

GENERALIZED COVERING SPACES AND THE GALOIS FUNDAMENTAL GROUP

CHRISTIAN KLEVDAL

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ABSTRACT. We introduce a functor π_1^{Gal} from the category of based, connected, locally path connected spaces to the category of complete topological groups. We then compare this groups to the fundamental group. In particular, we show that there is a topological group $\pi_1^\sigma(X, x)$ whose underlying group is $\pi_1(X, x)$ so that $\pi_1^{\text{Gal}}(X, x)$ is the completion of $\pi_1^\sigma(X, x)$.

1. INTRODUCTION

In this paper, we give define a topological group $\pi_1^{\text{Gal}}(X, x)$ called the Galois fundamental group, which is associated to any based, connected, locally path connected space X . As the notation suggests, this group is intimately related to the fundamental group $\pi_1(X, x)$. The motivation of the Galois fundamental group is to look at automorphisms of certain covers of a space instead of homotopy classes of loops in the space. It is well known that if X has a universal cover, then $\pi_1(X, x)$ is isomorphic to the group of deck transformations of the universal cover. However, this approach breaks down if X is not semi locally simply connected. The Galois fundamental group has the advantage that instead of looking at the automorphisms of a specific cover, it looks at automorphisms of all covers of a space. In the case where X has a universal cover, the Galois fundamental group and the usual fundamental group are isomorphic. We show that in general, they are not isomorphic, but are related. In particular, we define the topological group $\pi_1^\sigma(X, x)$, whose underlying group is $\pi_1(X, x)$, and show that $\pi_1^{\text{Gal}}(X, x)$ is the completion $\pi_1^\sigma(X, x)$. This topology on $\pi_1(X, x)$ is obtained by the open subgroups of the topology of $\pi_1^\tau(X, x)$ as introduced by Brazas in [4, 3.11]. We do not know if $\pi_1^\tau(X, x) = \pi_1^\sigma(X, x)$.

The definition of the Galois fundamental group uses the notion of an infinite Galois theory as defined by Bhatt and Scholze in [1, Definition 7.2.1]. An infinite Galois theory consists of a category \mathcal{C} and a functor $F: \mathcal{C} \rightarrow \mathbf{Sets}$ called the fiber functor. These of course are required to satisfy some axioms. For our purposes, \mathcal{C} will be a category of covers of our space X and F will be pull back along inclusion of the base point. Since the category of covering spaces is not well enough behaved, it is necessary to use a more general notion of covers known as semicovers, due¹ to Brazas [3, Definition 3.1].

2. A CATEGORY OF COVERS

In the classical theory, covering spaces are fibrations with discrete fibers that are locally trivial in the base. Covering spaces are useful because they provide a geometric object for the fundamental group to act on, and for particularly nice spaces, there is a close connection between covers and subgroups of the fundamental group. By particularly nice, we mean path connected, locally path connected and semi-locally simply connected. If X is a space that satisfies these conditions, then there is a cover $p: \tilde{X} \rightarrow X$ called the universal cover. There is a faithful and transitive action of $\pi_1(X, x)$ on $p^{-1}(x)$ given by monodromy, and \tilde{X} enjoys the property that all path connected covers of X are quotients of \tilde{X} by subgroups of the fundamental group. The fundamental group can be recovered as the group of deck transformations of \tilde{X} .

However there is no such correspondence for spaces that are not semi-locally simply connected. In order to study these spaces, we need to find a suitable replacement for traditional covers.

We begin by examining some of the properties of traditional covering spaces. It is a classical result that if Z is a covering space over X then homotopies of maps in X can be lifted uniquely to Z . We make this precise.

Definition 1. (*Unique homotopy lifting property*) A map $p: Z \rightarrow X$ satisfies the unique homotopy lifting property with respect to a class of spaces \mathcal{T} if given a map $f: Y \times I \rightarrow X$ where $Y \in \mathcal{T}$ and a lift $\tilde{f}_0: Y \times \{0\} \rightarrow Z$ such that $p \circ \tilde{f}_0 = f|_{Y \times \{0\}}$, there is a unique map $\tilde{f}: Y \times I \rightarrow Z$ such that $\tilde{f}|_{Y \times \{0\}} = \tilde{f}_0$ and $p \circ \tilde{f} = f$.

The homotopy lifting property can be summarized in the following diagram:

$$\begin{array}{ccc}
 Y \times \{0\} & \xrightarrow{\tilde{f}_0} & Z \\
 \downarrow & \nearrow \exists! & \downarrow p \\
 Y \times I & \xrightarrow{f} & X
 \end{array}$$

Here we are given that the outer square commutes, \tilde{f} is the unique map that makes the whole diagram commute. Note that if $Z \rightarrow X$ satisfies the homotopy lifting property, a path γ that starts at $x \in X$ can be lifted to a path $\tilde{\gamma}$ starting at s for any $s \in p^{-1}(x)$ by taking Y to be a point.

¹We remove a redundant axiom from Brazas' definition.

We call a space $p: Z \rightarrow X$ over X étale (or an étale cover) if p is a local homeomorphism. A nice property of covering spaces is that they are étale. If $p: Z \rightarrow X$ is a local homeomorphism, then the fiber must be discrete over each point. To see this, let $x \in X$ and $z \in p^{-1}(x)$ (we assume $p^{-1}(x)$ is non empty, otherwise we are done trivially). Choose an open neighborhood U of z such that $p|_U: U \rightarrow p(U)$ is a homeomorphism. In particular, $p^{-1}(x) \cap U = \{z\}$, since otherwise $p|_U$ is not injective. Thus $\{z\}$ is open in the fiber, which shows discreteness. If in addition p satisfies the unique homotopy lifting property, we can define an action of $\pi_1(X, x)$ on $p^{-1}(x)$. This is just the usual monodromy action.

Proposition 2.1. (*Monodromy*) *Let $p: Z \rightarrow X$ be a map that satisfies the unique homotopy lifting property and is a local homeomorphism. Pick $x \in X$ and let $S = p^{-1}(x)$. Then there is a well defined action of $\pi_1(X, x)$ on S given by*

$$[\gamma].s = \tilde{\gamma}(1),$$

where $\tilde{\gamma}$ is a lift of γ starting at s .

Proof. As we noted before, we can clearly lift a path, however, our path lift depended on our choice of representative of $[\gamma] \in \pi_1(X, x)$. We must show that if $[\alpha] = [\gamma]$ and $\tilde{\alpha}$ is a lift of α starting at s then $\tilde{\alpha}(1) = \tilde{\gamma}(1)$. Choose a homotopy $h: I \times I \rightarrow X$ from α to γ so that $h(t, 0) = \alpha(t)$. Since p satisfies the unique homotopy lifting property and $\tilde{\alpha}$ is a lift of $h(t, 0)$, we get a lift $\tilde{h}: I \times I \rightarrow Z$. The map $f: I \rightarrow Z$ defined by $t \mapsto (1, t)$ gives a continuous map onto the fiber $p^{-1}(x)$ (with the subspace topology). But the fiber is discrete, so it must be constant. Thus the action is well defined. \square

We know that covering spaces satisfy the homotopy lifting property and are local homeomorphisms, but there are other spaces that satisfy these properties, which motivates the following definition.

Definition 2. (*Semicovers*) A map $p: Z \rightarrow X$ is called a semicovering map if it is an étale cover and it satisfies the unique homotopy lifting property with respect to $\mathcal{T} = \{I, I \times I\}$, where I is the interval $[0, 1]$. A semicover of a space X is a pair (Z, p) where $p: Z \rightarrow X$ is a semicovering map. A morphism $(Z, p) \rightarrow (Y, q)$ of semicovers over X consists of a continuous map $f: Z \rightarrow Y$ such that $p = q \circ f$, i.e. that makes the following diagram commute.

$$\begin{array}{ccc} Z & \xrightarrow{f} & Y \\ & \searrow p & \swarrow q \\ & & X \end{array}$$

Semicovers over X and morphisms of semicovers over X form a category, denoted $\mathbf{SCov}(X)$.

By abuse of notation, we say that Z is a semicover when p is understood. In the definition, we choose to lift with respect to $I, I \times I$ because this is all that is necessary to have a well defined action of the fundamental group on a semicover.

In [3, Definition 3.1], Brazas defines semicovers as well. We will show definitions are equivalent, but first we must give Brazas' definition.

Definition 3. (*Continuous lifting of paths*) For a space X , let $\mathcal{P}X$ be the space of paths in X (with the compact open topology). For any $x \in X$ define $\mathcal{P}X_x, \mathcal{P}X^x$ to be the space of paths starting at x and the space of paths ending at x respectively. Given a continuous map $f: Y \rightarrow X$, there is a continuous map $\mathcal{P}f_y: \mathcal{P}Y_y \rightarrow \mathcal{P}X_{f(y)}$ obtained by composition. We say that f has *continuous lifting of paths* if $\mathcal{P}f_y$ is a homeomorphism.

Definition 4. (*Continuous lifting of homotopies*) For a space X , let \mathcal{H} denote the space of homotopies in X , i.e. the set of continuous maps $I \times I \rightarrow X$, again with the compact open topology. For any $x \in X$ we let $\mathcal{H}X_x$ be the set of homotopies beginning at x , i.e. maps $h: I \times I \rightarrow X$ such that $h|_{\{0\} \times I}$ is the constant map to $\{x\}$. We define $\mathcal{H}X^x$ similarly, with $\{1\} \times I$ in place of $\{0\} \times I$. If $f: Y \rightarrow X$ is continuous, we get a continuous map $\mathcal{H}f_y: \mathcal{H}Y_y \rightarrow \mathcal{H}X_{f(y)}$ by composition. We say that f has *continuous lifting of homotopies* if $\mathcal{H}f_y$ is a homeomorphism.

If $f: Y \rightarrow X$ is a semicovering map and $y \in Y$, then surjectivity of $\mathcal{P}f_y: \mathcal{P}Y_y \rightarrow \mathcal{P}X_{f(y)}$ follows from the path lifting property, and injectivity follows from uniqueness of path lifts. Consequently $\mathcal{P}f_y$ is bijective, and the same reasoning shows $\mathcal{H}f_y$ is bijective.

For any space X , we form a category $\mathbf{SCov}_{Br}(X)$ with objects being local homeomorphisms $p: Z \rightarrow X$ that have continuous lifting of paths and homotopies and morphisms being the obvious commuting triangles. Brazas shows ([3, Theorem 7.19]) that there is a categorical equivalence between $\mathbf{SCov}_{Br}(X)$ and $\pi_1^r(X, x)$ -**Sets**, where $\pi_1^r(X, x)$ is the topologized fundamental group introduced in [4, 3.11], and $\pi_1^r(X, x)$ -**Sets** are the discrete sets with a continuous action. We will explore this later, but for now we show that any semicover has continuous lifting of paths and homotopies, i.e. that our notion of semicovers agrees with that of Brazas. Proposition 3.7 of [3] shows that any covering space is a semicover. However, the proof only uses that covering spaces are local homeomorphisms that satisfy the unique homotopy lifting property, so it extends to semicovers. We recall the proof here.

For any space X with basis \mathcal{B} , a convenient sub basis for the compact open topology on $\mathcal{P}X$ are sets of the form

$$\langle K, U \rangle = \{\gamma \in \mathcal{P}X: \gamma(K) \subseteq U\},$$

where $K \subseteq I$ is compact and $U \in \mathcal{B}$. We can then form a basis for the topology on $\mathcal{P}X$ by taking sets of the form

$$\cap_{j=1}^n \langle K_n^j, U_j \rangle$$

where $K_n^j = [\frac{j-1}{n}, \frac{j}{n}]$ and $U_j \in \mathcal{B}_p$.

Proposition 2.2. (*Brazas*) *If $p: Z \rightarrow X$ is a semicover, then p has continuous lifting of paths and homotopies.*

Proof. Let

$$\mathcal{B}_p = \{U \subseteq Z: p|_U: U \rightarrow p(U) \text{ is a homeomorphism}\}.$$

Since p is a local homeomorphism, this is a basis of Z . A basic open set in $\mathcal{P}Z_z$ is of the form

$$\mathcal{U} = \bigcap_{j=1}^n \langle K_n^j, U_j \rangle \cap \mathcal{P}Z_z,$$

where $U_j \in \mathcal{B}_p$. Let

$$\mathcal{V} = \bigcap_{j=1}^n \langle K_n^j, p(U_j) \rangle \cap \mathcal{P}X_{p(z)}.$$

Since p is an open map, $p(U_j)$ is open for all j . It is clear that $\mathcal{P}p(\mathcal{U}) \subseteq \mathcal{V}$, and if we can show equality, it will follow that $\mathcal{P}p$ is a homeomorphism. If $\gamma \in \mathcal{V}$, since p satisfies the homotopy lifting property, we can find a lift $\tilde{\gamma} \in \mathcal{P}Z_z$. Suppose $t \in K_n^j$. Since $p|_{U_j}$ is a homeomorphism and $p|_{U_j}(t) = \gamma(t)$, it follows that $\tilde{\gamma}(t) \in U_j$. This shows that $\tilde{\gamma} \in \mathcal{U}$ and $\gamma = \mathcal{P}p(\tilde{\gamma})$, hence $\mathcal{P}p(\mathcal{U}) = \mathcal{V}$.

We now show that p has continuous lifting of homotopies. Suppose $\mathcal{U} \subseteq \mathcal{H}Z_z$ is a basic open set of the form

$$\mathcal{U} = \bigcap_{0 < i, j \leq n} \langle K_n^{i,j}, U_{i,j} \rangle,$$

where $K_n^{i,j} = K_n^i \times K_n^j$ and $U_{i,j} \in \mathcal{B}_p$. Let

$$\mathcal{V} = \bigcap_{0 < i, j \leq n} \langle K_n^{i,j}, p(U_{i,j}) \rangle.$$

It is clear that $\mathcal{H}p(\mathcal{U}) \subseteq \mathcal{V}$. If $h \in \mathcal{H}X_{p(z)}$ then we can lift h to some $\tilde{h} \in \mathcal{H}Z_z$. If $t \in K_n^{i,j}$ then using the homeomorphism $p|_{U_{i,j}}$ and identity $p|_{U_{i,j}} \circ \tilde{h}(t) = h(t)$ we see that $\tilde{h}(t) \in U_{i,j}$. As this holds for any t in any $K_n^{i,j}$ it follows that $\mathcal{H}p(\mathcal{U}) = \mathcal{V}$, so \mathcal{H} is an open map. \square

Corollary 2.1. For a topological space X , there is an equivalence of categories between $\mathbf{SCov}_{Br}(X)$ and $\mathbf{SCov}(X)$.

Proof. The proposition above shows that every semicovering in the sense of definition 2 is a semicovering in the sense of Brazas. For the converse, one can easily see that continuous lifting of paths and homotopies implies the unique homotopy lifting property with respect to I and $I \times I$. Thus the two definitions are equivalent, and since the morphisms in both categories are just continuous maps that commute over the base the categories are equivalent. \square

We now discuss a few features of semicovering maps. One nice feature that fails for regular covering spaces is the ‘two out of three’ property, as illustrated by [3, §3 figure 1].

Proposition 2.3. (*Two out of three property*) For spaces X, Y, Z and maps $f: X \rightarrow Y, g: Y \rightarrow Z$ with f surjective, if two of $f, g, h = g \circ f$ are semicovers so is the third.

Proof. It is easy to show that if two of the three maps are local homeomorphisms, then the third is also. If either both g, h or f, g have continuous lifting of paths, then for any $x \in X$ two of the three maps $\mathcal{P}f_x, \mathcal{P}g_{f(x)}, \mathcal{P}h_x$ are homeomorphisms so the third must be. On the other hand suppose f, h both have continuous lifting of paths. For any $y \in Y$ we can write $y = f(x)$ for some $x \in X$. In this case $\mathcal{P}f_x, \mathcal{P}h_x$ are both homeomorphisms,

hence $\mathcal{P}g_y$ is a homeomorphism. A similar argument will work for continuous lifting of homotopies. Then corollary 2.1 proves the two of three property for semicovers. \square

Another property worth mentioning is that semicovers are stable under pullback.

Proposition 2.4. *Let $f: Y \rightarrow X$ be a continuous map of spaces. Then there is a functor*

$$f^*: \mathbf{SCov}(X) \rightarrow \mathbf{SCov}(Y)$$

defined by pullback.

The proof follows from the universal property of the fiber product and the well know fact that the pullback of a local homeomorphism is again a local homeomorphism. Alternatively, this is proven by Brazas [3, Proposition 3.9].

3. UNIFORM SPACES AND TOPOLOGICAL GROUPS

In this section, we go over some of the basic facts about uniform structures that will be used later on the paper. Most of the proofs will be skipped, all proofs can found in [2]. We begin with some definitions.

Definition 5. (*Filters*) Let X be a set. A filter \mathcal{F} on X is a set of subsets of X that satisfy the following axioms:

- F1 If $U \in \mathcal{F}$ and $U \subseteq V$ then $V \in \mathcal{F}$.
- F2 If $U, V \in \mathcal{F}$ then $U \cap V \in \mathcal{F}$.
- F3 $\emptyset \notin \mathcal{F}$.

Definition 6. (*Filter Bases*) If \mathcal{B} is a non empty set of subsets of a set X then the set of $A \subseteq X$ that contained some element of \mathcal{B} forms a filter if and only if the following hold:

- B1 If $A, B \in \mathcal{B}$, there is some $C \in \mathcal{B}$ such that $C \subseteq A \cap B$.
- B2 $\emptyset \notin \mathcal{B}$.

The set \mathcal{B} is said to be a filter base of the filter it generates.

Proof. Bourbaki [2, Ch. 1 §6.3 proposition 2]. \square

If X is a topological space and $x \in X$, the set of (not necessarily open) neighborhoods of x form a filter $\mathcal{N}(x)$. This is called the *neighborhood filter* of x .

Recall that for any subset $U \subseteq X \times X$, the inverse of U is

$$U^{-1} = \{(x, y) : (y, x) \in U\}.$$

If $V \subseteq X \times X$ also, then the composition of U with V is

$$U \circ V = \{(x, z) : \text{there exists } y \in X \text{ such that } (x, y) \in U \text{ and } (y, z) \in V\}.$$

We may now give the definition of a uniform space.

Definition 7. (*Uniform structures, spaces, and continuity. Fundamental system of entourages*) Let X be a set. A uniform structure Φ on X is a filter on $X \times X$ that satisfies the following axioms.

- U1 For all $U \in \Phi$, we have $\Delta X \subseteq U$.
- U2 If $U \in \Phi$ then $U^{-1} \in \Phi$.
- U3 If $U \in \Phi$ then there exists $V \in \Phi$ such that $V \circ V \subseteq U$.

The pair (X, Φ) is called a uniform space, and the elements of Φ are called entourages. If (Y, Ψ) is another uniform space a map, a uniformly continuous map $f: (X, \Phi) \rightarrow (Y, \Psi)$ is a function $f: X \rightarrow Y$ so that if $U \in \Psi$ then $(f \times f)^{-1}(U) \in \Phi$. A set of entourages $\{V_i\}_{i \in I}$ of a uniform space is said to be a fundamental system of entourages if every entourage contains some V_i .

Let (X, Φ) be a uniform space. For any $V \in \Phi$ and $x \in X$, let

$$V(x) = \{y: (x, y) \in V\}.$$

The associated topology is defined so that a set $U \subseteq X$ is open if and only if for each $x \in U$, there is an entourage $V \in \Phi$ such that $V(x) \subseteq U$. Note that this gives a topology on X , where the open sets are the sets $V(x)$ for $V \in \Phi$ and $x \in X$. This is called the topology induced by the uniform structure Φ .

We come to some important examples of uniform structures that are associated to topological group.

Definition 8. (*The left, right and two sided uniformities*) Let G be a topological group. The right (resp. left) uniformity has a fundamental system of entourages given by sets of pairs $\{(g, h) \in G \times G: hg^{-1} \in V \text{ (resp } g^{-1}h \in V)\}$ as V runs over neighborhood of the identity $e \in G$. Denote the uniform structure of the left and right uniformity as \mathcal{L}, \mathcal{R} respectively.

The two sided uniformity \mathcal{T} is the coarsest uniformity on G so that the identity maps $(G, \mathcal{T}) \rightarrow (G, \mathcal{L})$ and $(G, \mathcal{T}) \rightarrow (G, \mathcal{R})$ are uniformly continuous.

Definition 9. (*Cauchy filters, limits, separated and complete uniform spaces*) Let X be a uniform space, and \mathcal{F} a filter on X . We say that \mathcal{F} is a Cauchy filter if for any entourage V there is some $A \in \mathcal{F}$ such that $A \times A \subseteq V$. We say that $x \in X$ is the limit of \mathcal{F} and that \mathcal{F} converges (to x) if for any entourage V there is some $A \in \mathcal{F}$ such that $A \subseteq V(x)$. A filter base for a Cauchy filter is called a Cauchy filter base. A uniform space is complete if every Cauchy filter converges. A uniform space X is separated if the intersection of all entourages is the diagonal ΔX .

Definition 10. (*Complete topological groups*) A topological group G is said to be complete if G with the two sided uniformity is a complete uniform space.

Proposition 3.1. *A Hausdorff topological group G is isomorphic to a dense subgroup of a complete group \hat{G} if and only if the image of a Cauchy filter base under the transformation $g \mapsto g^{-1}$ is again a Cauchy filter base. In this case \hat{G} is unique up to isomorphism.*

Proof. Bourbaki Chapter 3 §3.4 theorem 1 [2]. □

We now come to some examples which will be useful later. First, if Φ is of a collection of equivalence relations on X , then the filter Φ generates is a uniform structure. Indeed if $V \in \Phi$ then $\Delta X \subseteq V$ by reflexivity, $V^{-1} = V$ by the symmetric property and $V \circ V = V$ by transitivity.

In the next section, we will be interested in putting a uniformity on the set of natural automorphisms of a functor. Given any functor $F: \mathcal{C} \rightarrow \mathbf{Sets}$, we can put a uniformity on $\text{Aut}(F)$ (provided $\text{Aut}(F)$ is actually a set). We do this by putting a uniformity on $\text{Aut}(F(C))$ for any $C \in \mathcal{C}$ and then putting the coarsest uniformity on $\text{Aut}(F)$ that makes the maps $\text{Aut}(F) \rightarrow \text{Aut}(F(C))$ uniformly continuous.

Suppose S is a set. For any set T we can form the following set

$$(1) \quad V_T = \{(\sigma, \tau): \sigma^{-1}\tau|_T = \sigma\tau^{-1}|_T = \text{id}_T\} \subseteq \text{Aut}(S) \times \text{Aut}(S)$$

This is an equivalence relation, and one easily sees that for finite sets $T_1, T_2 \subseteq S$, that $V_{T_1} \cap V_{T_2} = V_{T_1 \cup T_2}$. Thus we define a uniformity on S by taking the entourages to be sets that contain some V_T , where T is finite. The topology we get from this uniformity is in fact the same as the compact open topology for $\text{Aut}(S)$ where S has been given the discrete topology, so we call this the compact open uniformity.

Lemma 3.1. *With the topology from the compact open uniformity, $G = \text{Aut}(S)$ is a topological group, and the compact open uniformity on G is the same as the two sided uniformity of G .*

Proof. If $\sigma \in G$ and $T \subseteq S$ is finite, then the image of the neighborhood $V_T(\sigma)$ under inversion is clearly just $V_T(\sigma^{-1})$. Thus inversion is continuous.

For multiplication, fix $(\sigma, \tau) \in G^2$ and let $V_T(\sigma\tau)$ be a neighborhood of their product. Consider the neighborhood

$$W = V_{T \cup \tau(T)}(\sigma) \times V_{T \cup \sigma^{-1}(T)}(\tau).$$

One can easily check that if $(\alpha, \beta) \in W$, then $\alpha\beta \in V_T(\sigma\tau)$. This shows continuity of multiplication.

The final statement is easy to verify and is left to the reader. \square

Lemma 3.2. *For a set S , the group $G = \text{Aut}(S)$ with the compact open uniformity is a complete, separated uniform space.*

Proof. Bhatt and Scholze [1, Lemma 7.1.4]. \square

The main result is the following.

Proposition 3.2. *Let \mathcal{C} be a category and $F: \mathcal{C} \rightarrow \mathbf{Sets}$ a functor such that the natural automorphisms $\text{Aut}(F)$ forms a set. Give $G = \text{Aut}(F)$ be the coarsest uniformity make the maps $\text{Aut}(F) \rightarrow \text{Aut}(F(X))$ uniformly continuous for each $X \in \mathcal{C}$, where $\text{Aut}(F(X))$ is given the compact open uniformity. Then the topology obtained by this uniformity makes G a separated complete topological group.*

Proof. For each $X \in \mathcal{C}$ the maps $G \rightarrow \text{Aut}(F(X)) \xrightarrow{\text{inv}} \text{Aut}(F(X))$ and $G \times G \rightarrow \text{Aut}(F(X)) \times \text{Aut}(F(X)) \xrightarrow{m} \text{Aut}(F(X))$ are continuous by definition of the uniformity put on G and lemma 3.1. It is clear the the two compositions are the same as the compositions $G \xrightarrow{\text{inv}} G \rightarrow \text{Aut}(F(X))$ and $G \times G \xrightarrow{m} G \rightarrow \text{Aut}(F(X))$ respectively. Since these maps are continuous, by definition of the uniformity on G a map from a space $Y \rightarrow G$ is continuous if and only if the maps $Y \rightarrow \text{Aut}(F(X))$ are continuous for each $X \in \mathcal{C}$. Thus multiplication and inversion are continuous in G showing that it is a topological group.

We now show G is separated and complete.

If $(\eta, \xi) \in G \times G$ is in the intersection of all entourages of G , then by lemma 3.2, $\eta_{F(X)} = \xi_{F(X)}$ for all $X \in \mathcal{C}$. Thus $\eta = \xi$ and G is separated.

Let \mathcal{F} be a Cauchy filter on G . This clearly induces a Cauchy filter on $F(X)$ for all $X \in \mathcal{C}$. By lemma 3.2, these have a limit, which we call $\eta_{F(X)}$. We need to show these maps together form a natural automorphism. Suppose that $f: X \rightarrow Y$ is an arrow in \mathcal{C} . Let $x \in F(X)$. Since \mathcal{F} is Cauchy, we can find some $\xi \in G$ such that $\eta_{F(Y)} \circ Ff(x) = \xi_{F(Y)} \circ Ff(x)$ and $\eta_{F(X)}(x) = \xi_{F(X)}(x)$. As ξ is a natural automorphism, we have $\xi_{F(Y)} \circ Ff(x) = Ff \circ \xi_{F(X)}(x)$. Thus

$$\eta_{F(Y)} \circ Ff(x) = \xi_{F(Y)} \circ Ff(x) = Ff \circ \xi_{F(X)}(x) = Ff \circ \eta_{F(X)}(x),$$

which shows that the maps are natural, and hence give a natural transformation η . Since all of the maps $\eta_{F(X)}$ are bijections, it is indeed a natural automorphism, hence it is in G .

Finally we show that η is the limit of \mathcal{F} . A neighborhood of η in G is of the form

$$V_{T_1, \dots, T_n}(\eta) = \{\xi \in G: \xi_{F(X_i)} \circ \eta_{F(X_i)}^{-1}|_{T_i} = \xi_{F(X_i)}^{-1} \circ \eta_{F(X_i)}|_{T_i} = \text{id}_{T_i} \text{ for } i = 1, \dots, n\}.$$

where X_1, \dots, X_n are objects of \mathcal{C} and $T_i \subseteq F(X_i)$. By definition of being Cauchy, there is some $A \in \mathcal{F}$ such that $A \times A \subseteq V_{T_1, \dots, T_n}$. Here V_{T_1, \dots, T_n} is the entourage of G given by pairs (η, ξ) such that $\xi \in V_{T_1, \dots, T_n}(\eta)$. From the construction of η it is clear that $A \subseteq V_{T_1, \dots, T_n}(\eta)$ so η is the limit of \mathcal{F} and G is complete. \square

4. GALOIS THEORY OF SEMICOVERS

In this section, we introduce categorical Galois theory in the framework of objects called infinite Galois theories as defined in [1, Definition 7.2.1]. Categorical Galois theory encompasses the usual Galois theory of fields, and as we shall see the Galois theory of covering spaces.

Choice of a base point x of a space X can be seen as a map $i: \{x\} \rightarrow X$. We get a functor $i^*: \mathbf{SCov}(X) \rightarrow \mathbf{Sets}$ that assigns to a semicover (Z, p) the set $i^*Z = p^{-1}(x)$. If $f: (Z, p) \rightarrow (Y, q)$ is a map of semicovers, then the associated map of sets $i^*f: i^*Z \rightarrow i^*Y$ is the restriction of f to $p^{-1}(x)$. In this section, we aim to show that the pair $(\mathbf{SCov}(X), i^*)$ constitute a tame infinite Galois theory. We recall the definition of an infinite Galois theory here.

Definition 11. (*Infinite Galois theory*) Let \mathcal{C} be a category and F a functor from \mathcal{C} to \mathbf{Sets} . The pair (\mathcal{C}, F) constitute an infinite Galois theory if it satisfies axioms IGT1 - IGT4. If \mathcal{C} has an initial object, we say that an object $X \in \mathcal{C}$ is GT connected if whenever there is a monomorphism $f: Y \rightarrow X$, either Y is initial or f is an isomorphism.

- IGT1 \mathcal{C} has small colimits and finite limits.
- IGT2 Each object in \mathcal{C} can be written as a coproduct of GT connected objects.
- IGT3 \mathcal{C} is generated under colimits by a set of GT connected objects.
- IGT4 F is faithful, conservative and commutes with colimits and finite limits.

Remark: The category \mathcal{C} is necessarily a tops.

In showing that (\mathbf{SCov}, i^*) is an infinite Galois theory, the hardest part is showing that colimits exist, and we would also like to know that GT connectedness agrees with our intuition. We begin with a few lemmas.

Lemma 4.1. *Let X be a connected, locally path connected space. For any $(Z, p) \in \mathbf{SCov}(X)$ the following are equivalent:*

- (1) Z is a connected topological space.
- (2) Z is path connected.
- (3) (Z, p) is GT connected.

Proof.

(1 \implies 2) Now Z is locally path connected since X is and since connected locally path connected spaces are path connected, Z must be path connected.

(2 \implies 3) Any map $f: Y \rightarrow Z$ must be a semicover by proposition 2.3. Suppose f is a monomorphism and Y is not empty. Let $y \in Y$ and $z \in Z$. Since Z is path connected, there is a path γ from $f(y)$ to z . If $\tilde{\gamma}$ is a lift of this in Y , then $\tilde{\gamma}(1)$ maps to z . Thus f is surjective, meaning that f is bijective. Since bijective maps of local homeomorphisms are homeomorphisms, Z is GT connected.

(3 \implies 1) We prove the contrapositive. Suppose Z is not path connected. Then there is a path component $Y \subsetneq Z$ that is open since Z is locally path connected. Then Y will be a semicover and the map $Y \hookrightarrow Z$ is an injective map. Since functions are determined by their values on points, injective maps are monomorphisms, so Z is not GT connected. \square

Let $\mathbf{\acute{E}t}(X)$ be the category of étale covers of X .

Lemma 4.2. *Small colimits exist in $\mathbf{\acute{E}t}(X)$.*

Proof. Suppose I is a small category and $D: I \rightarrow \mathbf{\acute{E}t}(X)$ is a diagram. There is a forgetful functor $F: \mathbf{\acute{E}t}(X) \rightarrow \mathbf{Top}$ that forgets about the map to X . Since \mathbf{Top} is complete, let Z be the colimit of $F \circ D$. For any $i \in I$ denote $D(i)$ by $p_i: Z_i \rightarrow X$, and let $f_i: Z_i \rightarrow Z$ be the map to the colimit in \mathbf{Top} . Then by the universal property of the colimit, we get a map $p: Z \rightarrow X$ so that $p \circ f_i = p_i$ for all $i \in I$.

Given $z \in Z$, we have that $z = f_i(z_i)$ for some i and some $z_i \in Z_i$. Since p_i is a local homeomorphism, there are open subsets U_i of Z_i and an open set $V \subseteq X$ so that each p_i restricts to give a homeomorphism $U_i \rightarrow V$. Let $U = p^{-1}(V) \cap f_i(U_i) \subseteq Z$. Then there is a commuting diagram

$$\begin{array}{ccc} U_i & \xrightarrow{f_i|_{U_i}} & U \\ & \searrow p_i|_{U_i} & \swarrow p|_U \\ & & V \end{array}$$

Since $p_i = p \circ f_i$, we have $p(f(U_i)) = p_i(U_i) = V$. This means that $p|_U$ and $f_i|_{U_i}$ are surjective. From this, it follows that $p|_U, f_i|_{U_i}$ are also injective, hence are bijective. Thus to show that $p|_U$ is a homeomorphism onto V , we only need to know that U is open. We can check that U is open by checking if $f_j^{-1}(U) \subseteq Z_j$ is open for all j . If $f_j^{-1}(U)$ is empty it is open, otherwise assume there is some $y \in f_j^{-1}(U)$. Let $x = p_j(y) \in V$. Since p_j is a local homeomorphism, we can find an open neighborhood W of y such that p_j induces a homeomorphism of W onto its image. By shrinking W if necessary, we may also assume that $p_j(W) \subseteq V$. Thus we get a commutative diagram

$$\begin{array}{ccc}
W & \xrightarrow{f_j|_W} & Z \\
p_j|_W \searrow & & \swarrow p \\
& p_j(W) &
\end{array}$$

Note that $p_j(W) \subseteq V = p(U)$. But $p|_U$ is bijective, so $f_j(W) = (p|_U)^{-1}(p_j(W)) \subseteq U$. This means that $p_j^{-1}(U)$ is open in Z_j , and hence U is open. Note that this implies that $f_j|_W$ is an isomorphism. This result holds for arbitrary $y \in Z_j$ (since each $y \in f_j^{-1}(U)$ for some open set $U \subseteq Z$). \square

Remark: The proof above shows that $f_j: Z_j \rightarrow Z$ is a local homeomorphism. This will be important to show the homotopy lifting property. The above lemma is equivalent to the fact that small colimits of sheaves on a space X exist.

Lemma 4.3. *Let $(W, r), (Y, q)$ be in $\mathbf{SCov}(X)$ and let $g, h \in \text{Hom}_{\mathbf{SCov}(X)}(W, Y)$. Let (Z, p) be the coequalizer in \mathbf{Top} . Then the map $Y \rightarrow Z$ is a semicover.*

Proof. Let f denote the map $Y \rightarrow Z$. From the remark above, we see that f is a local homeomorphism.

The maps g, h induce a map $W \rightarrow Y \times_X Y$. This gives a relation which generates an equivalence relation, and it is easy to see that Z is the quotient of Y by the corresponding equivalence relation. To give an explicit description of the equivalence relation, $a, b \in Y$ are related if there is a chain w_1, \dots, w_n so that $g(w_1) = a, h(w_n) = b$ and $g(w_{i+1}) = h(w_i)$ for $1 \leq i < n$.

We first show that f has unique path lifting, and for this sake let $\gamma: I \rightarrow Z$ be any path. To show uniqueness, we suppose α, β are liftings of γ . Then α, β are both lifts of $g \circ \gamma$ and since q is a semicovering map, we must have $\alpha = \beta$.

Now we show existence, with γ as above. Since p_2 is a semicover, for each $y \in f^{-1}(\gamma(0))$ we can lift $p \circ \gamma$ (a path in X) to some loop $\tilde{\gamma}_y$ in Y starting at y . For any fixed $y_0 \in f^{-1}(\gamma(0))$, we must show that $f \circ \tilde{\gamma}_{y_0} = \gamma$. Since $\gamma(I)$ is compact and f is a local homeomorphism, we may find open sets $V_1, \dots, V_n \subseteq Y$ so that $f|_{V_i}$ is a homeomorphism onto its image, and the sets $f(V_i)$ cover $\gamma(I)$. By shrinking the V_i if necessary and reordering them, we may assume there exist $a_1, \dots, a_{n+1} \in [0, 1]$ with $0 = a_1 < a_2 \dots < a_{n+1} = 1$ and $\gamma([a_i, a_{i+1}]) \subseteq f(V_i)$. We may lift $\gamma|_{[a_0, a_1]}$ to some path β via the homeomorphism $f|_{V_1}$. Now $f(\beta(a_0)) = f(y_0)$, so there is some chain w_1, \dots, w_m such that $g(w_1) = \beta(a_0), h(w_m) = y_0$ and $g(w_{i+1}) = h(w_i)$. Since the maps g, h are semicovers, we can lift α via g to a path β_1 starting at w_1 . Inductively, we let β_i be the lift of $h \circ \beta_{i-1}$ via g . We must have that $h \circ \beta_m(a_0) = y_0$. By the property of the coequalizer, the compositions of the paths $g \circ \beta_1, h \circ \beta_2, \dots, h \circ \beta_m$ with f and with q all agree. Now $h \circ \beta_m$ is a lift of $p \circ \gamma|_{[a_0, a_1]}$ starting at y_0 so by uniqueness of path lifting, $h \circ \beta_m = \tilde{\gamma}_{y_0}|_{[a_0, a_1]}$. We also know that $f \circ g \circ \beta_1$ is a lift of $\gamma|_{[a_0, a_1]}$ and $g \circ \beta_1$ agrees with $h \circ \beta_m = \tilde{\gamma}_{y_0}|_{[a_0, a_1]}$. Thus $f \circ \tilde{\gamma}(t) = \gamma(t)$ for $t \in [a_0, a_1]$. We may inductively repeat this procedure for $[a_i, a_{i+1}]$ to show that $f \circ \tilde{\gamma}(t) = \gamma(t)$ for $t \in [a_i, a_{i+1}]$. This shows that $\tilde{\gamma}_{y_0}$ is a lift of γ , as required. \square

Proposition 4.1. *Let X be a connected, locally path connected space, and let $i: \{x\} \rightarrow X$ be a base point. Then $(\mathbf{SCov}(X), i^*)$ is an infinite Galois theory.*

Proof. We must show that $(\mathbf{SCov}(X), i^*)$ satisfy IGT1–IGT4.

(IGT1) We show that small colimits exist by showing small coproducts and coequalizers exist. To show existence of finite limits, we need only to show finite products and equalizers exist. Since $\text{id}: X \rightarrow X$ is terminal in $\mathbf{SCov}(X)$, products are already fiber products. Since any category with products and fiber products has equalizers, we need only to show the existence of finite fiber products.

Suppose that $D: I \rightarrow \mathbf{SCov}(X)$ is a small diagram of semicovers with $D(i) = (Z_i, p_i: Z_i \rightarrow X)$. Let Z be the colimit in \mathbf{Top} , as described in lemma 4.2. By the same lemma, we get that $p: Z \rightarrow X$ is a local homeomorphism. If p satisfies the unique homotopy lifting property, then (Z, p) will satisfy the correct universal property to be the colimit in $\mathbf{SCov}(X)$.

If Z is the disjoint union of the Z_i , then it is clear that it satisfies the unique homotopy lifting property. For coequalizers, lemma 4.3 tells us that the maps from the diagram to the coequalizer are semicovering. Since the map to the coequalizer is always surjective, the two out of three property shows that the coequalizer is a semicover.

It is easy to see that the unique homotopy lifting property is preserved by limits. To show this, suppose we are given a diagram (Z_i, p_i) in $\mathbf{SCov}(X)$ and Z is the limit in \mathbf{Top} . Note that the following diagram commutes for each i .

$$\begin{array}{ccccc} Y & \longrightarrow & Z & \longrightarrow & Z_i \\ \downarrow & & \downarrow & & \downarrow \\ Y \times I & \longrightarrow & X & \xrightarrow{=} & X \end{array}$$

So if we are given that the inner square commutes, we get that the outer square commutes and by the homotopy lifting property, there are unique maps $Y \times I \rightarrow Z_i$ for each I making the diagram commute. By the universal property of limits, this means that there is a unique map $Y \times I \rightarrow Z$ making the diagram commute. This is exactly the unique homotopy lifting property.

We move on to finite fiber products. Given maps of semicovers

$$f_1: (Z_1, p_1) \rightarrow (W, q) \quad f_2: (Z_2, p_2) \rightarrow (W, q),$$

let $Z = Z_1 \times_W Z_2$ be the fiber product in \mathbf{Top} . We get a map $s: Z \rightarrow W$ and define $p: Z \rightarrow X$ by $p = q \circ s$. If (Z, p) is a semicover, then it will be the fiber product in $\mathbf{SCov}(X)$ of $(Z_1, p_1), (Z_2, p_2)$ over (W, q) . If one of Z_1 or Z_2 is empty, then Z is empty and is trivially a semicover. Otherwise let $(z_1, z_2) \in Z$. We may find neighborhoods U_1, U_2 of z_1, z_2 respectively and an open set $V \subseteq W$ such that $f_i|_{U_i}: U_i \rightarrow V$ for $i = 1, 2$. One can easily see that for any subspace $T \subseteq Y$, the fiber product $T \times_Y T$ is homeomorphic to T . From this, we deduce that $(U_1 \times_W U_2) \cap Z$ is open in Z and isomorphic to V . This shows that s is a local homeomorphism. Since q is also a local homeomorphism, the composition $p = q \circ s$ is a local homeomorphism, as required.

(IGT2) We know that space Z splits up into path components, which are open since X (hence Z) is locally path connected and are GT connected by lemma 4.1.

(IGT3) In light of (2), we need to only show that the class of path connected semicovers is essentially small. Fixing $x \in X$, the cardinality of a semicover Z is bounded by the set of paths starting at x and the set of points of X by the path lifting property. For any given set Z , there are only a sets worth of topologies on Z , and only a sets worth of maps $Z \rightarrow X$. Thus up to isomorphism, there is only a sets worth of connected semicovers, bounded in terms of the cardinality of $\Omega(X, x)$.

(IGT4) Suppose $(Z, p), (Y, q)$ are semicovers and f, g are maps $(Z, p) \rightarrow (Y, q)$. Suppose $i^*f = i^*g$ and let $z_0 \in Z$. Fix some path γ such that $\gamma(0) = z_0$ and $z = \gamma(1) \in i^*Z$. Since $f(z) = i^*f(z) = i^*g(z) = g(z)$ the loop $\alpha = \overline{g\gamma} * f\gamma$ (where $\overline{g\gamma}$ is the reverse of $g\gamma$ and $*$ represents path concatenation) is a lift of the nullhomotopic loop $\overline{p\gamma} * p\gamma$, so $\alpha(0) = \alpha(1)$. Consequently $f(z_0) = \alpha(0) = \alpha(1) = g(z_0)$. Hence $f = g$ and i^* is faithful.

To show that i^* is conservative, suppose i^*f is a bijective map. Using the same technique as above, it is clear that f is an injection since i^*f is an injection. Suppose that i^*f is a surjection. Given any $y \in Y$ we may find a path γ in Y so that $\gamma(0) \in i^*Y$ and $\gamma(1) = y$. Since there is some $z_0 \in i^*Z$ that maps to $\gamma(0)$ we lift γ to a path $\tilde{\gamma}$ in Z starting at z_0 . This path has the property that $f(\tilde{\gamma}(1)) = y$, hence f is surjective. Thus f is homeomorphism, as it is a bijective local homeomorphism.

From the construction given for the colimits and finite limits, it is easy to see that i^* preserves colimits and finite limits. \square

Remark: We have shown that i^* preserves injections and surjections, which is stronger than being conservative. It is now possible to define the Galois fundamental group.

Definition 12. (*Fundamental group of an infinite Galois theory*) Let (\mathcal{C}, F) be an infinite Galois theory. Then the fundamental group of (\mathcal{C}, F) is defined as

$$\pi_1(\mathcal{C}, F) := \text{Aut}(F),$$

where $\text{Aut}(F)$ is the set of natural automorphisms of F . We give the uniform structure of proposition 3.2 to $\pi_1(\mathcal{C}, F)$.

Remark: We have to justify that $\text{Aut}(F)$ forms a set. This follows from the fact that a natural transformation $\eta: F \rightarrow F$ is determined by it's action on connected objects, since every object is a disjoint union of connected objects by assumption, and up to isomorphism these form a set.

Proposition 3.2 shows that $\pi_1^{\text{Gal}}(X, x)$ is a complete and separated topological group.

Definition 13. (*Galois fundamental group*) Let X be a connected, locally path connected space, and let $i: \{x\} \rightarrow X$ be a base point. We define the Galois fundamental group of (X, x) as $\pi_1^{\text{Gal}}(X, x) := \pi_1(\mathbf{SCov}(X), i^*)$.

As we mentioned earlier, we aim to show that $(\mathbf{SCov}(X), i^*)$ is a tame infinite Galois theory. This gives an equivalence of categories between $\mathbf{SCov}(X)$ and $\pi_1^{\text{Gal}}(X, x)$ -**Sets**,

the category of discrete sets with a continuous action of $\pi_1^{\text{Gal}}(X, x)$ on them. In this next portion, we present the idea of a tame infinite Galois theory.

If (\mathcal{C}, F) is an infinite Galois theory and X is an object of \mathcal{C} , there is an action of $\pi_1(\mathcal{C}, F)$ on $F(X)$. If $\eta \in \pi_1(\mathcal{C}, F)$, then the action on $x \in F(X)$ is given by $\eta \cdot x = \eta_{F(X)}(x)$.

Definition 14. (*Tame infinite Galois theory*) Let (\mathcal{C}, F) be an infinite Galois theory. We say that (\mathcal{C}, F) is tame if it satisfies axiom IGT5.

IGT5 For any connected object $X \in \mathcal{C}$, the action of $\pi_1(\mathcal{C}, F)$ on $F(X)$ is transitive.

Proposition 4.2. *Let X be a connected, locally path connected space, with base point $i: \{x\} \rightarrow X$. Then there is a group homomorphism $\nu: \pi_1(X, x) \rightarrow \pi_1^{\text{Gal}}(X, x)$, where $\nu(\gamma)$ is the natural transformation defined by the monodromy action of γ , i.e.*

$$\nu(\gamma)_{i^*Z}(z) = \gamma \cdot z.$$

Proof. Given $\gamma \in \pi_1(X, x)$ we must show that $\nu(\gamma)$ is a natural transformation. Suppose $f: (Z, p) \rightarrow (Y, q)$ is a map of semicovers over X . In other words, the square

$$\begin{array}{ccc} i^*Z & \xrightarrow{i^*f} & i^*Y \\ \nu(\gamma)_{i^*Z} \downarrow & & \downarrow \nu(\gamma)_{i^*Y} \\ i^*Z & \xrightarrow{i^*f} & i^*Y \end{array}$$

commutes. We also can clearly see that if e is the constant loop at x , then $\nu(e)$ is the natural automorphism that is the identity on all i^*Z .

To show that ν is homomorphism, let $\gamma, \alpha \in \pi_1(X, x)$. Then

$$\nu(\gamma * \alpha)_{i^*Z}(z) = (\gamma * \alpha) \cdot z = \gamma \cdot (\alpha \cdot z) = \gamma \cdot \nu(\alpha)_{i^*Z}(z) = \nu(\gamma)_{i^*Z} \circ \nu(\alpha)_{i^*Z}(z).$$

Thus $\nu(\gamma * \alpha) = \nu(\gamma) \circ \nu(\alpha)$ so ν is a group homomorphism. \square

Corollary 4.1. With X, i as above, $(\mathbf{SCov}(X), i^*)$ is a tame infinite Galois theory.

Proof. If $Z \in \mathbf{SCov}(X)$ is path connected, then the action of $\pi_1(X, x)$ is transitive on i^*Z . From the construction of ν , we see that $\pi_1^{\text{Gal}}(X, x)$ will also act transitively on i^*Z . \square

From now on, if G is a topological group, then $G\text{-Sets}$ refers to the category of discrete sets with a continuous action of G . A basic fact about infinite Galois theories is that their fundamental groups are Noohi. We recall the definition of a Noohi group, as introduced in [1, Definition 7.1.1].

Definition 15. (*Noohi groups*) For a topological group G , let $F_G: G\text{-} \rightarrow \mathbf{Sets}$ be the forgetful functor. Then G is said to be Noohi if the natural map $G \rightarrow \text{Aut}(F_G)$ is an isomorphism of topological groups, where $\text{Aut}(F_G)$ is given the topology of proposition 3.2.

Proposition 4.3. *If (\mathcal{C}, F) is an infinite Galois theory, then $\pi_1(\mathcal{C}, F)$ is a Noohi group.*

Proof. Bhatt and Scholze [1, Theorem 7.2.5] \square

Theorem 7.2.5 in [1] also states that if (\mathcal{C}, F) is a tame infinite Galois theory, then the category \mathcal{C} is equivalent to $\pi_1(\mathcal{C}, F)$ -**Sets**, where $\pi_1(\mathcal{C}, F)$ have the topology from proposition 3.2. In our case, this gives

Corollary 4.2. If X is a connected locally path connected space with base point x , then the categories $\mathbf{SCov}(X)$ and $\pi_1^{\text{Gal}}(X, x)$ -**Sets** are equivalent.

The functor $\mathbf{SCov}(X) \rightarrow \pi_1^{\text{Gal}}(X, x)$ -**Sets** is just i^* . For a semicover Z the action of $\eta \in \pi_1^{\text{Gal}}(X, x)$ on $z \in i^*Z$ is $\eta \cdot z = \eta_{i^*Z}(z)$. If $f: Z \rightarrow Y$ is a map of semicovers then $i^*f: i^*Z \rightarrow i^*Y$ is a map of $\pi_1^{\text{Gal}}(X, x)$ -sets, since $\pi_1^{\text{Gal}}(X, x)$ acts by natural transformations.

5. FUNCTORIALITY OF THE GALOIS FUNDAMENTAL GROUP

We have introduced $\pi_1^{\text{Gal}}(X, x)$ for based space (X, x) that is connected and locally path connected and shown that this is a complete separated topological group. In this section, we will show that π_1^{Gal} is actually a functor from the category of based connected, locally path connected spaces to the category of complete topological groups. This requires that for any map $f: (X, x) \rightarrow (Y, y)$, we get a map $f_*: \pi_1^{\text{Gal}}(X, x) \rightarrow \pi_1^{\text{Gal}}(Y, y)$, satisfying the composition laws and preserving identities. Recall from proposition 2.4 that pullback along f defines a functor $f^*: \mathbf{SCov}(Y) \rightarrow \mathbf{SCov}(X)$. Let $i: \{x\} \rightarrow X, j: \{y\} \rightarrow Y$ be the choice of base point. Then $f \circ i = j$ hence $i^* \circ f^* = j^*$. That is, the diagram

$$\begin{array}{ccc} \mathbf{SCov}(Y) & \xrightarrow{f^*} & \mathbf{SCov}(X) \\ & \searrow j^* & \swarrow i^* \\ & \mathbf{Sets} & \end{array}$$

commutes. Thus if we are given a natural automorphisms of i^* , we obtain a natural automorphism of j^* by precomposition with f^* . More explicitly, given $\eta \in \pi_1^{\text{Gal}}(X, x)$ we define $f_*(\eta)$ by

$$f_*(\eta)_Z = \eta_{f^*Z}$$

for any $Z \in \mathbf{SCov}(Y)$. One can check under these assumptions that f_* is a homomorphism.

Recall that for a space X and base point $i: \{x\} \rightarrow X$, there are maps $\pi_1^{\text{Gal}}(X, x) \rightarrow \text{Aut}(i^*Z)$ for all $Z \in \mathbf{SCov}(X)$. The uniform structure on $\pi_1^{\text{Gal}}(X, x)$ is the one that is pulled back from the uniform structure on $\text{Aut}(i^*Z)$ for all $Z \in \mathbf{SCov}(X)$. The uniformity on i^*Z is generated by entourages of the form

$$V_{Z,T} = \{(\eta, \xi): \eta^{-1}\xi \cdot t = \eta\xi^{-1} \cdot t = t \text{ for all } t \in T\},$$

where $Z \in \mathbf{SCov}(X)$ and $T \subseteq i^*Z$ is finite. A fundamental system of entourages is given by taking finite intersections of such sets. In fact, this can be simplified.

Proposition 5.1. *Entourages of the form $V_{Z,\{t\}}$ where Z is connected form a fundamental system of entourages.*

Proof. To begin, we can assume that T consists of only a single element in the above entourage. To see this, we replace Z with $Z' = Z \times_X \cdots \times_X Z$, where there are $|T|$ many copies (we will show later that $\mathbf{SCov}(X)$ is closed under finite limits so that the fiber

product is indeed a semicovering). Let $t \in i^*Z'$ be any tuple that contains all members of T . The condition for $\eta, \xi \in \pi_1^{\text{Gal}}(X, x)$ to agree on T is the same as requiring that they agree on t .

Now suppose that we are given Z_1, \dots, Z_n and $t_i \in i^*Z_i$. We can assume that the Z_i are connected. In this case, we let $Z = Z_1 \times_X \cdots \times_X Z_n$ and $t = (t_1, \dots, t_n)$. The intersection of the entourages $V_{Z_i, \{t_i\}}$ is then clearly the same as $V_{Z, \{t\}}$. If Z is not connected, we take Z' to be the connected component of Z containing t . Any natural transformation must map i^*Z' into i^*Z' , so $V_{Z, \{t\}} = V_{Z', \{t\}}$. \square

Remark: If X has a universal cover, then this uniformity induces the discrete topology on $\pi_1(X, x)$. This is because for any finite subset $T \subseteq i^*\tilde{X}$, we have $V_T = \Delta X$.

We will write $V_{Z, t}$ instead of $V_{Z, \{t\}}$ from now on.

Proposition 5.2. *Suppose $f: (X, x) \rightarrow (Y, y)$ is a continuous map of based spaces. Let $G = \pi_1^{\text{Gal}}(X, x), H = \pi_1^{\text{Gal}}(Y, y)$. Then $f_*: G \rightarrow H$ is a continuous homomorphism of topological groups. Further, if $g: (Y, y) \rightarrow (Z, z)$ is another map of based spaces, then $(g \circ f)_* = g_* \circ f_*$.*

Proof. Let $i: \{x\} \rightarrow X$ be the inclusion of x and similarly for $j: \{y\} \rightarrow Y$. Suppose $V_{Z, t}(h)$ is a neighborhood of h , with Z connected. Now f^*Z is in $\mathbf{SCov}(X)$ and since $i^*f^*Z = j^*Z$, we have $t \in i^*f^*Z$. So it at least makes sense to talk about the neighborhood $V_{f^*Z, t}(g)$. If $g' \in V_{f^*Z, t}(g)$, then $g' \cdot t = g \cdot t$ and $(g')^{-1} \cdot t = g^{-1} \cdot t$. Applying f_* shows that $f_*(g') \cdot t = f_*(g) \cdot t = h \cdot t$ and likewise $f_*(g')^{-1} \cdot t = h^{-1} \cdot t$. In other words, $V_{f^*Z, t}(g)$ is a neighborhood of g in G , and it maps into $V_{Z, t}(h)$, thus f_* is continuous. \square

This leads to the following corollary.

Corollary 5.1. π_1^{Gal} is a functor from based topological spaces into topological groups.

6. UNIVERSAL COVERS

We can compare the previous construction of the fundamental group to the construction of based loops up to homotopy, which we denote π_1^{top} .

Proposition 6.1. *If X is a space with a universal cover \tilde{X} , and $x \in X$ is a base point, then $\pi_1^{\text{Gal}}(X, x)$ and $\pi_1^{\text{top}}(X, x)$ are isomorphic as topological groups if $\pi_1^{\text{top}}(X, x)$ is given the discrete topology.*

Proof. Suppose $p: \tilde{X} \rightarrow X$ is a universal cover. It is well known that if X has a universal cover, then $\pi_1^{\text{top}}(X, x) \simeq \text{Aut}(\tilde{X})$. By Yoneda's lemma and the fact that $\pi_1^{\text{top}}(X, x)$ is isomorphic to the automorphisms of \tilde{X} over X , showing that \tilde{X} represents i^* is sufficient to complete the proof. Let $p: \tilde{X} \rightarrow X$ be the universal cover of X . Fix a point $s \in i^*\tilde{X}$ and let $(Z, q) \in \mathbf{SCov}(X)$. For any $z \in i^*Z$, construct a map $f_z: \tilde{X} \rightarrow Z$ in the following way: For $t \in \tilde{X}$, choose a path γ from s to t . This projects down to a path in X , which we lift up to a path $f_z\gamma$ in Z starting at z . Let $f(t)$ be the endpoint of this path. We must show that this is well defined, so suppose γ' is another path from s to t . As \tilde{X} is simply connected, these paths are homotopic, so they map to homotopic paths in X . By the homotopy lifting property and local homeomorphism, we see that the lifts $f_z\gamma, f_z\gamma'$ have the same endpoint. Thus f_z is well defined.

To show continuity, suppose $x \in \tilde{X}$ and $U \subseteq Z$ is an open neighborhood of $f_z(x)$. We will show that there is a neighborhood of x that maps into U . Assume $q|_U: U \rightarrow X$ is an open embedding and U is path connected. Let $V \subseteq \tilde{X}$ be the set of points $y \in \tilde{X}$ so that there exists a path γ from x to y where $p\gamma$ is a path in $q(U)$. For all $y \in V$ there is an open path connected neighborhood W of y . Then $y \in p^{-1}(q(U)) \cap W \subseteq V$ which is open since $q(U)$ is open, so V is open. Finally for any $y \in V$, let γ be a path from x to y so that $p\gamma$ is a path in $q(U)$. Lift $p\gamma$ to a path $\tilde{p}\gamma$ starting at $f(x)$. This will be a path in U , hence $\tilde{p}\gamma(1) \in U$. All that is left is to show $f(y) = \tilde{p}\gamma(1)$ to prove continuity. Let α is a path in \tilde{X} from s to x , and $\tilde{p}\alpha$ a lift in Z of $p\alpha$. Then $\alpha * \gamma$ is a path from s to y and $\tilde{p}\alpha * \tilde{p}\gamma$ is a lift of $p\alpha * p\gamma$, so $f(y) = \tilde{p}\alpha * \tilde{p}\gamma(1) = \tilde{p}\gamma(1)$. Consequently f is continuous.

This defines a map $\eta_Z: i^*Z \rightarrow \text{Hom}(\tilde{X}, Z), z \mapsto f_z$. It is injective since $f_z(s) = z$. We show that this map is surjective. Suppose $f: \tilde{X} \rightarrow Z$. In order to commute over X , for any $t \in \tilde{X}$ the element $f(t)$ must be the same as the element we get by constructing a path from s to t , projecting it into X , lifting it to a path in Z starting at $f(s)$ and looking at the endpoint.

Finally all that is left is to show that the maps η_Z are natural. Let $g: (Z, q) \rightarrow (Y, r)$ be a map of semicovers and let $z \in i^*Z$. For any $t \in \tilde{X}$, we need to show $g(f_z(t)) = f_{g(z)}(t)$. Suppose α is a path from s to t in \tilde{X} and let $\tilde{p}\alpha$ be the lift of $p\alpha$ in Z starting at z , so that $f_z(t) = \tilde{p}\alpha(1)$. Note $g(\tilde{p}\alpha(t)) = g(f_z(t))$. However, $g\tilde{p}\alpha$ is a lift of $p\alpha$ in Y starting at $g(z)$, so $g(\tilde{p}\alpha(1)) = f_{g(z)}(t)$. This means $gf_z = f_{g(z)}$, showing the η_Z give a natural transformation $\eta: i^* \rightarrow \text{Hom}(\tilde{X}, -)$. Since each η_Z is a bijection η is a natural isomorphism. Consequently, $\pi_1^{\text{Gal}}(X, x)$ and $\pi_1^{\text{top}}(X, x)$ are isomorphic as groups.

The remark after proposition 5.1 shows that $\pi_1^{\text{Gal}}(X, x)$ has the discrete topology, which shows the two groups are isomorphic as topological groups. \square

Thus for a space X with a universal cover, we may write $\pi_1(X, x)$ without confusion.

7. THE TOPOLOGIZED FUNDAMENTAL GROUP

The contents of this section are a summary of a few of the ideas from section 4 of [3].

The fundamental group $\pi_1(X, x)$ for a space X is a quotient of the loop space $\Omega(X, x) = \mathcal{P}_x \cap \mathcal{P}^x$. Since $\Omega(X, x)$ is a topological space under the compact open topology, $\pi_1(X, x)$ inherits a topology via the quotient map. While in general this does not make $\pi_1(X, x)$ into a topological group as seen in [7], it does make it into a quasitopological group in the sense that the inversion map is continuous and multiplication is continuous in each variable.

The category **TopGrp** of topological groups is a full sub category of **qTopGrp**, the category of quasitopological groups, so there is a forgetful functor $F: \mathbf{TopGrp} \rightarrow \mathbf{qTopGrp}$. We will show that this functor has a left adjoint using Freyd's adjoint functor theorem. In order to use this theorem the following lemma is necessary.

Lemma 7.1. *The categories **TopGrp** and **qTopGrp** are complete and F preserves all small limits.*

Proof. Suppose there is a diagram G_i of quasitopological groups. Since the category of groups is complete, let G be the limit of the diagram G_i in this category. We put the

coarsest topology on G that make the homomorphisms $f_i: G \rightarrow G_i$ continuous for all i . If H is a quasitopological group that maps in a compatible way into the diagram G_i , then there is a unique group homomorphism $H \rightarrow G$ by the universal property of limits of groups. The open sets of G are finite intersections of open sets pulled back from the G_i , and since $H \rightarrow G_i$ is continuous for each i , we must have $H \rightarrow G$ continuous. Thus G satisfies the correct universal property. We need now to show that G is in fact a quasitopological group.

Now we must show that inversion is continuous and multiplication is continuous in each variable. For each group G_i in the diagram, let inv_i be the inversion map and $\mu_i: G_i \times G_i \rightarrow G_i$ the multiplication map for G_i . We also let $\text{inv}: G \rightarrow G$ be the inversion map and $\mu: G \times G \rightarrow G$ the multiplication map. Now for each i , the following diagram of topological spaces commutes

$$\begin{array}{ccc} G & \xrightarrow{\text{inv}} & G \\ f_i \downarrow & & \downarrow f_i \\ G_i & \xrightarrow{\text{inv}_i} & G_i \end{array}$$

It follows that if $U \subseteq G_i$ is open, then $(f_i \circ \text{inv})^{-1}(U) = (\text{inv}_i \circ f_i)^{-1}(U)$ is open. This means that inv is continuous since the sets $f_i^{-1}(U)$, where U is an open subset of some G_i , form a sub basis for the topology on G .

Now to see that multiplication is continuous in each variable, we fix some $g \in G$. For each G_i , the following diagram of topological spaces commutes

$$\begin{array}{ccc} G \times \{g\} & \xrightarrow{\mu} & G \\ f_i \times f_i \downarrow & & \downarrow f_i \\ G_i \times \{f_i(g)\} & \xrightarrow{\mu_i} & G_i \end{array}$$

The map $f_i \times f_i|_{\{g\}}$ is continuous since the products of continuous maps are continuous. For similar reasons as inversion, this implies that the top map is continuous. An analogous argument shows that fixing an element on the left will give a continuous map $\{g\} \times G \rightarrow G$. Thus G is a quasitopological group.

Now suppose that all of the G_i are topological groups, and let G denote the same construction. Then G will satisfy the correct universal property, and we also have that the following diagram commutes for each G_i .

$$\begin{array}{ccc} G \times G & \xrightarrow{\mu} & G \\ f_i \times f_i \downarrow & & \downarrow f_i \\ G_i \times G_i & \xrightarrow{\mu_i} & G_i \end{array}$$

For analogous reasons, the map μ must be continuous. This shows that G is intact a topological group.

All of this shows that the limit of a diagram of topological groups is the same whether this limit is taken in the category of quasitopological or the category of topological groups. In other words, the forgetful functor F preserves limits. \square

Proposition 7.1. *The forgetful functor $F: \mathbf{TopGrp} \rightarrow \mathbf{qTopGrp}$ has a left adjoint $\tau: \mathbf{qTopGrp} \rightarrow \mathbf{TopGrp}$.*

Proof. In light of the previous lemma, all we need to show is that for any quasitopological group G , there is a small collection of topological groups G_i indexed by I and arrows $f_i: G \rightarrow G_i$ such that any continuous morphism $h: G \rightarrow H$ whose target is a topological group can be written as $h = t \circ f_i$ for some $t: G_i \rightarrow H$. First, if G is a topological group and K is a normal subgroup, a topology \mathcal{T} on G/K is said to be a G -compatible topology for K if G/K is a topological group and $G \rightarrow G/K$ is continuous. Now let

$$I = \{(K, \mathcal{T}) : K \subseteq G \text{ is a normal subgroup, } \mathcal{T} \text{ is a } G\text{-compatible topology for } K\}$$

Then if $i = (K, \mathcal{T})$, let $G_i = G/K$ with the topology \mathcal{T} and $f_i: G \rightarrow G/K = G_i$ be the usual quotient map for groups. Any map from $h: G \rightarrow H$ where H is a topological group can be factored as $G \rightarrow \text{im}(h) \hookrightarrow H$. The underlying group of $\text{im}(h)$ is isomorphic to $G/\ker(h)$ which is isomorphic to G_i for some $i \in I$. The inclusion $\text{im}(h) \hookrightarrow H$ is a morphism of topological groups. Hence we apply Freyd's adjoint functor theorem (Theorem 2 §6 Chapter 5 [9]) which shows that τ exists. \square

Note that if G is a quasitopological group, then $\tau(G)$ has the same underlying group as G . Indeed, if we let $\text{in}(G)$ be the same underlying group as G with the indiscrete topology then the identity $G \rightarrow \text{in}(G)$ is continuous. We get a map $\tau(G) \rightarrow \text{in}(G)$ that makes the following diagram commute

$$\begin{array}{ccc} G & \longrightarrow & \tau(G) \\ & \searrow \text{id} & \downarrow \\ & & \text{in}(G) \end{array}$$

This means that $G \rightarrow \tau(G)$ is injective. It is clear to see that the the image of G in $\tau(G)$ satisfies the correct universal property to be the adjoint, hence by uniqueness of the adjoint, the image of G must be $\tau(G)$.

Definition 16. (*The topological fundamental group*) Given any based space (X, x) we define $\pi_1^\tau(X, x)$ to be $\tau(\pi_1(X, x))$.

It is a result of [3] that for a connected, locally path connected based space (X, x) there is an equivalence between the category $\pi_1^\tau(X, x)\text{-Sets}$ (where the sets are considered as discrete spaces and the action is continuous) and the category $\mathbf{SCov}_{Br}(X)$. In fact, the conditions of being connected and locally path connected may be weakened, but for

our purposes it is enough. From 2.2, we know there is a categorical equivalence between $\mathbf{SCov}_{Br}(X)$ and $\mathbf{SCov}(X)$. These remarks prove the following proposition.

Proposition 7.2. *If (X, x) is a based topological space that is connected and locally path connected, there is a categorical equivalence between $\pi_1^\tau(X, x)$ -Sets and $\mathbf{SCov}(X)$.*

If $\pi_1^\tau(X, x)$ had a basis of open subgroups of the neighborhood filter of the identity, the proposition above combined with proposition 7.1.5 of [1] would show that $\pi_1^{\text{Gal}}(X, x)$ is the completion of $\pi_1^\tau(X, x)$ with respect to the two sided uniformity. Unfortunately, we don't know whether or not this is true, so instead we introduce a new topology on $\pi_1(X, x)$ that lets us use proposition 7.1.5 of [1].

Definition 17. (*The σ -topology*) Let G be a quasitopological group. The σ -topology on G is the unique topology where the neighborhoods of the identity are sets that contain an open subgroup of $\tau(G)$. Let $\pi_1^\sigma(X, x)$ denote $\pi_1(X, x)$ with the σ -topology.

Proposition 7.3. *For any based topological space, $\pi_1^\sigma(X, x)$ is a topological group.*

Proof. The neighborhood filter of the identity is generated by certain subgroups of π_1 so this filter satisfies GV1-GV2 as defined in Chapter 3, §2 of [2]. If $N \subseteq \pi_1^\tau(X, x)$ is an open subgroup and $g \in \pi_1^\tau(X, x)$, then gNg^{-1} is an open subgroup because conjugation is a homeomorphism in a topological group. Thus the neighborhood satisfies GV3. Therefore by proposition 1 of Chapter 3, §2 in [2], $\pi_1^\sigma(X, x)$ is a topological group. \square

The σ -topology is coarser than the τ topology, so the identity $\pi_1^\tau(X, x) \rightarrow \pi_1^\sigma(X, x)$ is continuous. We will now show that $\pi_1^{\text{Gal}}(X, x)$ is the completion of $\pi_1^\sigma(X, x)$, but first we need a lemma. For the rest of the paper, whenever G is a topological group, G -Sets denotes the category of discrete sets with a continuous action of G .

Lemma 7.2. *Suppose that G, H are topological groups so that there is a bijective continuous map $\varphi: G \rightarrow H$. If G and H have the same open subgroups, then there is a categorical equivalence between G -Sets and H -Sets.*

Proof. We need to show that if S is a set with an action of the underlying group of G, H on it, the action of G on S is continuous if and only if the action of H is continuous. In this case, the equivalence will be given by the functor G -Sets \rightarrow H -Sets which doesn't change the actions, sets or morphisms.

Without loss of generality we assume that S is transitive, since every G -set decomposes as a disjoint union of transitive G -sets (the orbits). Recall that for any topological group M , a transitive action of M on a discrete set T is continuous if and only if for any $t \in T$, the stabilizer of t is an open subgroup of M . Since G, H have the same open subgroups, it follows that if the action of G on S is continuous if and only if the action of H on S is continuous. \square

Corollary 7.1. The categories $\pi_1^\tau(X, x)$ -Sets and π_1^σ -Sets are equivalent.

Proof. By construction, $\pi_1^\tau(X, x)$ and $\pi_1^\sigma(X, x)$ have the same open subgroups, hence by lemma 7.2, the categories are equivalent. \square

For a topological group G , denote the completion of G with respect to the two sided uniformity by G^* .

Corollary 7.2. The completion $\pi_1^\sigma(X, x)^*$ of $\pi_1^\sigma(X, x)$ with respect to the two sided uniformity is $\pi_1^{\text{Gal}}(X, x)$.

Proof. Let $F: \pi_1^\sigma(X, x)\text{-Sets} \rightarrow \mathbf{Sets}$ be the forgetful functor. Then $\pi_1^\sigma(X, x)^* \simeq \text{Aut}(F)$ by proposition 7.1.5 of [1]. On the other hand, we have equivalences

$$\pi_1^{\text{Gal}}(X, x)\text{-Sets} \simeq \mathbf{SCov}(X) \simeq \mathbf{SCov}_{Br}(X) \simeq \pi_1^\tau(X, x)\text{-Sets} \simeq \pi_1^\sigma(X, x)\text{-Sets}.$$

The first equivalence is corollary 4.2, the second is corollary 2.1, the third is theorem 7.19 in [3] and the last follows from lemma 7.2. Since $\pi_1^{\text{Gal}}(X, x)$ is Noohi (proposition 4.3) and $\pi_1^{\text{Gal}}(X, x)\text{-Sets} \simeq \pi_1^\sigma(X, x)\text{-Sets}$ we have $\pi_1^{\text{Gal}}(X, x) \simeq \text{Aut}(F) \simeq \pi_1^\sigma(X, x)^*$. \square

8. COVERS OF THE EARRING

Let C_n (for $n > 0$) be the circle of radius $1/n$ centered at $(0, 1/n)$ in the plane. Then the hawaiian earring is $\mathbf{E} = \bigcup_{n=1}^{\infty} C_n$ and is given the subspace topology. Let \mathbf{Y} be the countably infinite wedge of circles. There is an obvious bijection $f: \mathbf{Y} \rightarrow \mathbf{E}$ which is continuous. Pullback by f defines a functor $\mathbf{SCov}(\mathbf{E}) \rightarrow \mathbf{SCov}(\mathbf{Y})$. For any $Z \in \mathbf{SCov}(\mathbf{E})$ the corresponding semicovering space f^*Z is the same set as Z (since f is a bijection), it just has a different topology which we call the \mathbf{Y} topology.

Proposition 8.1. *The functor $f^*: \mathbf{SCov}(\mathbf{E}) \rightarrow \mathbf{SCov}(\mathbf{Y})$ is fully faithful.*

Proof. It is faithful since f^* doesn't change the points and maps are determined by what they do on points. Now suppose $g: (W, q) \rightarrow (Z, p)$ is continuous in the \mathbf{Y} topology and $U \subseteq Z$ is open in the \mathbf{E} topology. If there is no point in U that maps onto the origin, then $g^{-1}(U)$ is open in the \mathbf{E} topology, as \mathbf{E} and \mathbf{Y} are homeomorphic away from the origin, so we assume that there is some $z \in U$ that maps to the origin in \mathbf{E} . By choosing a smaller neighborhood if necessary, we can assume that U is homeomorphic to a connected open neighborhood of the origin in \mathbf{E} . For any point $w \in g^{-1}(z)$ we can find an open neighborhood V of w that is homeomorphic to a connected open neighborhood of the origin. Now $g^{-1}(U) \cap V$ is open in the \mathbf{Y} topology. One sees that $g^{-1}(U) \cap V$ will be open in the \mathbf{E} topology if $q(g^{-1}(U) \cap V)$ contains all but finitely many of the circles. As $p(U) \cap q(V)$ is open, it contains all but finitely many of the circles, and since $p(U) \cap q(V) \subseteq q(g^{-1}(U) \cap V)$, we see that $g^{-1}(U) \cap V \subseteq g^{-1}(U)$ is open in the \mathbf{E} topology. From this it is clear that g is continuous in the \mathbf{E} topology. Thus f^* is full. \square

Let $e \in \mathbf{E}$ denote the origin of the hawaiian earring, $y \in \mathbf{Y}$ be the point that maps to e , and $G = \pi_1(\mathbf{Y}, y)$. Using the equivalence of $G\text{-Sets}$ and $\mathbf{SCov}(\mathbf{Y})$, we can classify the essential image of $\pi_1^{\text{Gal}}(\mathbf{E}, e)$ in terms of $G\text{-Sets}$. Now G is isomorphic to a free group on countably a infinite set of generators, so a giving a G -set is the same as giving a set S and a bijection $\varphi_i: S \rightarrow S$ for each integer i such that $\varphi_{-i} = \varphi_i^{-1}$ and φ_0 is the identity.

Proposition 8.2. *A G -set S is in the essential image of $\mathbf{SCov}(\mathbf{E}) \rightarrow G\text{-Sets}$ if and only if the following two conditions hold:*

- (1) *For any $s \in S$, all but finitely many of the φ_i fix s .*
- (2) *If there is a sequence $\varphi_{i_1}, \varphi_{i_2}, \dots$ where any φ_k appears only finitely many times, then for any $s \in S$, the sequence $\varphi_{i_1}(s), \varphi_{i_2}\varphi_{i_1}(s), \dots$ eventually stabilizes.*

Proof. In the equivalence of $\mathbf{SCov}(\mathbf{Y})$ and $G\text{-Sets}$ the automorphism φ_k is the action of the fiber that we get from the loop c_k that goes around the $|k|$ -th circle clockwise if k is positive or counterclockwise otherwise (and is constant if $k = 0$). Thus if $Z \in \mathbf{SCov}(\mathbf{E})$ then for each $z \in i^*Z$, there is a neighborhood U of z that is homeomorphic to its image in E , and hence all but finitely many of the circles are unwound. This means that all but finitely many of the φ_k act trivially on z . Now a sequence $\varphi_{i_1}, \varphi_{i_2}, \dots$ where φ_k only appears finitely many times corresponds to an element of $\pi_1^{\text{top}}(\mathbf{E}, \mathbf{e})$ as seen in [6]. Let γ be a representative of this loop. The set S corresponds to i^*Z for some $Z \in \mathbf{SCov}(\mathbf{E})$. Since we produce a path $\tilde{\gamma}$ that is a lift of γ to Z , we see that the sequence stabilizes to $\tilde{\gamma}(1)$.

Conversely, suppose we are given a set S and automorphisms φ_k that satisfy the two conditions. Let Z' be the corresponding cover of \mathbf{Y} . Let $p': Z' \rightarrow \mathbf{E}$ be the composition $Z' \rightarrow \mathbf{Y} \rightarrow \mathbf{E}$. The first condition ensures that all but finitely many of the loops at any point in the fiber are unwound. We make a new cover $p: Z \rightarrow \mathbf{E}$ so that $Z = Z'$ as a set and $p = p'$. A subset $U \subseteq Z$ is open if and only if U is open in Z' and for all $s \in S \cap U$, all but finitely many of the wound loops based at s are contained in U . Already $p: Z \rightarrow \mathbf{E}$ is locally bijective, and the topology is defined to make it a local homeomorphism, thus we only need to show the unique homotopy lifting property.

Let $\gamma: I \rightarrow \mathbf{E}$ be a path, $z \in Z$ and $F = \gamma^{-1}(e)$. We assume without loss of generality that F contains no closed intervals. Let $0 = x_1 = \inf F$ and $x_n = \inf F - \{x_1, \dots, x_{n-1}\}$. If there are only finitely many x_i then γ is continuous when considered as a loop $I \rightarrow \mathbf{Y}$, so we may lift it to a loop in Z' starting at z and compose the lift with $Z \rightarrow Z'$ to get a lift starting at z . Otherwise, there is an $x \in F$ that is a limit of the sequence. Let $\gamma_n = \gamma|_{[x_n, x_{n+1}]}$. By definition of the x_i , γ_n must be a loop that traverses some circle $C_{|k(n)|}$ once, either counter clockwise or clockwise. This means it is homotopic to $c_{k(n)}$, where $k(n)$ is signed according to the direction γ_n traverses $C_{|k(n)|}$. We have shown we can lift γ_n . Let $\tilde{\gamma}_1$ be a lift of γ_1 starting at z and $\tilde{\gamma}_n$ be a lift of γ_n starting at $\tilde{\gamma}_{n-1}(1)$. Note that $\gamma_n(1) = \varphi_{k(1)} \cdots \varphi_{k(n)}(z)$. The sequence $\varphi_{k(1)}(z), \varphi_{k(2)}\varphi_{k(1)}(z), \dots$ must have any fixed k appear finitely many times, otherwise γ traverses $C_{|k|}$ infinitely many times violating the fact that γ has compact image. Hence the sequence $\gamma_1(1), \gamma_2(1), \dots$ eventually stabilizes by the second condition. That means we may find some N and some $U \subseteq Z$ so that $\gamma_n(1) = \gamma_m(1)$ for all $m > n > N$, $\gamma|_{[x_n, x_m]} \subseteq U$ and $p|_U$ is an open embedding. We have shown $\gamma|_{[0, x_N]}$ can be lifted, and since the image of $\gamma|_{[x_N, x]}$ is contained in $p(U)$, this portion can be lifted using the homeomorphism $p|_U: U \rightarrow p(U)$. □

9. THE GALOIS FUNDAMENTAL GROUP OF THE HARMONIC ARCHIPELAGO

The harmonic archipelago \mathbf{A} is the space that is obtained by filling in the gaps between the loops of the hawaiian earring and adding a bump of height one between loops c_n, c_{n+1} . A more explicit description can be found in [8]. In particular, there is a continuous inclusion $f: \mathbf{E} \rightarrow \mathbf{A}$. We view \mathbf{E} as a subset of \mathbf{A} and let e denote the shared origin of the harmonic archipelago and the hawaiian earring.

Proposition 9.1. $\pi_1^{\text{Gal}}(\mathbf{A}, \mathbf{e})$ is trivial.

Proof. We prove this by showing that \mathbf{A} has no non trivial connected semicovers.

Suppose $p: Z \rightarrow \mathbf{A}$ is a non trivial connected semicover. Then $W = p^{-1}(E)$ is a non trivial connected semicover of \mathbf{E} . Since W is non trivial, there is some $z \in i^*W$ and some n so that $c_n \cdot z \neq z$. Since for all k , c_k is homtopic to c_n in \mathbf{A} we have $c_k \cdot z = c_n \cdot z$. This contradicts proposition 8.2, thus there are no non trivial connected semicovers of \mathbf{A} . \square

In [8] it is shown that $\pi_1(\mathbf{A}, \mathbf{e}) \neq \mathbf{0}$, so in general the groups $\pi_1(X, x)$ and $\pi_1^{\text{Gal}}(X, x)$ are not isomorphic.

10. THE GALOIS FUNDAMENTAL GROUP OF THE EARRING

Let $i: \{e\} \rightarrow \mathbf{E}$ be the inclusion of the origin into the hawaiian earring, \mathbf{Y} the infinite wedge of circles and $j: \{y\} \rightarrow \mathbf{Y}$ be the inclusion of the point at which all of the circles meet in the infinite wedge of circles. The bijection $f: \mathbf{Y} \rightarrow \mathbf{E}$ takes e to y , so there is a group morphism $\pi_1(\mathbf{Y}, \mathbf{y}) \rightarrow \pi_1^{\text{Gal}}(\mathbf{E}, \mathbf{e})$. To show that this map is injective, we use the covers \mathbf{E}_n , where the first n circles of the hawaiian earring are unwound completely and the rest are still wound. Now $\alpha, \beta \in \pi_1(\mathbf{Y}, \mathbf{y})$ correspond to words in an alphabet that is indexed by \mathbf{N} . Since words are necessarily finite, we can choose some n such that no letters corresponding to $k > n$ appear in the words α, β . Then α and β will act differently on \mathbf{E}_n , which means that they correspond to different natural transformations. Thus $\pi_1(\mathbf{Y}, \mathbf{y})$ is naturally a subgroup of $\pi_1^{\text{Gal}}(\mathbf{E}, \mathbf{e})$.

We want to show that $\pi_1(\mathbf{Y}, \mathbf{y})$ is dense in $\pi_1^{\text{Gal}}(\mathbf{E}, \mathbf{e})$, and for this we will use the following lemma.

Lemma 10.1. *Let $Z \in \mathbf{SCov}(\mathbf{E})$ be a connected semicover. Then for any $s, t \in i^*Z$ there exists some $\alpha \in \pi_1(\mathbf{Y}, \mathbf{y})$ such that $\alpha \cdot s = t$.*

Proof. Since Z is connected, there is a path $\tilde{\gamma}$ that starts s and ends at t . If we project this down, we get a loop γ in E , which gives an element of the fundamental group. Let $F = \tilde{\gamma}^{-1}(i^*Z)$. We can assume that F does not contain any closed intervals, since if it did we could adjust $\tilde{\gamma}$ by gluing the endpoints together and still have a continuous loop that is homtopic to $\tilde{\gamma}$. It is clear that if F is finite, then $\gamma \in \pi_1(\mathbf{Y}, \mathbf{y})$, and we are done. Otherwise F is infinite and we consider the set of limit points $L(F)$ of the subspace $F \subseteq [0, 1]$. Since F is infinite and bounded, $L(F)$ is not empty. Let $x \in L(F)$. Thus there is a sequence $x_1, x_2, \dots \in F$ that converges to x . Now $\tilde{\gamma}$ induces a continuous map $F \rightarrow i^*Z$ with their respective subspace topologies, and thus the sequence $\tilde{\gamma}(x_1), \tilde{\gamma}(x_2), \dots$ converges. Since i^*Z is discrete, this means there is some N such that $\tilde{\gamma}(x_n) = \tilde{\gamma}(x_m)$ for all $n, m \geq N$. This means that we can glue x_N to x to get a new loop that acts the same way as $\tilde{\gamma}$ on the fiber. This gets rid of the limit point x , and as all limit points can be removed this way, we can find a loop α that acts the same way on the fiber, but such that $\alpha^{-1}(i^*Z)$ is finite. Thus α projects down to a loop in $\pi_1(\mathbf{Y}, \mathbf{y})$. \square

Proposition 10.1. *$\pi_1(\mathbf{Y}, \mathbf{y})$ is dense in $\pi_1^{\text{Gal}}(\mathbf{E}, \mathbf{e})$.*

Proof. Let $\eta \in \pi_1^{\text{Gal}}(\mathbf{E}, \mathbf{e})$, and consider the neighborhood $V = V_{Z,t}(\eta)$ (where $z \in i^*Z$) of η . By Proposition 5.1 we may assume Z is connected. Let $Z' = Z \times_{\mathbf{E}} Z$. The elements $(\eta_Z^{-1}(t), t), (t, \eta_Z(t))$ are in the same connected component of Z' . Thus by Lemma 10.1, there is some $\alpha \in \pi_1(\mathbf{Y})$ such that $\alpha \cdot (\eta_Z^{-1}(t), t) = (t, \eta_Z(t))$. It follows that $\alpha \in V_{Z,t}$. \square

Corollary 10.1. *$\pi_1^{\text{Gal}}(\mathbf{E}, \mathbf{e})$ is the completion of $\pi_1(\mathbf{Y}, \mathbf{y})$.*

Proof. Since $\pi_1^{\text{Gal}}(\mathbf{E}, \mathbf{e})$ is complete, and $\pi_1(\mathbf{Y}, \mathbf{y})$ is dense in $\pi_1^{\text{Gal}}(\mathbf{E}, \mathbf{e})$, this result follows from the uniqueness of the completion of a topological group. \square

Recall that the completion of a topological group can be described as the set of minimal Cauchy filters on the group. We now describe how to go from a Cauchy filter \mathcal{F} on $\pi_1(\mathbf{Y}, \mathbf{y})$ to an element of $\pi_1^{\text{Gal}}(\mathbf{E}, \mathbf{e})$. Let $Z \in \mathbf{SCov}(\mathbf{E})$ and define $\varphi_Z(t) = \alpha \cdot t$, where $\alpha \in A$ and $A \in \mathcal{F}$ is $V_{Z, \{t\}}$ close. As in Lemma 3.2, we see that this is a bijection. The argument used in proposition 3.2 shows that the φ_Z give a natural transformation. A topological group G has a completion if and only if the image of a Cauchy filter base under the inversion map is a Cauchy filter, base, hence the image of \mathcal{F} under inversion generates a Cauchy filter, and the natural transformation associated to this is clearly the inverse the the natural transformation associated to \mathcal{F} . Thus we get a natural automorphism.

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