Exploration 20

Math 2001–002, Fall 2016

November 4, 2016

Theorem (Well-ordering principle)**.** *Every nonempty subset of* $\mathbb{N}$ *has a least element.*

Theorem. *Every integer is threeven or throdd like* 1 *or throdd like* 2*.*

1 *Proof.* Suppose that n is an integer.

2 We will show that n is threeven or throdd like 1 or throdd like 2.

3 Define a set S by the following formula:

$$S = \{n - 3q \,:\, q \in \mathbb{Z} \wedge n - qd > 0\}$$

4 We will apply the well-ordering principle to S, but before we can do so, we need to verify that $S \subseteq \mathbb{N}$ and $S \neq \varnothing$.

5 To check that S is a subset of $\mathbb{N}$, suppose that z is an element of S.

6 Therefore $z = n - 3q$ for some integer q and $n - 3q > 0$.

7 Since n and q are integers, z is made from integers using subtraction and multiplication, so z is an integer.

8 Therefore z is an integer that is > 0, so $z \in \mathbb{N}$, as required.

9 To apply the well-ordering principle, we also need to show that $S \neq \varnothing$.

10 We will show $S \neq \varnothing$ by illustrating an element of S.

11 Since n is an integer, either $n > 0$ or $n \leq 0$.

12 If $n > 0$ then $n = n - 3 \times 0$, so n is in S.

13 This proves $S \neq \varnothing$ is this case.

14 If $n \leq 0$, then let $z = n - 3(n - 1)$.

15 This is an integer because it is built from integers using addition, subtraction, and multiplication.

16 Furthermore, $z = n - 3n + 3 = -2n + 3$.

17 Since $n \leq 0$, $-2n \geq 0$, so $-2n + 3 > 0$.

18 Therefore z is an integer and $z > 0$, so $z \in \mathbb{N}$.

19 Either way, we have seen that $S \neq \varnothing$, so we can apply the well-ordering principle.

20 Let r be the smallest element of S.

21 I claim that $r \leq 3$.

22 We will prove this by contradiction.

23 That is, we will assume that $r > 3$ and derive an absurd conclusion.

24 Suppose that $r > 3$.

25 Then, by the definition of S, we could write $r = n - 3q$ for some integer q.

26 But then $n - 3(q+1)$ would also be in S, because $q+1$ is an integer and

$$n - 3(q+1) = n - 3q - 3 = r - 3 > 3 - 3 = 0.$$

27 But $n - 3(q+1) < n - 3q = r$, so r couldn't have been the smallest element of S.

28 Since r actually is the smallest element of S, the assumption that $r > 3$ must have been false.

29 This means that $r \leq 3$, as we required.

30 Since $r \in \mathbb{N}$ and $r \leq 3$, we know $r = 1$ or $r = 2$ or $r = 3$.

31 If $r = 1$ then $n - 3q = 1$, so $n = 3q + 1$ and n is throdd like 1.

32 If $r = 2$ then $n - 3q = 2$, so $n = 3q + 2$ and n is throdd like 2.

33 If $r = 3$ then $n - 3q = 3$, so $n = 3q + 3 = 3(q+1)$, so n is threeven. Q.E.D.

Question 1. Label the sentences in the proof above as *claims*, *assumptions*, or *assertions*. Remember, claims are statements that will be justified. Assumptions set up the discussion for a direct proof. Assertions are statements that have been justified.

Question 2. For each sentence you labelled as a claim, indicate the line number of the assertion where that claim was justified.

Question 3. For each sentence you labelled as an assumption, indicate the last line where that assumption is still in force.

Question 4. For each sentence you labelled as as assertion, indicate which previous lines were used in the justification of that assertion.

Question 5. Prove the following statement using the well-ordering principle:

> **Theorem.** Suppose that n and d are integers, with $d > 0$. Then there are integers q and r such that $n = qd + r$ and $0 \leq r < d$.