Exam 2 Solutions

Math 2001 Fall 2026

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1. A natural number n is called *irreducible* if

$$\neg(n = 1) \wedge \forall a, b \in \mathbb{N},\ (a \times b = n) \implies (a = 1) \vee (b = 1).$$

   (a) *According to this definition*, is 0 irreducible? Justify your answer.

   Solution. No, 0 is not irreducible. This is because it is possible to find $a, b \in \mathbb{N}$ such that $ab = 0$ but neither a nor b is 1. For example, we could take $a = 0$ and $b = 2$.

   (b) Write a *maximally negated* formal proposition meaning n *is not irreducible*. 'Maximally negated' means that the $\neg$ symbol is only applied to propositions that cannot be broken down into simpler propositions.

   Solution. $(n = 1) \vee \exists a, b \in \mathbb{N},\ (a \times b = n) \wedge \neg(a = 1) \wedge \neg(b = 1)$

   (c) Set up a proof by strong induction for the following theorem :

   Theorem. *If $n \in \mathbb{N}$ and $n \geq 1$ then n can be written as a product of irreducible natural numbers.*

   You should make the induction hypothesis and goal clear. You do not have to prove the theorem !

   Proof. We prove this by strong induction on n. The induction hypothesis is that, for every natural number n, and every natural number $m < n$, if $m \geq 1$ then m can be written as a product of irreducible natural numbers. We must show that, if $n \geq 1$, then n can be written as a product of irreducible natural numbers. QED

   (d) Explain why a proof by ordinary induction would fail for this theorem.

   Solution. Proving $\forall n \in \mathbb{N},\ P(n)$ by ordinary induction requires finding a way to turn a proof for $P(n)$ into a proof for $P(s(n))$. But understanding how to write n as a product or irreducible numbers does not help to give a way to write $n + 1$ as a product of irreducible numbers because $n + 1$ is (usually) not a multiple of n.

2. What is wrong with the following induction proof ? Be specific. For full credit, you should identify the flaw that makes the proof incorrect. If you identify multiple errors, you should indicate which you believe is the fundamental error that cannot be fixed.

   Theorem. *Every number is both odd and even.*

   Proof. We will prove by induction on n that every natural number $n \in \mathbb{N}$ is both odd and even. We have to show that, for every $n \in \mathbb{N}$, if n is both odd and even then $s(n)$ is both odd and even. Let n be any natural number and assume that n is both odd and even. To see that $s(n)$ is even, we notice that $s(n) = n + 1$ by definition of addition. Since 1 is odd and n is odd, this means $s(n)$ is the sum of two odd numbers, hence is even. To see that $s(n)$ is odd, we again use the fact that $s(n) = n + 1$ and observe that becuase n is even and 1 is odd, $s(n)$ is the sum of an even number and an odd number, hence $s(n)$ is odd. Thus $s(n)$ is both even and odd, so by induction, we conclude that every natural number is both even and odd. QED

   Solution. The error is omitting the base case. Induction shows that the statement at $s(n)$ follows from the statement at n, but this gives no information if there isn't a place to start!

3. A function $f : \mathbb{N} \to \mathbb{N}$ is defined recursively by the following formula :

$$f(0) = 0 \qquad \forall n \in \mathbb{N},\ f(s(n)) = 2f(n) + 1$$

   (a) Compute $f(0)$, $f(1)$, $f(2)$, and $f(3)$ using the definition.

   Solution. We have $f(0) = 0$ by the base case of the definition of f. The step of f gives $f(1) = 2f(0)+1 = 2\times 0+1 = 1$. The step of f then gives $f(2) = 2f(1)+1 = 2 \times 1 + 1 = 3$. Finally, the step of f gives $f(3) = 2(f(2) + 1 = 2 \times 3 + 1 = 7$.

   (b) Prove the following theorem :

   Theorem. *For all $n \in \mathbb{N}$, we have $f(n) + 1 = 2^n$.*

   Proof. We prove this by induction on n. The base case asserts that $f(0)+1 = 2^0$. By definition, $f(0) = 0$, so $f(0) + 1 = 1$. On the other hand, $2^0 = 1$ by the base case of the definition of exponentiation. Therefore we have $f(0) + 1 = 1 = 2^0$, as required.

   For the inductive step, we have to show that $\forall n \in \mathbb{N}$, $(f(n) + 1 = 2^n) \implies (f(s(n)) + 1 = 2^{s(n)})$. Assume that n is a natural number and that $f(n)+1 = 2^n$. We have to show that $f(s(n)) + 1 = 2^{s(n)}$. By the step in the definition of f, we have $f(s(n)) = 2f(n) + 1$ so

$$\begin{aligned} f(s(n)) + 1 &= (2f(n) + 1) + 1 && \text{step of } f \\ &= 2f(n) + 2 && \text{algebra} \\ &= 2(f(n) + 1) && \text{left distributivity} \\ &= 2 \times 2^n && \text{inductive hypothesis} \\ &= 2^{s(n)} && \text{step of } 2^{(-)}. \end{aligned}$$

This completes the proof of the inductive step. We conclude that $f(n) + 1 = 2^n$ for every $n \in \mathbb{N}$, by induction. QED

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4. Prove the following statement :

   Theorem. *If n is a natural number and $n \geq 8$ then there are natural numbers a and b such that $3a + 5b = n$.*

   Proof. We prove this statement by strong induction on n. We assume that, for all $m < n$, if $m \geq 8$ then there are natural numbers a and b such that $3a + 5b = m$. We need to show that if $n \geq 8$ then there are natural numbers c and d such that $n = 3c + 5d$. There are three cases :

   CASE I: If $n < 8$ then we have a contradiction with $n \geq 8$, so there is nothing to show.

   CASE II: If $8 \leq n < 11$ then we argue by inspection. When $n = 8$, we may take $c = d = 1$ and then $8 = 3 \times 1 + 5 \times 1$. If $n = 9$, we may take $c = 3$ and $d = 0$ and then $9 = 3\times3+5\times0$. If $c = 10$, we may take $c = 0$ and $d = 2$ and then $10 = 3\times0+5\times2$.

   CASE III: If $n \geq 11$ then there is a natural number m such that $m \geq 8$ and $n = m + 3$. Since $m < n$, the induction hypothesis says that there are natural numbers a and b such that $m = 3a + 5b$. We take $c = a + 1$ and $d = b$. Then

   $$3c + 5d = 3(a + 1) + 5b = 3a + 3 + 5b = (3a + 5b) + 3 = m + 3 = n$$

   as required. Therefore there are natural numbers c and d such that $n = 3c + 5d$.

   We conclude by strong induction that $\forall n \in \mathbb{N}$, $(n \geq 8) \implies \exists a, b \in \mathbb{N}$, $3a + 5b = n$. QED

5. In this problem, we will consider *connected planar graphs*. A connected planar graph starts with one vertex and can be enlarged by the following two rules : Rule A adds a new vertex and a new edge connecting the new vertex to a vertex that was already present ; Rule B adds a new edge connecting two vertices that are already present. If G is a connected planar graph, we write $V(G)$ for the number of *vertices* in G, we write $E(G)$ for the number of *edges* in G, and we write $F(G)$ for the number of *faces* (or regions) cut out by G.

   (a) The accompanying figure shows a connected planar graph build using a sequence of applications of Rules A and B. In each picture G, count $V(G)$, $E(G)$, and $F(G)$. Verify that $V(G) + F(G) = E(G) + 2$ in each picture.

   Solution.

   (a) $V = 1$, $E = 0$, $F = 1$, $V + F = 1 + 1 = 2$, $E + 2 = 0 + 2 = 2$
   (b) $V = 2$, $E = 1$, $F = 1$, $V + F = 2 + 1 = 3$, $E + 2 = 1 + 2 = 3$
   (c) $V = 3$, $E = 2$, $F = 1$, $V + F = 3 + 1 = 4$, $E + 2 = 2 + 2 = 4$

(d) $V = 3$, $E = 3$, $F = 2$, $V + F = 3 + 2 = 5$, $E + 2 = 3 + 2 = 5$
(e) $V = 3$, $E = 4$, $F = 3$, $V + F = 3 + 3 = 6$, $E + 2 = 4 + 2 = 6$
(f) $V = 4$, $E = 5$, $F = 3$, $V + F = 4 + 3 = 7$, $E + 2 = 5 + 2 = 7$

(b) Prove, for every connected planar graph G, that $V(G) + F(G) = E(G) + 2$.

Proof. We prove this by induction on the number of rules applied making G. The base case is where no rules are applied. Then the graph is just one vertex. In this case, there is one face, and there are no edges. Thus $V(G) + F(G) = 1 + 1 = 2$ and $E(G) + 2 = 0 + 2 = 2$. This proves the base case.

In the induction step, we assume that when G is built in n steps we have $V(G) + F(G) = E(G) + 2$. We have to show that if G' is build in $s(n)$ steps then $V(G') + F(G') = E(G') + 2$. Since G' is built in $s(n)$ steps, it is build by applying either Rules A or Rule B to a graph G built in n steps. We separate two cases :

CASE I: If G' is built from G by Rule A then G' has one more edge and one more vertex than G. There are no new faces. Thus $V(G') = V(G)+1$ and $E(G') = E(G) + 1$. This gives us

$$\begin{aligned} V(G') + F(G') &= V(G) + 1 + F(G) && F(G') = F(G) \text{ and } V(G') = V(G) + 1 \\ &= V(G) + F(G) + 1 && \text{commtuativity and associativity of addition} \\ &= E(G) + 2 + 1 && \text{induction hypothesis} \\ &= E(G) + 1 + 2 && \text{commutativity and associativity of addition} \\ &= E(G') + 2 && E(G') = E(G) + 1 \end{aligned} \tag{1}$$

so $V(G') + F(G') = E(G') + 2$, as required.

CASE II: If G' is built from G by Rule B then G' has one more edge but no more vertices than G. Therefore the new edge must divide one of the faces of G into two. Thus $V(G') = V(G)$ and $E(G') = E(G)+1$ and $F(G') = F(G)+1$. Thus

$$\begin{aligned} V(G') + F(G') &= V(G) + F(G) + 1 && F(G') = F(G) + 1 \text{ and } V(G') = V(G) \\ &= E(G) + 2 + 1 && \text{induction hypothesis} \\ &= E(G) + 1 + 2 && \text{commutativity and associativity of } + \\ &= E(G') + 2 && E(G') = E(G) + 1, \end{aligned} \tag{2}$$

so $V(G') + F(G') = E(G') + 2$, as required.

We conclude by induction that $V(G) + F(G) = E(G) + 2$ for every connected planar graph G. QED