting q. Thus $V \subset U$ so we conclude $U = X$. $\square$

Exercise 45.14. Suppose that $\mathscr{G}$ acts on a sheaf of sets $\mathscr{P}$.

(i) Show that there is an exact sequence

$$0 \to \Gamma(X, \mathscr{G}) \to \Gamma(X, \mathscr{P}) \to \Gamma(X, \mathscr{P}/\mathscr{G}) \to H^1(X, \mathscr{G}).$$

(Hint: Given a section $\sigma \in \Gamma(X, \mathscr{P}/\mathscr{G})$, consider its preimage in $\mathscr{P}$.)

(ii) Show that the last arrow is surjective if $\mathscr{P}$ is flaccid.

Solution. Let $\mathscr{Q}$ be a $\mathscr{G}$-torsor. Consider the sheaf of $\mathscr{G}$-equivariant morphisms $\underline{\mathrm{Hom}}(\mathscr{Q}, \mathscr{P})$. This is flaccid. Indeed, suppose $f : \mathscr{Q}\big|_V \to \mathscr{P}\big|_U$ and $V \subset U$. Let $U' \subset U$ be the largest open subset to which f extends. We argue that $U' = U$.

For any $x \in X$, pick an open neighborhood W of x in X such that $\mathscr{Q}\big|_W \simeq \mathscr{G}\big|_W$. Pick $q \in \mathscr{Q}(W)$ and restrict to $W \cap U'$. Then $f(q\big|_{W\cap U'}) = y\big|_{W\cap U'}$ for some $y \in \mathscr{P}(W)$. For any $W' \subset W$, any $z \in \mathscr{Q}(W')$ can be written as $gq\big|_{W'}$ for a unique $g \in \mathscr{G}(W')$. Define $f'(z) = gy\big|_{W'}$. If $W' \subset W \cap U'$ then $f'(g.q\big|_{W'}) = g.y\big|_{W'} = g.f(q\big|_{W'}) = f(g.q\big|_{W'})$ so $f\big|_{W\cap U'} = g'\big|_{W\cap U'}$. Therefore f and f' extend to $W \cap U'$. We conclude that $W \subset U'$ so $U' = U$, as required.

Since $\underline{\mathrm{Hom}}(\mathscr{Q}, \mathscr{P})$ is flaccid, it follows that $\mathrm{Hom}(\mathscr{Q}, \mathscr{P}) \neq \varnothing$. Consider the map $\mathscr{Q} \to \mathscr{P} \to \mathscr{P}/\mathscr{G}$. The image is a section of $\mathscr{P}/\mathscr{G}$ whose image in $H^1(X, \mathscr{G})$ is $\mathscr{Q}$. $\square$

Exercise 45.15. Prove that our two definitions of $H^1(X, \mathscr{G})$ coincide when $\mathscr{G}$ is a sheaf of abelian groups.

Solution. Choose an embedding $\mathscr{G} \subset \mathscr{I}$ where $\mathscr{I}$ is flaccid. Then we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X,\mathscr{G}) \arrow[r] \arrow[d, equal] & \Gamma(X,\mathscr{I}) \arrow[r] \arrow[d, equal] & \Gamma(X,\mathscr{I}/\mathscr{G}) \arrow[r] \arrow[d] & H^1_{\mathrm{tors}}(X,\mathscr{G}) \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & \Gamma(X,\mathscr{G}) \arrow[r] & \Gamma(X,\mathscr{I}) \arrow[r] & \Gamma(X,\mathscr{I}/\mathscr{G}) \arrow[r] & H^1_{\mathrm{der}}(X,\mathscr{G}) \arrow[r] & 0
\end{tikzcd}$$

The last vertical arrow exists and is an isomorphism because the other vertical arrows are isomorphisms. $\square$

Exercise 45.16. Show that the sheaf of isomorphisms between two $\mathscr{G}$-torsors is naturally equipped with the structure of a $\mathscr{G}$-torsor by $(g.f)(x) = g.(f(x))$. Show that $[\mathscr{Q}] - [\mathscr{P}] = [\underline{\mathrm{Hom}}(\mathscr{P}, \mathscr{Q})]$ in the additive structure of $H^1(X, \mathscr{G})$ induced from the derived functor construction.

45.5 Line bundles

Exercise 45.17. Construct an equivalence of categories between the category of line bundles on a scheme X and the category of torsors on X under the group $\mathbf{G}_m$.

46 Čech cohomology

Exercise 46.1. Let $\mathscr{F}$ be a sheaf of abelian groups on X. For each open $U \subset X$, define $\mathscr{H}^p\mathscr{F}(U) = H^p(U, \mathscr{F})$.

(i) Show that $\mathscr{H}^p\mathscr{F}$ is naturally a presheaf on X. (Hint: The restriction of a flaccid sheaf to an open subset is still flaccid, and restriction of sheaves is exact.)

(ii) Show that the sheafification of $\mathscr{H}^p\mathscr{F}$ is the zero sheaf for all $p > 0$.

Solution. It's enough to show that the sheafification of $\mathscr{H}^1\mathscr{F}$ is the zero sheaf. We must show that every section is locally zero. But a section of $\mathscr{H}^1\mathscr{F}$ over U corresponds to the isomorphism class of an $\mathscr{F}$-torsor over U, which is trivial over some open cover of U. To put it another way, if we choose an embedding $\mathscr{F} \subset \mathscr{I}$ with $\mathscr{I}$ injective then a section of $\mathscr{H}^1\mathscr{F}$ over U can be represented by a section of $\mathscr{I}/\mathscr{F}$ over U. Any such section is locally in the image of $\mathscr{I}$ by the definition of a surjection of sheaves, so the section represents the zero element of $\mathscr{H}^1\mathscr{F}$ locally in U. □

Definition 46.2. For each n, let $\mathfrak{U}_n$ be the set of all symbols $U_1 \wedge \cdots \wedge U_n$. Define $C^p(\mathfrak{U}, \mathscr{F})$ to be the set of functions σ on $\mathfrak{U}_n$ with

$$\sigma(U_1 \wedge \cdots \wedge U_n) \in \mathscr{F}(U_1 \cap \cdots \cap U_n)$$
$$\sigma(U_{f(1)} \wedge \cdots \wedge U_{f(n)}) = \mathrm{sgn}(f)\sigma(U_1 \wedge \cdots \wedge U_n).$$

These are the *Čech p-cochains.* Define a *coboundary map*

$$C^p(\mathfrak{U}, \mathscr{F}) \to C^{p+1}(\mathfrak{U}, \mathscr{F})$$

by defining $d(\sigma)(U_1 \wedge \cdots \wedge U_{p+1}) = \sum(-1)^i \sigma_{U_1,\ldots,\hat{U}_i,\ldots,U_{p+1}}\big|_{U_1 \cap \cdots \cap U_{p+1}}$. The cohomology of this complex is called the *Čech cohomology* of $\mathscr{F}$ with respect to $\mathfrak{U}$ and is denoted $H^*(\mathfrak{U}, \mathscr{F})$.[3]

Exercise 46.3. (i) Construct a map $H^1(\mathfrak{U}, \mathscr{G}) \to H^1(X, \mathscr{G})$.

(ii) Show that the image consists of all $\mathscr{G}$-torsors $\mathscr{P}$ such that $\mathscr{P}\big|_U$ trivial for all U in $\mathfrak{U}$.

(iii) Conclude that $\check{H}^1(X, \mathscr{G}) = H^1(X, \mathscr{G})$ for all sheaves of groups $\mathscr{G}$ on X.

Exercise 46.4. Suppose that $\mathscr{I}$ is a flaccid sheaf. Show that $H^p(\mathfrak{U}, \mathscr{I}) = 0$ for all $p > 0$. (Hint: Realize the Čech complex as the global sections of an exact sequence of sheaves.) Conclude that Čech cohomology agrees with sheaf cohomology for flaccid sheaves.

46.1 Affine schemes

Exercise 46.5. Prove $H^1(X, \mathscr{G}) = 0$ for all quasicoherent sheaves $\mathscr{G}$ on all affine scheme X.

Soon we will see that $H^n(X, \mathscr{G}) = 0$ for $n > 0$ and all quasicoherent sheaves $\mathscr{G}$ on all affine schemes X.

`ex:affine-cech` **Exercise 46.6.** Prove that the Čech complex is exact for any quasicoherent sheaf on an affine scheme and any cover by distinguished open affines. Conclude that $\check{H}^n(X, \mathscr{F}) = 0$ for all $n > 0$ when X is affine and $\mathscr{F}$ is quasicoherent.

[3]The *Čech cohomology* is defined by taking a colimit of $H^*(\mathfrak{U}, \mathscr{F})$ over all covers, ordered by refinement. In order for this to make sense and be a filtered colimit, it is best to define a cover to be a choice of open neighborhood of each point.

Chapter 15

Lines on a cubic surface

47 Čech cohomology II

47.1 The Čech spectral sequence

Should be simple

Exercise 47.1. Suppose $\mathscr{I}$ is a flaccid sheaf on X. Show that $\mathscr{I}\big|_U$ is also flaccid, for all open $U \subset X$.

Let $\mathscr{F}$ be a sheaf on X and let $\mathscr{I}^\bullet$ be a flaccid resolution of $\mathscr{F}$. Fix a cover $\mathfrak{U}$ of X and write $C^\bullet(\mathfrak{U}, \mathscr{F})$ for the Čech complex of $\mathscr{F}$. We can form a *double complex*:

$$C^\bullet(\mathfrak{U}, \mathscr{I}^\bullet)$$

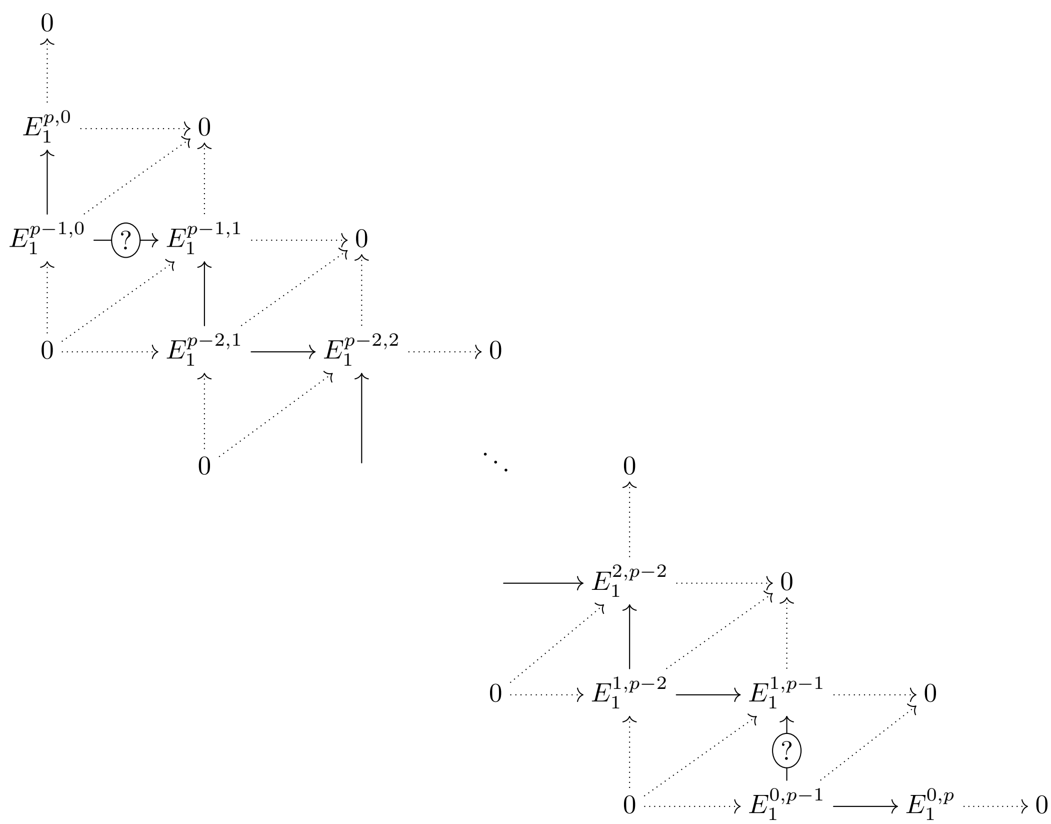
If we compute the cohomology first with respect to $C^\bullet$, we get $\Gamma(X, \mathscr{I}^\bullet)$, whose cohomology is $H^q(X, \mathscr{F})$. If we compute it first with respect to the $\mathscr{I}^\bullet$ differential, we get a complex whose (p, q)-entry is

$$\prod_{U_1,\ldots,U_p \in \mathfrak{U}} H^q(U_1 \cap \cdots \cap U_p, \mathscr{F}\big|_{U_1 \cap \cdots \cap U_p}).$$

In particular, we find the Čech cohomology as the $q = 0$ column. Suppose that $H^q(U_1 \cap \cdots \cap U_p, \mathscr{F}) = 0$ for all p and all $q > 0$. Then everything vanishes but the Čech cohomology with respect to $\mathfrak{U}$.

Exercise 47.2. Suppose that E is a double complex in the first quadrant. Assume that for all $p > 0$ we have $H^{p,\bullet}E = 0$ and that for all $q > 0$ we have $H^{\bullet,q}E = 0$. (In other words, the columns and rows are all exact, except in degree 0.) Conclude that $H^p H^{0,\bullet}E = H^p H^{\bullet,0}E$ (in a natural way) for all p.

Solution. The diagram chasing can be simplified by the following calculus: Keep track of how E changes if we replace an entry by 0 and replace all arrows out of that entry by cokernels and all arrows into that entry by kernels. Modify E according to the following diagram:



A dotted arrow into or out of an entry indicates that a cokernel or kernel has been taken (respectively). Except for those marked with question marks, all arrows in the diagram above are isomorphisms. The arrows marked with question marks are nevertheless surjections. Now divide both $E_1^{p,0}$ and $E_1^{p-1,0}$ by the kernel of $E_1^{p-1,0} \to E_1^{p-1,1}$; likewise, divide both $E_1^{0,p-1}$ and $E_1^{0,p}$ by the kernel of $E_1^{0,p-1} \to E_1^{1,p-1}$. Then $E_1^{p,0}$ is replaced by $H^p H^{\bullet,0} E$ and $E_1^{0,p}$ is replaced by $H^p H^{0,\bullet} E$. All of the solid arrows remain isomorphisms. $\square$

Theorem 47.3 ([Gro57, Théorème 3.8.1])**.** *Suppose $\mathfrak{U}$ is a cover of X and $H^q(U_1 \cap \cdots \cap U_p, \mathscr{F}) = 0$ for all $p, q > 0$ and all $U_1, \ldots, U_p \in \mathfrak{U}$. Then Čech cohomology of $\mathscr{F}$ agrees with derived functor cohomology.*

A more careful analysis of the proof of the exercise above gives a refinement of this theorem. Observe that to get an isomorphism

$$H^p(\mathfrak{U}, \mathscr{F}) \simeq H^p(X, \mathscr{F})$$

we needed the following vanishing:

$$\begin{gathered} H^1(U_1 \cap \cdots \cap U_p, \mathscr{F}) = 0 \\ H^1(U_1 \cap \cdots \cap U_{p-1}, \mathscr{F}) = H^2(U_1 \cap \cdots \cap U_{p-1}, \mathscr{F}) = 0 \\ H^2(U_1 \cap \cdots \cap U_{p-2}, \mathscr{F}) = H^3(U_1 \cap \cdots \cap U_{p-2}, \mathscr{F}) = 0 \\ \vdots \end{gathered}$$

for all $U_1, \dots, U_k \in \mathfrak{U}$. In particular, we can make the following conclusion:

Theorem 47.4. *If $H^i(U_1 \cap \cdots \cap U_q, \mathscr{F}) = 0$ for $i + q + 1 \leq p$ and $i > 0$ then*

$$H^p(\mathfrak{U}, \mathscr{F}) = H^p(X, \mathscr{F}).$$

Corollary 47.4.1. *The cohomology of a quasicoherent sheaf on an affine scheme is trivial in positive degrees.*

Proof. We build the double complex $E_0^{p,q} = C^p(\mathfrak{U}, \mathscr{I}^q)$ for each pointed open cover $\mathfrak{U}$ and then take a direct limit to get the Čech complex. We have exactness in the $\mathscr{I}^\bullet$ direction, except in degree zero, automatically, by the exactness of filtered colimits and the exactness at each stage. Assume for the sake of induction that $H^i(Y, \mathscr{G}) = 0$ for all affine schemes Y and all quasicoherent sheaves $\mathscr{G}$ on Y and all $0 < i \leq p$. Then we get $H^i(U_1 \cap \cdots \cap U_q, \mathscr{F}) = 0$ for $i + q + 1 \leq p + 1$ and $i \leq p$. In other words, we have the desired vanishing for $q > 1$. The last thing we want is $H^i(U, \mathscr{F}) = 0$ for all $U \in \mathfrak{U}$. Of course, this doesn't look any easier, but remember that we are taking a colimit over all open covers $\mathfrak{U}$. We have $\varinjlim \prod_{U \in \mathscr{U}} H^i(U, \mathscr{F}) = 0$, as required. ☐

Corollary 47.4.2. *The Čech cohomology of a quasicoherent sheaf with respect to an affine cover of a separated scheme agrees with the derived functor cohomology.*

47.2 Cohomology and dimension

Theorem 47.5 ([Gro57, Théorme 3.6.5], [Har77, Theorem III.2.7])**.** *Let X be a noetherian topological space of dimension n. Show that $H^p(X, \mathscr{F}) = 0$ for all sheaves of abelian groups $\mathscr{F}$ on X and all $p > 0$.*

Proof. We can assume by noetherian induction that the statement holds for all closed subsets of X other than X itself. If X is reducible, suppose $X = \bigcup Y_i$ with neither Y_i distinct and irreducible. Then we have an exact sequence:

$$0 \to \mathscr{F} \to \prod \mathscr{F}_{Y_i} \to \prod \mathscr{F}_{Y_i \cap Y_j} \to 0$$

This induces an exact sequence

$$\prod H^{p-1}(Y_i \cap Y_j, \mathscr{F}) \to H^p(X, \mathscr{F}) \to \prod H^p(Y_i, \mathscr{F})$$

and we have the desired vanishing on the ends already. We can therefore assume X is irreducible.

As $\mathscr{F}$ is a filtered colimit of finitely generated subsheaves and filtered colimits commute with cohomology, we can assume $\mathscr{F}$ is finitely generated. By induction on the number of generators, we can assume $\mathscr{F}$ has a single generator, i.e., it is a quotient of some $\mathbf{Z}_U$. We therefore have an exact sequence

$$0 \to \mathscr{R} \to \mathbf{Z}_U \to \mathscr{F} \to 0$$

and we already hve $\square$

Exercise 47.6. Assume the result holds for all closed subsets of X other than X itself.

(i) Show that the result holds when X is irreducible and $\mathscr{F} = \mathbf{Z}_U$ for some open $U \subset X$.

Solution. Let V be the complement of U. We have an exact sequence

$$0 \to \mathbf{Z}_U \to \mathbf{Z}_X \to \mathbf{Z}_V \to 0$$

whence

$$H^{p-1}(V, \mathbf{Z}) \to H^p(X, \mathbf{Z}_U) \to H^p(X, \mathbf{Z}_X).$$

But $\mathbf{Z}_X$ is flaccid and $\dim V \leq \dim X - 1$ so $H^{p-1}(V, \mathbf{Z}) = 0$ for $p > \dim X$. $\square$

(ii) Show that the result holds when X is irreducible and $\mathscr{F}$ is a quotient of $\mathbf{Z}_U$ for some open $U \subset X$.

Solution. Let $\mathscr{R}$ be the kernel of the projection $\mathbf{Z}_U \to \mathscr{F}$. Then $\mathscr{R}$ contains $d\mathbf{Z}_V$ for some open $V \subset U$. Indeed, we may take d so that $d\mathbf{Z}$ generates the kernel at the generic point. Then we have an exact sequence

$$0 \to d\mathbf{Z}_V \to \mathscr{R} \to \mathscr{Q} \to 0$$

and $\mathscr{Q}$ is supported on a proper closed subset and we already know the result for $d\mathbf{Z}_V \simeq \mathbf{Z}_V$. Hence we know the result for $\mathscr{R}$. But now we have the sequence

$$0 \to \mathscr{R} \to \mathbf{Z}_U \to \mathscr{F} \to 0$$

so we know the result as well for $\mathscr{F}$. $\square$

(iii) Show that the result holds when X is irreducible and $\mathscr{F}$ is finitely generated.

Solution. If $x_i \in \mathscr{F}(U_i)$ are generators then we have an exact sequence

$$0 \to \sum_{i=1}^{m-1} x_i \mathbf{Z}_{U_i} \to \mathscr{F} \to \mathbf{Z}_{U_m} x_m \to 0$$

and by induction we can reduce to the case where $\mathscr{F} = \mathbf{Z}_{U_m} x_m$. But then $\mathscr{F}$ is quotient of $\mathbf{Z}_{U_m}$. $\square$

(iv) Show that the result holds when X is irreducible.

Solution. Realize $\mathscr{F}$ as a filtered colimit of finitely generated sheaves and use the fact that cohomology commutes with filtered colimits. $\square$

(v) Show that the result holds for all X.

Solution. If X is reducible, say $X = Y \cup Z$. Then we have an exact sequence

$$0 \to \mathscr{F} \to \mathscr{F}_Y \times \mathscr{F}_Z \to \mathscr{F}_{Y\cap Z} \to 0$$

Then we have

$$H^{p-1}(Y \cap Z, \mathscr{F}) \to H^p(X, \mathscr{F}) \to H^p(Y, \mathscr{F}) \times H^p(Z, \mathscr{F}).$$

We have vanishing on the left because $\dim Y \cap Z \leq \dim X - 1$ and vanishing on the right because Y and Z are closed subsets not equal to X. □

47.3 Cohomology of invertible sheaves on projective space

Exercise 47.7. Compute $H^p(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1})$ for all p.

Solution. We know it vanishes for all $p < 0$ (definition) and $p > 1$ (dimension). We can compute using Čech cohomology:

$$H^0(U_0, \mathcal{O}) \times H^0(U_1, \mathcal{O}) \to H^0(U_0 \cap U_1, \mathcal{O})$$

becomes

$$A[x_{1/0}] \times A[x_{0/1}] \to A[x_{0/1}, x_{1/0}]/(x_{0/1}x_{1/0} - 1).$$

We can easily see this is surjective with kernel A, so $H^1(\mathbf{P}^1, \mathcal{O}) = 0$. □

Exercise 47.8. Compute $H^p(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(n))$ for all integers n and p. (Hint: There is a map $\mathcal{O}(n) \to \mathcal{O}(n+1)$ by 'multiplication by x'.)

Exercise 47.9. Repeat the above calculation for $\mathbf{P}^2$ and then for all $\mathbf{P}^n$. (Hint: There should be an induction on n going on here. You'll have to compute the cohomology of $\mathcal{O}_{\mathbf{P}^n}$ by hand, though.)

Recall that quasicoherent sheaves on $\mathbf{P}^n$ are equivalent to graded modules on $\mathbf{A}^{n+1}$ such that if $j : \mathbf{A}^{n+1} \smallsetminus \{0\} \to \mathbf{A}^{n+1}$ is the inclusion then we have $\mathscr{F} \to j_* j^* \mathscr{F}$ is an isomorphism. We compute the derived functors of this inclusion. Under this equivalence, global sections on $\mathbf{P}^n$ correspond to $\Gamma(\mathbf{A}^{n+1}, \mathscr{F})_0$. Regarding $\mathscr{F} = \widetilde{M}$ for some graded module M, we form the following resolution:

$$0 \to M \to \prod_{1 \leq i \leq n} M_{x_i} \to \prod_{1 \leq i < j \leq n} M_{x_i x_j} \to \cdots \to M_{x_0 \cdots x_n} \to 0.$$

48 Čech cohomology III

49 Lines on a cubic surface

For any scheme S, let $G(S)$ be the set of lines in $\mathbf{P}^3$. Equivalently, $G(S)$ is the set of equivalence classes of 2-dimension vector subspaces $V \subset \mathbf{A}^4_S$. This is the Grassmannian.

Exercise 49.1. Show that $\mathbf{Grass}(k,n)$ is proper. (Hint: Use the valuative criterion. For the existence part, let R be a valuation ring with field of fractions K. Represent an element of $\mathbf{Grass}(k,n)(K)$ by an $k \times n$ matrix with entries in R such that not all $k \times k$ minors are zero. Multiply by the inverse of the $k \times k$-submatrix whose determinant has minimal valuation. Argue that the result has entries in R.)

Suppose that $V \subset \mathbf{A}^4_S$ represents a line. This gives an embedding

$$\mathbf{P}_S V \to \mathbf{P}^4_S$$

where $\mathbf{P}_S V$ is the space of lines in the rank 2 vector bundle V over S. Indeed, any 1-dimensional subspace of $\mathbf{P}_S V$ is also a 1-dimensional subspace of $\mathbf{A}^4_S$. If we choose an isomorphism $V \simeq \mathbf{A}^2_S$ then we get an isomorphism $\mathbf{P}_S V \simeq \mathbf{P}^1_S$.

This construction has an inverse: If we have a closed embedding of S-schemes $f : P \subset \mathbf{P}^3_S$ then it is given by a tuple $(\mathscr{L}, x_0, \ldots, x_3)$ with the x_i generating $\mathscr{L}$. Let $\pi : P \to S$ be the projection. Then $x_0, \ldots, x_3$ give a map $\mathcal{O}^4_S \to \pi_* \mathscr{L}$. If P is isomorphic to $\mathbf{P}^1_S$ locally in S and $\mathscr{L}$ is locally isomorphic to $\mathcal{O}_{\mathbf{P}^1_S}(1)$ then $\pi_* \mathscr{L}$ is a locally free sheaf of rank 2 on S. If we show the map $\mathcal{O}^4_S \to \pi_* \mathscr{L}$ is surjective then $\mathbf{V}(\pi_* \mathscr{L}) \to \mathbf{V}(\mathcal{O}^4_S) \simeq \mathbf{P}^3_S$ will be a closed embedding.

To see that $\mathcal{O}^4_S \to \pi_* \mathscr{L}$ is a surjection, it is sufficient to treat the case when S is a point. (A morphism of finite rank vector bundles that is a surjection fiberwise is a surjection, by Nakayama's lemma.) Now this corresponds to a degree 1 morphism of graded modules $k[x,y] \to k[x,y]$ that is surjective modulo x and modulo y. It follows that the image contains both x and y so it is surjective on global sections.

Exercise 49.2. Let $H(S)$ be the set of closed embeddings of S-schemes $f : P \subset \mathbf{P}^n_S$ such that locally in S, we have $P \simeq \mathbf{P}^k_S$ and $f^* \mathcal{O}_{\mathbf{P}^n_S}(1) = \mathcal{O}_{\mathbf{P}^k_S}(1)$. Prove that $H \simeq \mathbf{Grass}(k+1, n+1)$.

Let $Y = \mathbf{A}^N$ be the space of all homogeneous cubics in 4 variables. Let X be the functor with $X(S)$ equal to the set of pairs (p, L) where p is a homogeneous cubic in 4 variables, and L is family of lines in $\mathbf{P}^3$ parameterized by S such that L lies on the cubic surface defined by p. If we view p as a morphism $\mathbf{A}^4_S \to \mathbf{A}^1_S$ and L as a 2-dimensional linear subspace $V \subset \mathbf{A}^4_S$ then the condition defining X is $p(V) = 0$.

Exercise 49.3. (i) Show that X is representable by a scheme.

(ii) Show that X is proper over Y. (Hint: Make use of the properness of the Grassmannian.)

Exercise 49.4. Give an example of a non-smooth cubic surface. Show that the number of lines on a non-smooth cubic surface does not have to be the same as the number of lines on a smooth cubic surface.

Let $Z(S)$ be the set of pairs (p, x) where p is a homogeneous cubic and x is a point of the cubic surface defined by p. In other words, $Z(S)$ consists of $p : \mathbf{A}^4_S \to \mathbf{A}^1_S$ and x is represented by a 1-dimensional subspace $W \subset \mathbf{A}^4_S$ with $p(W) = 0$. Let $Z_0 \subset Z$ be the set of all points (p, x) such that Z is not smooth over Y at x.

Exercise 49.5. (i) Show that Z fails to be smooth at (p, x) if and only if $p'(x) = 0$. (You might need to work locally in Z to make sense of this condition.)

(ii) Conclude that $Z_0 \subset Z$ is closed.

(iii) Conclude that the $p \in Y$ defining smooth cubic surfaces form an open subset, denoted $Y^\circ \subset Y$.

Let X° be the preimage of Y°. Points of X° correspond to lines on smooth cubic surfaces.

Exercise 49.6. Let X' be a cubic surface over S' and $L \subset X$ a line. Suppose that $S \subset S'$ is a square-zero extension with ideal J and $\pi : X \to S$ is the restriction of X' to S. Let $\tau : L \to S$ be the restriction of π. Show that there is an obstruction to extending L to a line $L' \subset X'$ over S' lying in $\mathrm{Ext}^1_{\mathscr{O}_L}(\mathscr{I}/\mathscr{I}^2, \pi^* J)$ and that extensions are parameterized by $\mathrm{Hom}_{\mathscr{O}_L}(\mathscr{I}/\mathscr{I}^2, \tau^* J)$.

Solution. We have an exact sequence:

$$0 \to \mathscr{I} \to \mathscr{O}_X \to \mathscr{O}_L \to 0.$$

Let $\mathscr{A}$ be the quotient of $\mathscr{O}_{X'}$ by the kernel of the map $J \otimes \mathscr{O}_X \to J \otimes \mathscr{O}_L$. Let $\mathscr{J}$ be the kernel of $\mathscr{A} \to \mathscr{O}_L$. Then we have an exact sequence:

$$0 \to J \otimes \mathscr{O}_L \to \mathscr{J} \to \mathscr{I} \to 0$$

This is a sequence of $\mathscr{O}_{X'}$-modules. Dividing by $\mathscr{J}^2$ gives:

$$0 \to \tau^* J \to \mathscr{J}/\mathscr{J}^2 \to \mathscr{I}/\mathscr{I}^2 \to 0$$

This is a sequence of $\mathscr{O}_L$-modules. As $\mathscr{J}^2 = \mathscr{I}^2$, splitting this sequence or the one before is the same. But if σ is a splitting then $\mathscr{O}_{L'} = \mathscr{O}_{X'}/\sigma\mathscr{I}$ gives the deformation of L. This proves that the class of the sequence above obstructs the existence of L' and that splittings of the sequence, which are in bijection with $\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \tau^* J)$ parameterize choices of L'. □

Exercise 49.7. With notation as in the last exercise, show that $\mathscr{I}/\mathscr{I}^2 = \mathscr{O}_L(1)$.

Solution. We have an exact sequence:

$$0 \to N_{L/X} \to N_{L/\mathbf{P}^3} \to N_{X/\mathbf{P}^3}\big|_L \to 0$$

We know $N_{X/\mathbf{P}^3} \simeq \mathscr{O}_X(3)$ and $N_{L/\mathbf{P}^3}$ is an extension of $\mathscr{O}_L(1)$ by $\mathscr{O}_L(1)$. Taking $\bigwedge^2$, we find

$$\mathscr{O}_L(2) \simeq N_{L/X}(3)$$

yielding $N_{L/X} \simeq \mathscr{O}_L(-1)$. Dually, $\mathscr{I}/\mathscr{I}^2 = \mathscr{O}_L(1)$. □

Exercise 49.8. Prove that $\mathrm{Ext}^i(\mathscr{I}/\mathscr{I}^2, \tau^* J) = 0$ for $i = 0, 1$. (Suggestion: The case where $S = \operatorname{Spec} k$ and $J = k$ is all we really need, so feel free to do just that.)

Solution. In the special case, we have to show $H^0(L, \mathscr{O}_L(-1)) = H^1(L, \mathscr{O}_L(-1)) = 0$, which we know because we know the cohomology of quasicoherent sheaves on $\mathbf{P}^1$. In general, one can reduce to this case using the upper semicontinuity of the ranks of $R^1\tau_*\tau^* J(-1)$. □

Solution. Conclude that X° is proper and étale over Y°, so its fibers all have the same size. Compute that size over a particular smooth cubic surface, like the one defined by $\sum x_i^3 = 0$. □

Part IV

Deleted material

Chapter 16

Deformation theory

50 Formal functions

51 Locally trivial deformation problems

51.1 Deforming morphisms to a smooth scheme

51.2 Deforming smooth schemes

51.3 Deforming vector bundles

52 Homogeneous functors and Schlessinger's criteria

B Lines on a cubic surface

53 Divisors and line bundles

54 Associated points and the field of fractions

Reading 54.1. [Vak14, §5.5], [Eis91, §3.1]

Every integral domain can be embedded in a field, but not every commutative ring can. We will see that there is a replacement for the field of fractions, called the total ring of fractions, obtained by localizing the ring at its associated primes, or equivalently by inverting all nondivisors of zero. The only associated prime of an integral domain is the zero ideal, so the total ring of fractions recovers the field of fractions in this case.

Definition 54.2. Let A be a commutative ring and M an A-module. For any subset $S \subset M$, the *annihilator*[1] of S in A is the set of all $a \in A$ such that $ax = 0$. It is denoted $\mathrm{Ann}(S)$.

[1] The French use the more evocative *assassin*.

Definition 54.3. Let A be a noetherian ring and let M be a finitely generated A-module.[2] A prime $\mathfrak{p}$ of A is said to be *associated to* M if there is an injection of A-modules $A/\mathfrak{p} \to M$. The set of primes assocated to M is denoted $\mathrm{Ass}(M)$.

Should be quick and easy and not worth writing down.

Exercise 54.4. Show that $\mathfrak{p}$ is associated to M if and only if there is some $x \in M$ such that $\mathrm{Ann}_A(x) = \mathfrak{p}$.

Exercise 54.5 ([Vak14, Exercise 5.5.J], [Eis91, Proposition 3.4])**.** Consider the collection of all proper ideals of A that are annihilators of elements of M, ordered by inclusion. Show that the maximal elements of this collection are associated primes of M.

Solution. Suppose that I annihilates $x \in M$. If $ab \in I$ then $abx = 0$. Since $\mathrm{Ann}(ax)$ and $\mathrm{Ann}(bx)$ both contain I, it follows that either $\mathrm{Ann}(ax) = I$ or $\mathrm{Ann}(ax) = A$. In the former case, we have $b \in I$ and in the latter, we have $ax = 0$, whence $a \in I$. Thus I is prime, hence is an associated prime of M. □

Exercise 54.6 ([Vak14, Exercise 5.5.K])**.** Show that $M \to \prod_{\mathfrak{p} \in \mathrm{Ass}(M)} M_{\mathfrak{p}}$ is injective.

Solution. Suppose that $x \in M$ lies in the kernel. Then for every associated prime $\mathfrak{p}$ there is some $f \in A \smallsetminus \mathfrak{p}$ such that $fx = 0$. Thus $\mathrm{Ann}(x)$ is not contained in any associated prime. Then $\mathrm{Ann}(x) = A$ so $x = 0$. □

Exercise 54.7. Show that the union of the associated primes of a noetherian commutative ring A is the set of zero divisors of A.[3]

Solution. Suppose that $x \in A$ is a zero divisor. Then there is a nonzero $y \in A$ such that $xy = 0$. Then $\mathrm{Ann}(y) \neq A$ so $x \in \mathrm{Ann}(y)$ is contained in some associated prime of A. Conversely, if $x \in \mathfrak{p} \in \mathrm{Ass}\, A$ then $\mathfrak{p} = \mathrm{Ann}\, y$ for some $y \neq 0$ so $xy = 0$ and x is a zero divisor. □

Exercise 54.8. Let A be a commutative noetherian ring, M an A-module, and $f \in A$. Show that $\mathrm{Ass}_{A_f} M_f = D(f) \cap \mathrm{Ass}_A M$.

Solution. Suppose $\mathfrak{p} \in D(f) \cap \mathrm{Ass}_A M$. Then $\mathfrak{p} = \mathrm{Ann}_A\, x$ for some $x \in M$. We argue $\mathrm{Ann}_{A_f}\, x = \mathfrak{p} A_f$. Certainly $\mathfrak{p} A_f \subset \mathrm{Ann}_{A_f}\, x$. Suppose $g \in A$ has image in $\mathrm{Ann}_{A_f}\, x$. Then $f^k g x = 0$ for some $k \geq 0$. Then $f^k g \in \mathrm{Ann}_A\, x = \mathfrak{p}$. But $f \notin \mathfrak{p}$ by assumption so $g \in \mathfrak{p}$.

Suppose $\mathfrak{p} \in \mathrm{Ass}_{A_f} M_f$. Then $\mathfrak{p} \in D(f)$ by definition and there is some $x \in M$ such that $\mathrm{Ann}_{A_f}\, x = \mathfrak{p}$. Consider the increasing chain of ideals $\mathrm{Ann}_A\, f^k x$. This stabilizes since A is noetherian, say to I. Suppose $ab \in I$. Then $abf^k x = 0$ so $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. In the former case, we have $ax = 0$ in A_f so $f^\ell ax = 0$ in A, whence $a \in I$. Thus I is prime. Moreover, $IA_f = \mathfrak{p}$. Indeed, if $a \in IA_f$ then $af^k x = 0$ in A so $ax = 0$ in M_f; if $a \in \mathfrak{p}$ then $ax = 0$ in A_f so $f^k ax = 0$ in A so $a \in I$. □

54.1 Normalization

54.2 Intuition from topology

Let k be a field. Fix a k-vector space V. The set of 1-dimensional subspaces of V is denoted $\mathbf{P}(V)$. When $V = k^{N+1}$ we also write $k\mathbf{P}^N$. This notation is usually only employed when $k = \mathbf{R}$ or $k = \mathbf{C}$.

[2]The definition makes sense even when A is not noetherian and M is not finitely generated. It is less clear how useful the definition is in this generality, however.

[3]By a zero divisor, we mean an element $a \in A$ such that there is a nonzero $b \in A$ with $ab = 0$.

Every non-zero vector in V spans a 1-dimensional subspace. This gives a surjection $V \smallsetminus \{0\} \to \mathbf{P}V$. Two vectors span the same 1-dimensional subspace if and only if one is a non-zero multiple of the other. That is, we get a bijection

$$(V \smallsetminus \{0\})/k^* \simeq \mathbf{P}V$$

where $(V \smallsetminus \{0\})/k^*$ is the set of orbits of the group k^* acting on $V \smallsetminus \{0\}$.

When V has a topology (for example $k = \mathbf{R}$ or $k = \mathbf{C}$) this allows us to put a topology on $\mathbf{P}V$. However, this doesn't explain why this topology is natural.

Suppose $W \subset V$ has codimension 1 and let W' be the translate of W by a vector not in W. We obtain a map $W \simeq W' \to \mathbf{P}V$. Its complement is the natural inclusion $\mathbf{P}W \subset \mathbf{P}V$.

Exercise 54.9. Show that, when $k = \mathbf{R}$ or $k = \mathbf{C}$, the inclusions $W \subset \mathbf{P}V$ constructed above are open embeddings and that they cover $\mathbf{P}V$.

We can weaken the assumption that V be a vector space in this construction, at least when $k = \mathbf{R}$ or $k = \mathbf{C}$. What we really need is for V to be an *cone*. That is V should carry a continuous action of the multiplicative monoid k. The *vertex* of V is $0.V$.

Exercise 54.10. The vertex of V is the same as the fixed locus of k^*.

Solution. Suppose $k^* v = v$. Then $0v = \lim_{\lambda \to 0} \lambda v = v$ so $v \in 0.V$. Conversely, if $v = 0w$ then $\lambda v = \lambda 0 w = 0w = v$ for all $\lambda \in k$. □

Exercise 54.11. There is a continuous retraction $V \to 0.V$ sending v to $\lim_{\lambda \to 0} \lambda v$. (Note we are assuming $k = \mathbf{R}$ or $k = \mathbf{C}$ here.)

Define $\mathbf{P}V$ to be the set of lines in V, equivariant closed embeddings $k \to V$.

54.3 Cones in algebraic geometry

54.4 Line bundles

If $\mathbf{P}V$ is supposed to be the space of lines in the vector space V then a map $X \to \mathbf{P}V$ should be a family of lines in V *parameterized by* X. In this section, we make sense of what a "family of lines" is supposed to be.

We give several definitions of a line bundle over a scheme. The first will be familiar to those with background in differential geometry.

Exercise 54.12. Show that the functor $A \mapsto \mathrm{GL}_r(A)$ is representable by an affine scheme. (Hint: Show $\mathrm{Mat}_{m \times n}$ is representable by an affine scheme and then construct GL_r as an principal affine open subscheme of $\mathrm{Mat}_{r \times r}$.)

Definition 54.13. A map $X \times \mathbf{A}^n \to X \times \mathbf{A}^m$ is *linear* if it is of the form $(x, t) \mapsto (x, \lambda(x)t)$ where $\lambda : X \to \mathrm{Mat}_{m \times n}(r)$ is a morphism of schemes.

Exercise 54.14. Give an equivalent definition of linearity for a map $\operatorname{Spec} A \times \mathbf{A}^n \to \operatorname{Spec} A \times \mathbf{A}^m$ in terms of the homomorphism of commutative rings

$$A[t_1, \ldots, t_m] \to A[t_1, \ldots, t_n].$$

Solution. There is a $M \in \mathrm{Mat}_{m\times n}(A)$ such that

$$(t_1, \dots, t_m) \mapsto (t_1, \dots, t_m)M.$$

$\square$

Definition 54.15 (Line bundles via charts)**.** A line bundle on a scheme X is a scheme L and a projection $\pi : L \to X$, together with a cover of X by affine open subschemes $U \subset X$ and isomorphisms $\phi_U : \pi^{-1}U \simeq X \times \mathbf{A}^1$ such that the transition maps

$$(U \cap V) \times \mathbf{A}^1 \xleftarrow{\phi_U|_{U\cap V}} \pi^{-1}(U \cap V) \xrightarrow{\phi_V|_{U\cap V}} (U \cap V) \times \mathbf{A}^1$$

are linear.

Definition 54.16 (Line bundles via the functor of points)**.** A line bundle on a scheme X is a scheme L over X with an action of $\mathbf{A}^1$ on the fibers of L over X that is locally isomorphic in X to the action of $\mathbf{A}^1$ on itself.

Exercise 54.17. Show that these two definitions of line bundles are equivalent.

Reading 54.18. [Vak14, §§5.4, 9.7]

Definition 54.19. Let $A \to B$ be an injective homomorphism of commutative rings. We say that A is *integrally closed* in B if every $x \in B$ that satisfies a monic polynomial with coefficients in A lies in A.

Definition 54.20. A scheme X is said to be *normal* if for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is an integrally closed domain.

54.5 Torsors under the multiplicative group

Definition 54.21 (Torsor)**.** Let G be an algebraic group. A G-torsor over a scheme X is a G-action on an X-scheme P such that there is a cover of X by open subschemes $U \subset X$ such that $P\big|_U$ is isomorphic to G_U as a G-scheme.

Exercise 54.22. (i) Suppose that L is a line bundle on X. Define $\Phi(L) = \underline{\mathrm{Isom}}(L, \mathbf{A}^1_X)$ to be the sheaf whose value on $U \subset X$ is the set of isomorphisms $L \simeq \mathbf{A}^1_U$. Show that $\underline{\mathrm{Isom}}(L, \mathbf{A}^1_X)$ is a $\mathbf{G}_m$-torsor.

(ii) Suppose that P is a $\mathbf{G}_m$-torsor on X. Define $\Psi(P) = \underline{\mathrm{Hom}}(P, \mathbf{A}^1_X)$ to be the sheaf whose value on $U \subset X$ is the set of $\mathbf{G}_m$-equivariant morphisms $P\big|_U \to \mathbf{A}^1_U$. Show that $\underline{\mathrm{Hom}}(P, \mathbf{P}^1_X)$.

(iii) Show that Φ and Ψ define inverse equivalences of categories between the category of $\mathbf{G}_m$-torsors and the category of line bundles on X.

55 Quasicoherent sheaves on projective space

C Chow's lemma

56 Morphisms to projective space

56.1 Blowing up

56.2 A criterion for closed embeddings

56.3 Ample line bundles

56.4 Another proof of Noether normalization

In this section we will find a more geometric construction of Noether normalization. This yields a slightly less general version of the theorem, but it is just as good for practical purposes.

Exercise 56.1. (i) Let S be a scheme and suppose that M is a $m \times n$ matrix with coefficients in $\Gamma(S, \mathcal{O}_S)$ such that for every point $\xi \in S$ the matrix $M(\xi)$ has rank m. Construct a morphism of S-schemes $\mathbf{P}^m_S \to \mathbf{P}^n_S$ sending $(\mathscr{L}, x)$ to $(\mathscr{L}, xM)$.

(ii) What goes wrong in the previous part when $\operatorname{rank} M < m$?

(iii) More generally, suppose that M is an $m \times n$ matrix as above. Let $U \subset \mathbf{P}^m_S$ be the subfunctor consisting of all $(\mathscr{L}, x) \in \mathbf{P}^m_S$ such that xM generates L. Show that the formula above gives a map $U \to \mathbf{P}^n_S$.

(iv) Show that U is open in $\mathbf{P}^m_S$.

Solution. To show U is open in $\mathbf{P}^m_S$ is a local problem. We can therefore assume $S = \operatorname{Spec} A$ and that $L = A$. Then we have elements $xM_1, \ldots, xM_n \in A$ and want to show that there is a universal A-scheme over which these generate A as a module. By definition, this is $D(xM_1, \ldots, xM_n)$, which is open by definition. □

Exercise 56.2. Let k be a field, $p \in \mathbf{P}^n_k$, and $H \subset \mathbf{P}^n_k$ a hyperplane. We make the following construction precise: For any point $q \in \mathbf{P}^n_k \smallsetminus \{p\}$, there is a unique line connecting p and q, denoted $L(p, q)$. This line intersects H in a unique point, hence determines a map $\mathbf{P}^n_k \smallsetminus \{p\} \to \mathbf{P}^{n-1}_k$.

(i) First we explain what we mean by a hyperplane. Fix a linear equation $f(x_0, \ldots, x_n)$. For any k-scheme S, we let $H(S)$ be the set of all $(\mathscr{L}, x_0, \ldots, x_n) \in \mathbf{P}^n_k(S)$ (here $\mathscr{L}$ is an invertible sheaf on S and $x_i \in \Gamma(S, \mathscr{L})$ generate $\mathscr{L}$) such that $f(x_0, \ldots, x_n) = 0$ as an element of $\mathscr{L}$. Show that H is representable by a closed subscheme of $\mathbf{P}^n_k$ and that $H \simeq \mathbf{P}^{n-1}_k$.

(ii) Suppose that p and q are two *disjoint* S-points of $\mathbf{P}^n_k$. Show that there is a unique linear map $g : \mathbf{P}^1_S \to \mathbf{P}^n_k$ such that $g(0) = p$ and $g(\infty) = q$. (Here 0 is the S-point $(\mathcal{O}_S, 0, 1) \in \mathbf{P}^1_S(S)$ and ∞ is the S-point $(\mathcal{O}_S, 1, 0) \in \mathbf{P}^1_S(S)$. 'Linear' means that the map must be of the form $(\mathscr{L}, x_0, x_1) \mapsto (\mathscr{L}, g(x_0, x_1))$ where g is a linear function.)

(iii) Let g and H be as in the last two parts. Show that $g^{-1}H$ consists of a single S-point of $\mathbf{P}^1_S$.

Exercise 56.3. Let $X = \operatorname{Spec} B$ and assume that $f : X \to \mathbf{A}^n_k$ is a finite map that is not surjective. We construct a finite map $X \to \mathbf{A}^{n-1}_k$.

(i) Embed $\mathbf{A}^n_k \subset \mathbf{P}^n_k$ by the map sending $(x_1, \ldots, x_n)$ to $(\mathcal{O}, 1, x_1, \ldots, x_n)$. Show that there is a finite map $\bar{f} : \overline{X} \to \mathbf{P}^n_k$ such that $\bar{f}^{-1}\mathbf{A}^n_k = X$ and $\bar{f}\big|_X = f$. (Hint: Take the 'integral closure' of $\mathbf{P}^n_k$ in X.) (Suggestion: You may want to skip this part of the problem, since it is not necessary to prove Noether normalization if you set up your induction carefully.)

(ii) Choose a point $p \in \mathbf{P}^n_k$ not on H or X.

57 The relative spectrum

Reading 57.1. [MO, §I.7], [Gil11]

Suppose that X is a locally ringed space and that $\mathscr{A}$ is a sheaf of $\mathcal{O}_X$-algebras on X. We can define a functor on locally ringed spaces: for any locally ringed space S, let $F(S)$ be the collection of all (f, φ) where

(i) $f : S \to X$ is a morphism of locally ringed spaces, and

(ii) $\varphi : f^{-1}\mathscr{A} \to \mathcal{O}_Y$ is a $f^{-1}\mathcal{O}_X$-algebra homomorphism.

Exercise 57.2. The second datum could be replaced with an $\mathcal{O}_X$-algebra homomorphism $\mathscr{A} \to f_*\mathcal{O}_Y$.

Our goal is to demonstrate that F is representable.

Exercise 57.3. Show that if $X = \operatorname{Spec} k$ is a point, where k is a field, in which case $\mathscr{A}$ is just a k-algebra A, then F is representable by $\operatorname{Spec} A$.

Exercise 57.4. Show more generally that if X is a point, in which case $\mathcal{O}_X$ is a local ring B, and $\mathscr{A}$ is a $\mathcal{O}_X$-algebra A, that F is representable by $\operatorname{Spec} A \times_{\operatorname{Spec} B} X$.

Exercise 57.5. Show that if $\mathscr{A}$ is a sheaf of *local* rings then F is representable by $(X, \mathscr{A})$.

In order to construct $\operatorname{Spec} \mathscr{A}$, we first construct its underlying topological space. For each $x \in X$, let $\mathscr{A}(x) = \mathscr{A}_x \otimes_{\mathcal{O}_{X,x}} \mathbf{k}(x)$. Define

$$\operatorname{Spec} \mathscr{A} = \coprod_{x \in X} \operatorname{Spec} \mathscr{A}(x) = \{(x, z) \mid x \in X, z \in \operatorname{Spec} \mathscr{A}(x)\}$$

There is a function $\pi : \operatorname{Spec} \mathscr{A} \to X$ sending (x, z) to x.

Exercise 57.6. Show that, equivalently, a point of $\operatorname{Spec} \mathscr{A}$ consists of a point $x \in X$ and a prime ideal $\mathfrak{p} \subset \mathscr{A}_x$ such that, if $\alpha : \mathcal{O}_{X,x} \to \mathscr{A}_x$ is the structural map, then $\alpha^{-1}\mathfrak{p}$ is the maximal ideal of the local ring $\mathcal{O}_{X,x}$.

Now we need to give $\operatorname{Spec} \mathscr{A}$ a topology. For each $(x, z) \in \operatorname{Spec} \mathscr{A}$, let $\mathbf{k}(x, z)$ be the residue field of $\mathscr{A}_x$ at the points $z \in \operatorname{Spec} \mathscr{A}_x$.

Exercise 57.7. For each open $U \subset X$, construct a map $\pi^{-1}U \to \operatorname{Spec} \mathscr{A}(U)$ such that the restriction to $\pi^{-1}x = \operatorname{Spec} \mathscr{A}(x)$ is induced by the composition

$$\mathscr{A}(U) \to \mathscr{A}_x \to \mathscr{A}(x).$$

We give $\operatorname{Spec} \mathscr{A}$ the coarsest topology such that all of the maps

$$\pi^{-1}U \to \operatorname{Spec} \mathscr{A}(U)$$

and $\pi : \operatorname{Spec} \mathscr{A} \to X$ are continuous.

It is also possible to construct this topology without prior knowledge of $\operatorname{Spec} \mathscr{A}(U)$. That is the content of the next exercise (which is also important for constructing the sheaf of rings on $\operatorname{Spec} \mathscr{A}$).

Suppose that $U \subset X$ is open and $f \in \mathscr{A}(U)$. If $x \in U$ and $z \in \operatorname{Spec} \mathscr{A}(x)$, define $f(x,z)$ to be the image of f under the homomorphism

$$\mathrm{ev}_{(x,z)} : \mathscr{A}(U) \to \mathscr{A}_x \to \mathscr{A}(x) \xrightarrow{\mathrm{ev}_z} \mathbf{k}(x,z).$$

Let $D_U(f)$ be the set of points (x,z) in $\pi^{-1}U$ such that $f(x,z) \neq 0$.

Exercise 57.8. Suppose that $\varphi : \pi^{-1}U \to \operatorname{Spec} \mathscr{A}(U)$ denotes the continuous function constructed above.

(i) Show that $\varphi^{-1} D_{\operatorname{Spec} \mathscr{A}(U)}(f) = D_U(f)$

(ii) Show that the sets $D_U(f)$ are a basis for the topology of $\operatorname{Spec} \mathscr{A}$.

Morally, we will construct the structure sheaf on $\operatorname{Spec} \mathscr{A}$ by sheafifying the presheaf $\mathscr{B}(D_U(f)) = \mathscr{A}(U)[f^{-1}]$. However, it is a little tricky to get this right, because it is possible for $D_U(f)$ to coincide with $D_V(g)$ for different open subsets U and V, and therefore for the rings $\mathscr{A}(U)[f^{-1}]$ and $\mathscr{A}(V)[g^{-1}]$ to be different. This makes it hard to say just what the presheaf in question should be. We could get around this by considering $\mathscr{B}$ as a presheaf on a basis in which a single open set has multiple representatives (an example of a Grothendieck topology), but we will take a more direct route and bake the sheafification step into the definition.

In any case, we can see what the stalks of $\mathscr{O}_{\operatorname{Spec} \mathscr{A}}$ will be. For each $(x,z) \in \operatorname{Spec} \mathscr{A}$, let $\mathfrak{p}_{(x,z)}$ be the kernel of the map

$$\mathrm{ev}_{(x,z)} : \mathscr{A}_x \to \mathscr{A}(x) \to \mathbf{k}(x,z).$$

Then define $\mathscr{A}_{(x,z)}$ to be the local ring of $\mathscr{A}_x$ at $\mathfrak{p}_{(x,z)}$.

This effectively tells us the underlying set of the espace étalé of $\mathcal{O}_{\operatorname{Spec} \mathscr{A}}$. We can characterize the sheafification of a presheaf as the space of continuous sections of the espace étalé, which allows us to construct the sheafification as a *subsheaf* of an ambient sheaf. We do the same thing here:

Definition 57.9. For each open $U \subset \operatorname{Spec} \mathscr{A}$, let $\mathcal{O}_{\operatorname{Spec} \mathscr{A}}(U)$ be the collection of tuples $s \in \prod_{(x,z) \in U} \mathscr{A}_{(x,z)}$ with the following property:

> for each $(x,z) \in U$ there is an open neighborhood V of x in X and a section $g \in \mathscr{A}(V)$ with $g(x,z) \neq 0$ such that, for every $(y,w) \in D_V(g)$, the component $s_{(y,w)}$ is the image of $\mathscr{A}(V)[g^{-1}] \to \mathscr{A}_{(y,w)}$.

Exercise 57.10. Show that $\mathcal{O}_{\operatorname{Spec}\mathscr{A}}$ is a sheaf. (Hint: it is a subpresheaf of the sheaf $\mathscr{F}$ where $\mathscr{F}(U) = \prod_{(x,z)\in U} \mathscr{A}_{(x,z)}$.)

By construction, the stalks of $\mathcal{O}_{\operatorname{Spec}\mathscr{A}}$ are local rings. Now we build the projection map $\pi : \operatorname{Spec}\mathscr{A} \to X$ and show it is a morphism of locally ringed spaces. Define $\pi(x,z) = x$. To get a map $\mathcal{O}_X \to \pi_*\mathcal{O}_{\operatorname{Spec}\mathscr{A}}$ consider an open $U \subset X$. If $f \in \mathcal{O}_X$, we get a map

$$\mathcal{O}_X(U) \to \mathscr{A}(U) \to \prod_{x\in U} \mathscr{A}_x \to \prod_{(x,z)\in\pi^{-1}U} \mathscr{A}_{(x,z)}.$$

If $f \in \mathcal{O}_X(U)$, its image can certainly be represented on $D_U(1)$ by the image of f in $\mathscr{A}(U)$, which gives us a well-defined map.

Exercise 57.11. Prove that $\operatorname{Spec}\mathscr{A}$, with the sheaf constructed above, represents the functor F.

Solution. Suppose we have a map of locally ringed spaces $u : S \to X$ and an $\mathcal{O}_X$-algebra map $\varphi : \mathscr{A} \to u_*\mathcal{O}_S$. We construct a $v : S \to \operatorname{Spec}\mathscr{A}$ such that $\pi v = u$ and show it is unique. If $s \in S$, we get a homomorphism $\mathscr{A}(x) \to \mathbf{k}(s)$. The kernel is a point $z \in \pi^{-1}x$, so we define $v(s) = (x,z)$.

To see that this is continuous, consider $D_U(f) \subset \operatorname{Spec}\mathscr{A}$. Its preimage in S is precisely $D_S(\varphi(f))$.

For each $s \in S$, we have a local homomorphism of $\mathcal{O}_{X,x}$-algebras

$$\mathscr{A}_{v(s)} \to \mathcal{O}_{S,s}$$

by the construction of v. For each open $U \subset \operatorname{Spec}\mathscr{A}$, this gives us a map

$$\psi : \prod_{(x,z)\in U} \mathscr{A}_{(x,z)} \to \prod_{s\in v^{-1}U} \mathcal{O}_{S,s}.$$

We have to check it carries $\mathcal{O}_{\operatorname{Spec}\mathscr{A}}(U)$ into $\mathcal{O}_S(v^{-1}U)$. Suppose that $f \in \mathcal{O}_{\operatorname{Spec}\mathscr{A}}(U)$. We want to prove $\psi(f) \in \mathcal{O}_S(v^{-1}U)$, which is a local question since $\mathcal{O}_S$ is a sheaf. Therefore we can replace U with an open set $D_V(g)$ such that f is in the image of $\mathscr{A}(V)[g^{-1}]$. But then $\psi(f)$ is in the image of the map

$$\mathscr{A}(V)[g^{-1}] \to \mathcal{O}_S(v^{-1}D_V(g)) = \mathcal{O}_S(D_{u^{-1}V}(\varphi(g))) \to \prod_{s\in D_{u^{-1}V}(\varphi(g))} \mathcal{O}_{S,s}$$

as required. □

Exercise 57.12. Adapt the construction of this section to show that, if X is a ringed space, the functor

$$F(S) = \operatorname{Hom}_{\mathbf{RS}}(S, X)$$

is representable by a locally ringed space.

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