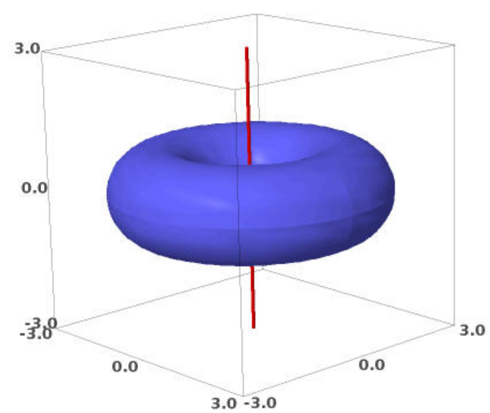
Math 52 – Spring 2012
Exam #2

You can use any resources you like—books, notes, internet, etc.—except other people. Note: talking to people on the internet (e.g., on question and answer sites) counts as talking to other people.

The exam is due by 5pm on Wednesday, May 23.

Problem 1. (11 points) The surface below is a torus that we will call S. The red line is the z-axis. You can see more visualizations of this torus at this webpage: http://nt.sagenb.org/home/pub/165/.



The following is a parameterization of this torus:

$$x = (2 + \cos(v))\cos(u) \qquad 0 \le u \le 2\pi$$
$$y = (2 + \cos(v))\sin(u) \qquad 0 \le v \le 2\pi$$
$$z = \sin(v)$$

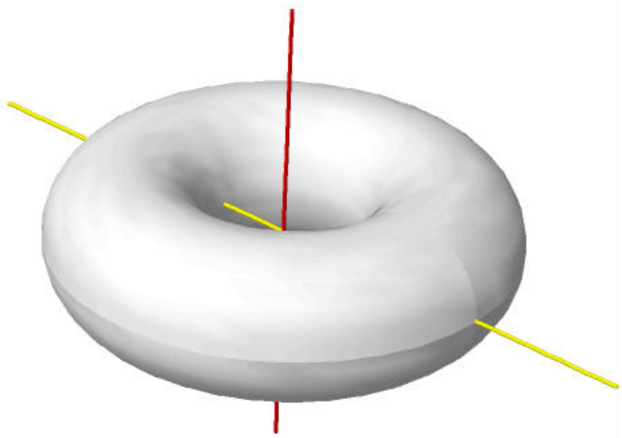
(a) (2 points) Find a normal vector to the surface of S at the point with uv-coordinates (u, v) (it does not have to be a unit normal vector).

(b) (2 points) Compute the Jacobian of the transformation from uv-coordinates to xyz-coordinates *on the torus*. In other words, compute the ratio between the infinitesimal area dA_S and dA_{uv} at a point with uv-coordinates (u, v).

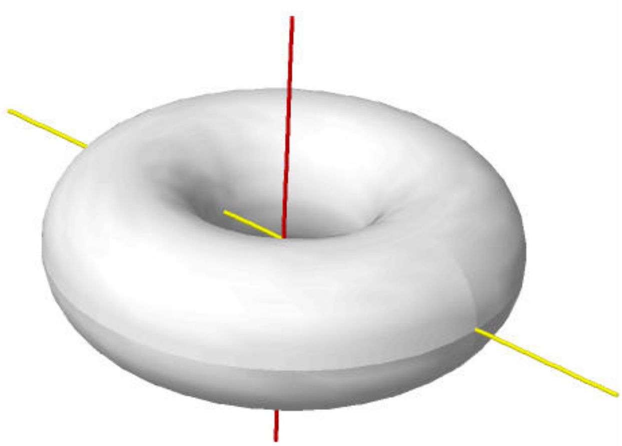
(c) (2 points) Find the surface area of S.

(d) (1 point) Find the curl of the vector field $(-y, x, 0)$. (Note that this corresponds to rotation at a constant rate around the z-axis.)

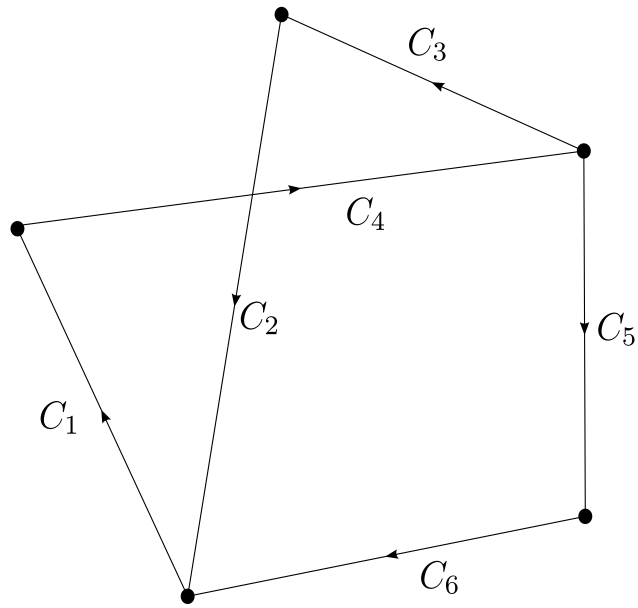
(e) (2 points) Suppose that the torus is rotating in the velocity vector field $(-y, x, 0)$ of the previous part. On a picture of the torus, such as the one below (you may also draw your own if you prefer), indicate the places where a person standing on the surface of the torus would experience no Coriolis effect. (The red line is the z-axis and the yellow line is the x-axis.)



(f) (2 points) Now suppose the torus is rotating in the velocity vector field $(0, -z, y)$. Indicate the points on the surface of the torus where there is no Coriolis effect.



Problem 2. (6 points) Suppose that F is a nice vector field on the entire plane. Consider curves $C_1, \ldots, C_6$ in the following configuration.



Assume that

$$\int_{C_1} F \cdot (dx, dy) = 1 \qquad \int_{C_2} F \cdot (dx, dy) = 4 \qquad \int_{C_3} F \cdot (dx, dy) = 2$$
$$\int_{C_4} F \cdot (dx, dy) = 4 \qquad \int_{C_5} F \cdot (dx, dy) = 3 \qquad \int_{C_6} F \cdot (dx, dy) = 1.$$

(a) (2 points) Could F be a conservative vector field?

(b) (2 points) Is it possible that $\text{curl}(F)$ is ≥ 0 at *all* points of the plane?

(c) (2 points) Is it possible that $\text{curl}(F)$ is $\neq 0$ at *all* points of the plane?

Problem 3. (8 points) Consider the region R defined by the inequalities,

$$0 \leq 4x + 3y \leq 25$$
$$0 \leq 4y - 3x \leq 25.$$

Suppose this region has density $\delta(x, y) = x^2 + y^2$.

In this problem you will have to write down various single variable integrals. Your score will be based not only on the correctness of your answer, but *also* on the number of single variable integrals you use (i.e., the number of integral signs in your formula): the fewer the better. (This is intended to discourage you from solving this problem by brute force.)

(a) (5 points) Compute the mass of R.

(b) (3 points) Write down an expression using single variable integrals (iterated integrals are okay, but multivariable integrals are not) for $E(f|R)$, where $f(x, y)$ is the distance of the point (x, y) from the origin.

Problem 4. (8 points) Suppose that a circle of radius 1 is *rolling* from left to right inside the plane so that the position of its center at time t is $(t, 0)$. Assume that a point on the edge of the circle has been marked—call this point P. You may want to imagine P as a piece of gum stuck to a rolling bicycle wheel. The curve traced out by C is called a *cycloid*.

Assume that when $t = 0$, the position of P is $(0, 1)$.

(a) (1 point) After $t = 0$, the y-coordinate of P next returns to 1 at $t = 2\pi$. Explain why this is.

(b) (1 point) Let C be the curve traversed by P between $t = 0$ and the value you found in the last part. Find a parameterization of C. Do not forget to indicate the values of t where your parameterization begins and ends.

(c) (2 points) Compute the distance travelled by P between $t = 0$ and the value you computed in the first part. (Hint: $1 + \cos(t) = 2\cos\left(\frac{t}{2}\right)^2$.)

(d) (1 point) Suppose that a force field $F = (x, 0)$ is acting on the wheel. Compute the work done by this field as P traverses the path from $t = 0$ to the value you computed in the first part. (Hint: is F conservative?)

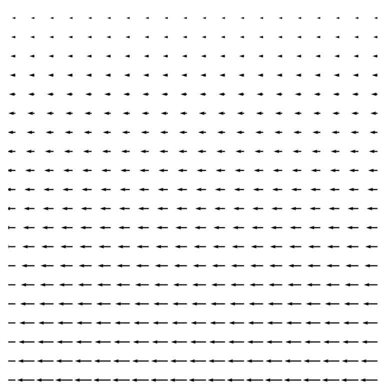
(e) (1 point) Let F be the vector field $(y, 0)$. Compute $\int_C F \cdot (dx, dy)$.

(f) (2 point) Use your answer from the last part to compute the area above C and below the line $y = 1$.

Problem 5. (4 points) Determine which of the following vector fields are gradients of functions on the region R that is indicated.

(a) (1 point) $F = (2x - 3y, 3x - y)$ and R is the whole plane.

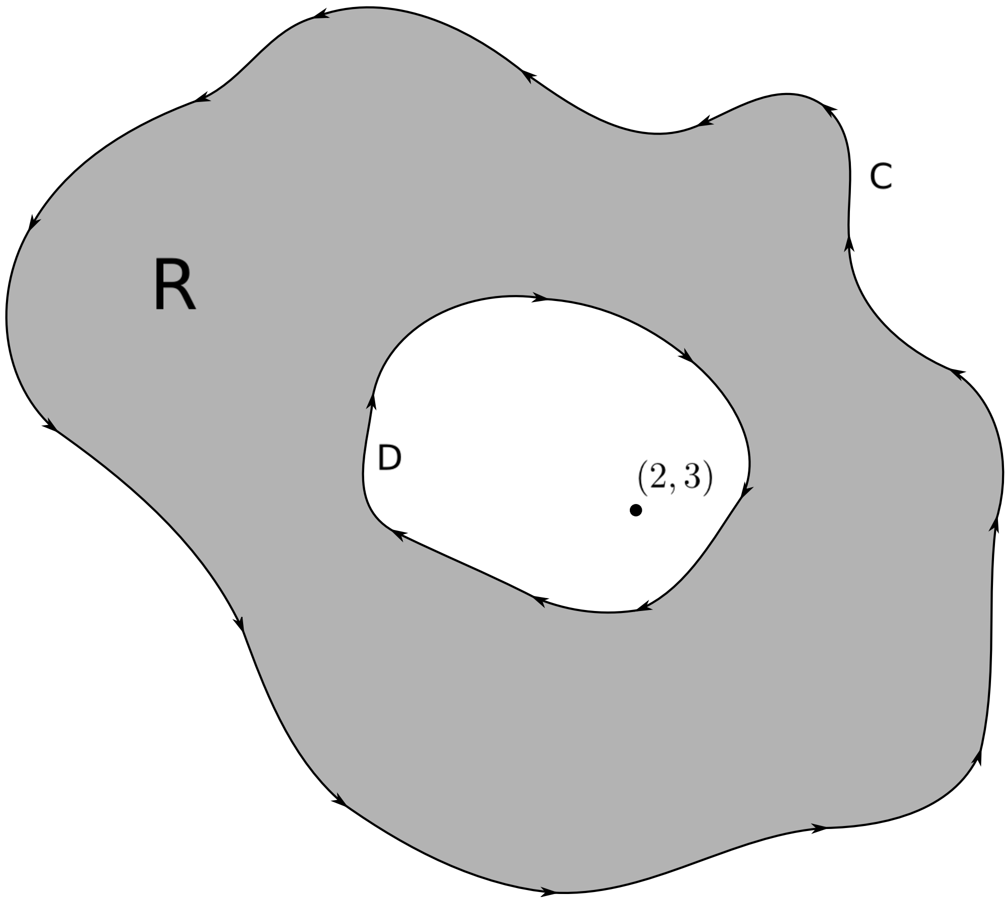
(b) (1 point) F is the vector field displayed below and R is the square region of the plane on which the vector field is shown.



(c) (2 points) $F = \left(\frac{-y}{(x-1)^2 + y^2}, \frac{x-1}{(x-1)^2 + y^2}\right) - \left(\frac{-y}{(x+1)^2 + y^2}, \frac{x+1}{(x+1)^2 + y^2}\right)$;

R is the annulus $4 \le x^2 + y^2 \le 9$.

Problem 6. (7 points) Let R be the region below.



(a) (3 points) Construct a vector field F on R and a closed curve E inside R such that $\mathrm{curl}(F) = 0$ and

$$\int_E F \cdot (dx, dy) \neq 0.$$

Indicate the curve E by drawing it on the region.

(b) (1 point) Using Green's theorem, conclude that R is not simply connected.

The boundary of R consists of the curves $C + D$, where C and D are as indicated in the image. The orientations on C and D are chosen so that R is always to the *left* as one traverses the curve in the indicated direction.

(c) (2 points) Demonstrate that if F is a vector field that is nice on R then

$$\int_R \mathrm{curl}(F)\, dA_R = \int_{\partial R} F \cdot (dx, dy) \qquad (*)$$

where $\partial R = C + D$ with the orientations as indicated. Notice that R is not simply connected, so you can't apply Green's theorem (in the form we discussed in class) directly! (Hint: divide R into pieces where you can use Green's theorem in the form we discussed.)

If R is any bounded region (possibly possessing holes) its boundary will be made up of multiple closed curves $C_1, \ldots, C_n$. We define the oriented boundary of R to be the sum $C_1 + \cdots + C_n$, with all curves oriented so that R is *to the left* of each curve when one traverses the curve according to its orientation.

(d) (1 point) Explain how you would use the ideas from this problem to show that Equation $(*)$ is true for *any* region R.

Problem 7. (extra credit: 3 points) Suppose that R is a region (that may or may not be simply connected) and F and G are vector fields on R with the following properties:

(i) both F and G are nice on R,

(ii) $\mathrm{curl}(F) = \mathrm{curl}(G) = 0$, and

(iii) every vector field on R that has curl zero is equal to $aF + bG + \mathrm{grad}(f)$ for some numbers a and b and some function f that is nice on R.

Justify your answer to the following question:

(a) What are the largest and smallest numbers of holes the region R could have?

Each of the following parts of this problem describes one or more *additional* assumptions about R, F, and G. In each situation, give the largest and smallest number of holes the region R could have, making sure to justify your answer.

(b) (iv) There is a closed curve C inside of R such that $\int_C (F + 3G) \cdot (dx, dy) \neq 0$.

(c) (iv) There is a closed curve C inside of R such that $\int_C F \cdot (dx, dy) \neq 0$, and
(v) there is a closed curve D inside of R such that $\int_D G \cdot (dx, dy) \neq 0$.

(d) (iv) For every closed curve C inside of R we have $\int_C (F + 3G) \cdot (dx, dy) = 0$.

(e) (iv) There is a closed curve C inside of R such that $\int_C F \cdot (dx, dy) = 1$ and $\int_C G \cdot (dx, dy) = 2$, and
(v) there is a closed curve D inside R such that $\int_D F \cdot (dx, dy) = 1$ and $\int_C G \cdot (dx, dy) = 3$.