Exam 2

Math 2001 Fall 2026

21 October 2026

Name _____

Instructions : work alone, using only a writing implement. Make sure your full name is written above. Justification is not required unless specifically requested, but justification may be helpful for partial credit in case of an incorrect answer.

AI will not be involved in grading unless you initial the box below. However, AI may be used for optical character recognition (OCR) to match your name to the class roster. If you approve AI grading checking, only grading corrections that improve your grade will be adopted.

☐ Initial in the box to give me permission to check my grading of your work using AI.

1. A natural number n is called *irreducible* if

$$\neg(n = 1) \wedge \forall a, b \in \mathbb{N},\ (a \times b = n) \implies (a = 1) \vee (b = 1).$$

(a) (2 points) *According to this definition*, is 0 irreducible? Justify your answer.

(b) (2 points) Write a *maximally negated* formal proposition meaning n *is not irreducible*. 'Maximally negated' means that the $\neg$ symbol is only applied to propositions that cannot be broken down into simpler propositions.

(c) (2 points) Set up a proof by strong induction for the following theorem :

Theorem. *If $n \in \mathbb{N}$ and $n \geq 1$ then n can be written as a product of irreducible natural numbers.*

You should make the induction hypothesis and goal clear. You do not have to prove the theorem !

(d) (2 points) Explain why a proof by ordinary induction would fail for this theorem.

2. (2 points) What is wrong with the following induction proof ? Be specific. For full credit, you should identify the flaw that makes the proof incorrect. If you identify multiple errors, you should indicate which you believe is the fundamental error that cannot be fixed.

Theorem. *Every number is both odd and even.*

Proof. We will prove by induction on n that every natural number $n \in \mathbb{N}$ is both odd and even. We have to show that, for every $n \in \mathbb{N}$, if n is both odd and even then $s(n)$ is both odd and even. Let n be any natural number and assume that n is both odd and even. To see that $s(n)$ is even, we notice that $s(n) = n + 1$ by definition of addition. Since 1 is odd and n is odd, this means $s(n)$ is the sum of two odd numbers, hence is even. To see that $s(n)$ is odd, we again use the fact that $s(n) = n + 1$ and observe that becuase n is even and 1 is odd, $s(n)$ is the sum of an even number and an odd number, hence $s(n)$ is odd. Thus $s(n)$ is both even and odd, so by induction, we conclude that every natural number is both even and odd. QED

3. A function $f : \mathbb{N} \to \mathbb{N}$ is defined recursively by the following formula :

$$f(0) = 0 \qquad \forall n \in \mathbb{N},\ f(s(n)) = 2f(n) + 1$$

(a) (2 points) Compute $f(0)$, $f(1)$, $f(2)$, and $f(3)$ using the definition.

(b) (6 points) Prove the following theorem :

Theorem. *For all $n \in \mathbb{N}$, we have $f(n) + 1 = 2^n$.*

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4. (6 points) Prove the following statement :

   Theorem. *If n is a natural number and $n \geq 8$ then there are natural numbers a and b such that $3a + 5b = n$.*

5. In this problem, we will consider *connected planar graphs*. A connected planar graph starts with one vertex and can be enlarged by the following two rules : Rule A adds a new vertex and a new edge connecting the new vertex to a vertex that was already present ; Rule B adds a new edge connecting two vertices that are already present. If G is a connected planar graph, we write $V(G)$ for the number of *vertices* in G, we write $E(G)$ for the number of *edges* in G, and we write $F(G)$ for the number of *faces* (or regions) cut out by G.

   (a) (1 point) The accompanying figure shows a connected planar graph build using a sequence of applications of Rules A and B. In each picture G, count $V(G)$, $E(G)$, and $F(G)$. Verify that $V(G) + F(G) = E(G) + 2$ in each picture.

   (b) (6 points) Prove, for every connected planar graph G, that $V(G) + F(G) = E(G) + 2$.

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