# Math 52 – Spring 2012
# Exam #3

No materials except pen or pencil and paper. Make sure to explain your answers fully.

First name:

Last name:

Student number:

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|---|---|
| Problem 1 | /8 |
| Problem 2 | /10 |
| Problem 3 | /10 |
| Problem 4 | /10 |
| Problem 5 | /10 |
| Problem 6 | /16 |
| Problem 7 | /10 |
| Problem 8 | /8 |
| Problem 9 | /8 |
| Problem 10 | /10 |
| Problem 11 | /10 |
| Total | / 100 |

**Problem 1.** (8 points) Let $C$ be the path

$$\begin{aligned} x(t) &= \cos(2\pi t) \\ y(t) &= \sin(2\pi t) \\ z(t) &= t \end{aligned}$$

Compute the length of the path traversed as $t$ increases from 0 to 1.

**Problem 2.** (10 points) Let $R$ be the square defined by the inequalities $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Let $f(x, y) = x + y - 1$. Suppose that $\delta$ is a function on $R$ such that $2 \leq \delta(x, y) \leq 4$ for all values of $x$ and $y$. What is the largest possible value of $\int_R \delta f \, dA$?

**Problem 3.** (10 points) Consider the vector field

$$\mathbf{F} = \left(\sqrt{x^3+x} + y^2, e^{-y^2} + 3x\right).$$

Compute

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where $C$ is the boundary of the rectangle $[0,2] \times [0,1]$ oriented counterclockwise. (Hint: $\sqrt{x^3+x}\,dx$ and $e^{-y^2}\,dy$ do not have closed form antiderivatives.)

**Problem 4.** (10 points) Rewrite the integral

$$\int_{x=-2}^{3} \int_{y=x^2-3}^{x+3} f(x, y)\, dy\, dx$$

as an integral (or sum of integrals) of the form

$$\int_{y=?}^{?} \int_{x=?}^{?} f(x, y)\, dx\, dy.$$

**Problem 5.** (10 points) Let $R$ be the 3-dimensional solid region defined by the inequalities

$$x^2 + \frac{y^2}{4} \leq z \leq 6x + y.$$

Compute the volume of $R$. (Hint: first make the change of coordinates $u = x - 3$, $v = \frac{y}{2} - 1$, and $w = z - 6x - y + 10$, then use cylindrical coordinates.)

**Problem 6.** (16 points) Consider the curve $C$ with equations and inequalities,

$$(x-2)^2 + z^2 - 1 = y = 0$$
$$x \geq 2$$
$$z \geq 0$$

(a) (6 points) Find the centroid of $C$.

Let $S$ be the surface obtained by rotating $C$ around the $z$-axis.

(b) (2 points) Find the surface area of $S$.

(c) (8 points) Compute $\int_S \mathbf{F} \cdot \mathbf{n}\, dA$ where $\mathbf{F}$ is the vector field $(0, 0, 1)$ and $S$ is given the orientation pointing away from the $y$-axis. (Hint: use Stokes's theorem or the divergence theorem.)

**Problem 7.** (10 points) Let $\mathbf{F}$ be a vector field such that

$$\begin{aligned} \operatorname{div}(\mathbf{F}) &= 0 \\ \mathbf{F}(x,y,0) &= (\ ?\ ,\ ?\ , xy) \\ \mathbf{F}(x,0,z) &= (\ ?\ , xz,\ ?\ ) \\ \mathbf{F}(0,y,z) &= (yz,\ ?\ ,\ ?\ ). \end{aligned}$$

(The question marks stand for things that you will not need to complete this problem.) Suppose that $S$ is the triangle with vertices $(0,0,1)$, $(0,1,0)$, and $(1,0,0)$ with normal vector $\mathbf{n} = \frac{1}{\sqrt{3}}(1,1,1)$. Compute $\int_S \mathbf{F} \cdot \mathbf{n}\, dA$.

**Problem 8.** (8 points)

(a) (4 points) Find all values of $a$, $b$, $c$, and $d$ such that the vector field

$$\mathbf{F} = \big(ax + by, cx + dy\big)$$

is conservative on the plane. **Justify your answer.**

(b) (4 points) Suppose that a surface $S$ is parameterized with coordinates $u$ and $v$ and

$$\frac{\partial(x,y)}{\partial(u,v)} = 2 \qquad \frac{\partial(z,x)}{\partial(u,v)} = 1 \qquad \frac{\partial(y,z)}{\partial(u,v)} = -2.$$

What is the surface area traced out by the parameters $-1 \leq u \leq 2$ and $-1 \leq v \leq 1$?

**Problem 9.** (8 points) Let $S$ be the paraboloid

$$z = x^2 + y^2 \leq 4.$$

Suppose that $S$ is rotating around the axis $(1, 1, 1)$. Find the points of $S$ that will not experience a Coriolis effect (recall that these are the points of $S$ where a normal vector is perpendicular to the axis of rotation). Indicate these points on a sketch of $S$.

**Problem 10.** (10 points) In order to discourage guessing, each of the multiple choice questions below is worth 2 points for a correct answer and −1 point for an incorrect answer. Problems that require justification have additional point values, as indicated.

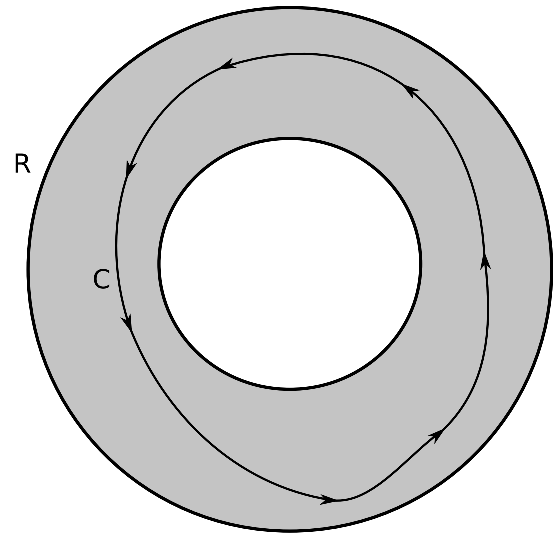
(a) (2 points) Let $R$ be the shaded region to the right and suppose that $\int_C \mathbf{F} \cdot (dx, dy) = 0$. Decide whether

 (i) there is a function $f$ on $R$ such that $F = \text{grad}(f)$,

 (ii) such a function may exist but is not guaranteed, or

 (iii) it is impossible that there is such a function.

Indicate your answer by circling the numeral of whichever response is true.



(b) (2 points) There is a closed curve $C$ inside the region $R$ at right such that

$$\int_C \mathbf{F} \cdot (dx, dy) \neq 0.$$

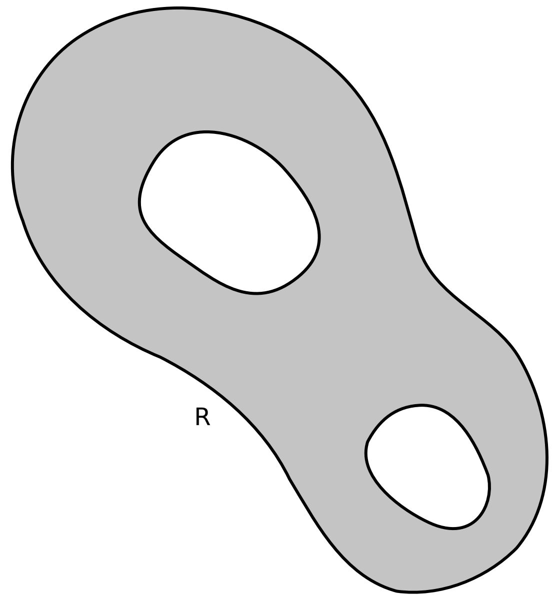
Decide whether

(i) $\text{curl}(\mathbf{F}) = 0$,

(ii) it is possible that $\text{curl}(\mathbf{F}) = 0$ but not guaranteed, or

(iii) it is impossible that $\text{curl}(\mathbf{F}) = 0$.

Indicate your answer by circling the numeral of whichever response is true.



(c) (6 points) Assume that **F** is a vector field that is nice (its components have all partial derivatives of all orders) on the whole plane except at the points $O$, $P$, and $Q$, that curl(**F**) = 0 where it is defined, and

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_D \mathbf{F} \cdot d\mathbf{r} = \int_E \mathbf{F} \cdot d\mathbf{r} = 0.$$

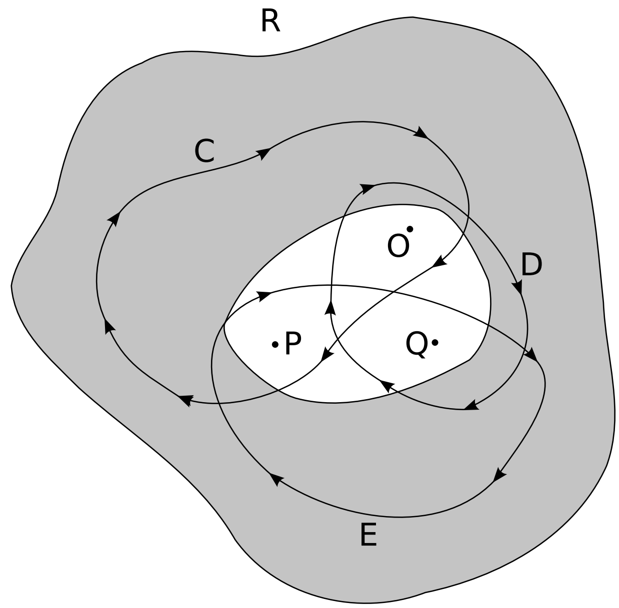
Decide whether

(i) on $R$, the vector field **F** is the gradient of a function,

(ii) **F** might or might not be the gradient of a function on $R$, or

(iii) **F** is not the gradient of a function on $R$.

Indicate your answer by circling the numeral of whichever response is true. **Then justify your answer below.**



**Problem 11.** (extra credit: 10 points) Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right).$$

Define a new vector field

$$\mathbf{G}(x_0,y_0) = \int_R \mathbf{F}(x_0-x, y_0-y)\, dA_R = \int_R \mathbf{F}(x_0-x, y_0-y)\, dx\, dy$$

where $R$ is the disc $x^2+y^2 \leq 1$.

(a) (8 points) Compute $\int_C \mathbf{G} \cdot (dx, dy)$ where $C$ is the loop $x^2+y^2=4$, oriented counterclockwise.

(b) (2 points) Find a number $a$ such that on the region $S$ defined by the inequalities $4 \leq x^2+y^2 \leq 9$, the vector field $\mathbf{G} + a\mathbf{F}$ is the gradient of a function.

**Extra space — do not detach.**

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