

Exam 1 Solutions

Math 2001 Fall 2026

16 September 2026

1. We will use V for the vertices (dots) of the diagram and W for the directed edges (arrows). Some of the problems below apply to the diagram on the accompanying page, and some apply to general diagrams built out of vertices and directed edges. A path is a sequence of arrows such that the head of each arrow in the sequence is the tail of the subsequent arrow. The questions will involve the following predicates, whose meanings are given in English below:

$S(a, x) \equiv$ Arrow a starts at x .

$T(a, x) \equiv$ Arrow a ends at x .

$P(x, y) \equiv$ There is a path from x to y .

- (a) (2 points) **In the figure**, for which values of $x \in V$ and $a \in W$ is the proposition $S(a, x) \wedge T(a, x)$ true? Explain your answer, making sure to explain why the proposition is true for everything in your list *and* why it is not true for anything outside your list.

Solution. This is only true for $x = v_5$ and $a = e_7$. The sentence says that the source and target of the arrow must be the same, and the only arrow with the same source and target is e_7 .

- (b) (2 points) Write a predicate $A(x, y)$ meaning, “There is an arrow from x to y .” This question is **not about the figure** specifically. Your answer should use quantifiers, logical operators, variables from the sets V and W , and the given predicates. A correct answer does not need further justification.

Solution.

$$\exists a \in W, S(a, x) \wedge T(a, y)$$

- (c) (2 points) Write a proposition that means “There is a path from every vertex of V to every other vertex of V .” This question is **not about the figure** specifically. (Hint : be careful about the word “other” !) Your answer should use quantifiers, logical operators, variables from the sets V and W , and the given predicates. A correct answer does not need further justification.

Solution.

$$\forall x \in V, \forall y \in V, (x \neq y) \Rightarrow P(x, y)$$

- (d) (3 points) For which $x, y \in V$ **in the figure** is the following sentence *false*? Explain your answer. A complete answer must not only explain why you included the x and y that you did, but also why you did not include any others.

$$A(x, y) \wedge A(y, x) \implies (x = y)$$

Solution. This statement can only be false if $A(x, y) \wedge A(y, x)$ is true and $x = y$ is false. There are only 3 ways we can have $A(x, y) \wedge A(y, x)$ be true : either $(x, y) = (1, 3)$ or $(x, y) = (3, 1)$ or $(x, y) = (5, 5)$. In the first two cases $x \neq y$, so these two the two where $A(x, y) \wedge A(y, x) \implies (x = y)$ is false.

- (e) (2 points) How can the predicate P be built using the predicate A ? You may find a few pieces of notation useful : $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of natural numbers, and if S is a set and n is a natural number, then S^n is the set of sequences $(s_1, \dots, s_n)$ of length n in S .

Solution.

$$\exists n \in \mathbb{N}, \exists (z_1, \dots, z_{n+1}) \in V^{n+1}, (z_1 = x) \wedge (z_{n+1} = y) \wedge \forall i \in \{1, \dots, n\}, A(z_i, z_{i+1})$$

- (f) (2 points) This question is about the following sentence:

$$\forall x, y, z \in V, P(x, y) \wedge P(y, z) \implies P(x, z)$$

Determine which of the following is true and **explain your answer**.

- (i) True for all graphs.
- (ii) True in the figure but not for all graphs.
- (iii) False in the figure but not for all graphs.
- (iv) False for all graphs.

Solution. This is true for all graphs. If there is a path from x to y and a path from y to z then the two paths can be concatenated to make a path from x to z .

- (g) (3 points) Formulate the logical opposite of the following proposition as a maximally negated logical formula (this means a formula where the negation symbol is only applied to propositions and predicates that cannot be broken into simpler propositions or predicates). Then determine if the proposition is true or false **in the figure**. Your answer should include a brief explanation.

$$\forall x \in V, \exists y \in V, (x \neq y) \wedge A(x, y)$$

Solution. The logical opposite is

$$\exists x \in V, \forall y \in V, (x = y) \vee \neg A(x, y).$$

The original proposition is false because we can choose $x = v_5$. Then for every $y \in V$ either $y = v_5$, in which case $x = y$, or $y \neq v_5$, in which case there is no arrow from v_5 to y , so $\neg A(x, y)$.

2. Label the following English proof with the introduction rule, elimination rule, or other principle being used in each step. Make sure to indicate how the rule is being used by showing which hypotheses allow it to be applied and which conclusion is being drawn. You can write your answer using the proof tree notation. No further justification is required (but explanation could be helpful in assigning partial credit).

Theorem. If $\forall x \in S, P(x) \Rightarrow Q(x) \vee R(x)$ and $\exists x \in S, \neg Q(x)$ then $\exists x \in S, P(x) \Rightarrow R(x)$.

Proof. We are assuming $\forall x \in S, P(x) \Rightarrow Q(x) \vee R(x)$ and that $\exists x \in S, \neg Q(x)$. We need to prove $\exists x \in S, P(x) \Rightarrow R(x)$.

- (a) (2 points) Let t be an element of S such that $\neg Q(t)$ is true.

Solution. Existential elimination :

$$\frac{\exists x \in S, \neg Q(x) \quad [t \in S], [\neg Q(t)] \rightsquigarrow \exists x \in S, P(x) \Rightarrow R(x)}{\exists x \in S, P(x) \Rightarrow R(x)}$$

- (b) (2 points) We need to find an $x \in S$ such that $P(x) \Rightarrow R(x)$ is true. We choose $x = t$. Now we need to show that $P(t) \Rightarrow R(t)$.

Solution. Existential introduction :

$$\frac{t \in S \quad P(t) \Rightarrow R(t)}{\exists x \in S, P(x) \Rightarrow R(x)}$$

- (c) (2 points) Suppose $P(t)$. Now we need to show that $R(t)$ is true.

Solution. Implication introduction :

$$\frac{[P(t)] \rightsquigarrow R(t)}{P(t) \Rightarrow R(t)}$$

(d) (2 points) Since $t \in S$, we know that $P(t) \Rightarrow Q(t) \vee R(t)$.

Solution. Universal elimination :

$$\frac{\forall x \in S, P(x) \Rightarrow Q(x) \vee R(x) \quad t \in S}{P(t) \Rightarrow Q(t) \vee R(t)}$$

(e) (2 points) Therefore we have $Q(t) \vee R(t)$.

Solution. Implication elimination :

$$\frac{P(t) \quad P(t) \Rightarrow Q(t) \vee R(t)}{Q(t) \vee R(t)}$$

(f) (2 points) We argue by cases.

Solution. Disjunction elimination :

$$\frac{Q(t) \vee R(t) \quad [Q(t)] \rightsquigarrow R(t) \quad [R(t)] \rightsquigarrow R(t)}{R(t)}$$

(g) (2 points) In the first case, $Q(t)$ is true and we have a contradiction.

Solution. Explosion :

$$\frac{Q(t) \quad \neg Q(t)}{R(t)}$$

In the second case, $R(t)$ is true then we have what we want.

QED

3. (12 points) Write a proof of the following statement (in English):

Theorem. *If $\forall x \in S, P(x) \Rightarrow Q(x)$ and $\exists x \in S, P(x) \wedge R(x)$ then $\exists x \in S, Q(x) \wedge R(x)$.*

Solution. We assume that $\forall x \in S, P(x) \Rightarrow Q(x)$ and that $\exists x \in S, P(x) \wedge R(x)$. We need to show that $\exists x \in S, Q(x) \wedge R(x)$.

Applying existential elimination to $\exists x \in S, P(x) \wedge R(x)$, we obtain a $t \in S$ such that $P(t) \wedge R(t)$ is true. By conjunction elimination, we get $P(t)$ and $R(t)$ from $P(t) \wedge R(t)$.

Since $t \in S$, we can apply universal elimination, to $\forall x \in S, P(x) \Rightarrow Q(x)$ to get $P(t) \Rightarrow Q(t)$.

Since we know $P(t)$, we can combine it with $P(t) \Rightarrow Q(t)$, by implication elimination, to get $Q(t)$.

Now we have $Q(t)$ and $R(t)$, so we can combine them to get $Q(t) \wedge R(t)$.

Finally, we apply existential introduction : we have $t \in S$ and $Q(t) \wedge R(t)$, so by letting $x = t$ we get $\exists x \in S, Q(x) \wedge R(x)$.