Math 2001 Assignment 14

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Problem 1. Write an English definition meaning "x is a perfect square". Then write a symbolic sentence meaning the same thing, using only quantifiers ($\forall$, $\exists$), variables, set membership ($\in$), logical connectives ($\wedge$, $\vee$, $\neg$, $\Rightarrow$, $\Leftrightarrow$), the set of integers ($\mathbb{Z}$), arithmetic (addition, subtraction, multiplication, division, exponentiation), and equality ($=$).

Problem 2. Use summation notation to rewrite the following expression without ellipses:

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2}$$

Problem 3. Compute $\frac{2001!}{1999!}$ and justify your answer.

Problem 4. Use summation notation to rewrite the following expression without ellipses:

$$n^2 = 1 + 3 + 5 + 7 + \cdots + (2n - 1)$$

Problem 5. Prove that the sum of two even integers is even.

Proof. Suppose that a and b are even integers. This means that there are integers x and y such that $a = 2x$ and $b = 2y$. Then $a + b = 2x + 2y = 2(x + y)$, by the distributive law. Since x and y are integers, so is $x + y$, so by the definition of divisibility, $a + b$ is divisible by 2. But to be even means precisely to be divisible by 2, so we conclude that $a + b$ is even.

As a and b were allowed to be arbitrary even integers in the paragraph above, this shows that the sum of two even integers is even. $\square$

Problem 6. Prove that the sum of two odd integers is even.

Problem 7. Use sum and product notation to rewrite the following expression without ellipses:

$$1 + (1 \times 3) + (1 \times 3 \times 5) + (1 \times 3 \times 5 \times 7) + \cdots + (1 \times 3 \times 5 \times 7 \times \cdots \times (2n + 1))$$

Problem 8. Write a symbolic sentence meaning "x is not divisible by any perfect square", using only quantifiers ($\forall$, $\exists$), variables, set membership ($\in$), logical connectives ($\wedge$, $\vee$, $\neg$, $\Rightarrow$, $\Leftrightarrow$), the set of integers ($\mathbb{Z}$), arithmetic (addition, subtraction, multiplication, division, exponentiation), and equality ($=$).

Problem 9. Suppose that $x_1, x_2, \ldots$ is a sequence of real numbers. We say that the sequence *converges* to a real number x if, for every positive real number ϵ, there is positive integer N such that, whenever n is a positive integer greater than N, the distance between x_n and x is less than ϵ.

Rewrite this sentence symbolically, using only quantifiers ($\forall$, $\exists$), variables, set membership ($\in$), logical connectives ($\wedge$, $\vee$, $\neg$, $\Rightarrow$, $\Leftrightarrow$), the set of real numbers ($\mathbb{R}$), arithmetic (addition, subtraction, multiplication, division, exponentiation), absolute value, and equality and inequalities ($=, \neq, <, \leq, >, \geq$).