# Pure Mathematics and Physical Reality

# (Continuity and Computability)

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ABSTRACT. We propose a thesis (**OT**) that the class of mathematical models of physical objects and processes is the class consisting of all real numbers and of all objects which can be coded (represented) by individual real numbers in a certain way. This class called here *the continuous realm* contains all finite and countable structures, and all continuous functions, functionals, operators, etc. We discuss several related issues: (1) similarities and differences between the continuous realm and the computable realm (consisting of computable reals, computable functions, computable functionals, etc.), and some reasons for which the latter cannot replace the former in the statement of **OT**; (2) the significance of **OT** for a correct view of the relationship between pure mathematics and its applications; and (3) the necessity of using concept or objects which are not in the continuous realm for the understanding and for efficient teaching of mathematics and sciences.

**1. Introduction.** There are excellent reasons for not reading this paper and they are explained at the end in Section **8**.3, however it is hoped that some readers may find it interesting to peruse the whole thing.

We define and study in mathematics many objects which represent physical objects or processes, and also some objects which have no known direct physical interpretations and are very unlikely to get any in the future. For example most of geometry, analysis and algebra can be viewed as studies of models of physical objets and processes, while well-orderings of the real line, sets of real numbers which are not Lebesgue-measurable or do not have the property of Baire have no such interpretations. Thus we have a natural ontological problem: *Characterize those objects of pure mathematics which could have direct physical interpretations!* We do not have in mind any very exact concept of such interpretations, but we try to achieve something similar to the definitions of convergence of a sequence or of continuity or of computability of a function. These concepts were vague before Cauchy, Heine, Weierstrass, Church, Gödel, Post and Turing gave their definitions. In a sense the relations between their precise definitions and physical reality still remain vague, but we have an extensive know-how concerning their applications. I hope to achieve something similar (even if much less important) with the currently vague concept of direct physical interpretability of mathematical objects. The examples mentioned at the beginning show that in some cases the distinction between interpretable an non-interpretable objects is obvious, but we shall see objects for which the question of interpretability is not so obvious and the precise definition given below may seem arbitrary (but I hope that it is not). Roughly speaking, I claim that all finite, countable and continuous structures, and only them, are interpretable. To illustrate this idea consider first a very elementary object, the characteristic function $f$ of the set of positive reals, that is $f(x) = 0$ for $x \leq 0$ and $f(x) = 1$ for $x > 0$. Thus $f$ is discontinuous at 0. Now, it appears that every physical object or process represented by $f$ can be also represented by the partial function $g$ such that $g(x) = f(x)$ for $x$ different from 0 but $g(0)$ is undefined. Of course g is a continuous function. We generalize this claiming that continuous partial functions always suffice (it is customary in mathematics, to omitt the word partial in this context). Thus we propose the following *Ontological Thesis.*

**OT.** *The class of those mathematical objects which have a potential for direct physical interpretations consists precisely of the real numbers and all objects which can be coded in a certain way by means of a real number, such as finite or countable structures, continuous real functions, functionals, operators, etc.*

That coding with real numbers will be fully defined in Sections **2** and **3**. We will see that it is somewhat artificial, but no more contrived than the usual coding (representation) of all mathematical structures by means of sets. We will clarify and discuss **OT** in Sections **3**, **4** and **5**.

Notice that our answer to the initial question is not in the form of an axiomatic theory but only in the form of a well defined class (or set), the class defined in **OT** which we will call *the realm of continuity.* In fact the realm of continuity admits some natural operations, so that it can be viewed as a model **M** in the sense of model theory (see Section **3**). But I do not know any interesting first-order axiomatization of this model **M**. Roughly speaking, **M** will be similar to such familiar concepts as a specific Hilbert space, say $L^2[0,1]$ with its functionals and operators. (And there is no interesting first-order description of $L^2[0,1]$ other than the universal interpretation of mathematics in set theory.)

Notice that **OT** has a fundamental significance for the ontology of mathematics and as such it is connected to many issues. I will discuss briefly several of these connections. Some of them will be purely philosophical, some more mathematical (discussing the mathematical structures suggested by our idea), and some of a practical or political nature.

We remark that **OT** is different from the well known idea of Edward Fredkin (adopted also by Stephen Wolfram) to represent physical reality as a giant cellular automaton. His view is more restrictive since it assumes that physical reality is countable (and locally finite). Most models of mathematical physics and other sciences use continua. I feel that these richer (uncountable) structures of the realm of continuity are strongly suggested by physical reality, and that, in the present state of science, they are very natural.

However, is it possible to manage with a countable realm? Do physicists really use continua? *The answer is not clear* and we examine this problem in Section **6** of this paper. Instead of accepting all real numbers we can restrict our realm to computable real numbers and objects coded by means of them. We call it *the realm of computability.* Of course this realm is countable. Let **OT'** be the modification of **OT**, where the realm of continuity is replaced by the realm of computability. (Recall that all algebraic numbers, and all constants of analysis, e.g. $\pi$ and $e$, are computable.) But we will propose some reasons to believe that there are non-computable reals with a direct physical significance. For example we will define a *computable* function $g$ whose domain consists of all non-zero real numbers, which can be extended to a continuous function $f$ whose domain includes 0, such that $f(0)$ is the maximum value of $f$, but $f(0)$ is not a computable number. Thus we can argue that $f(0)$ has a direct physical interpretation. Hence at present **OT** seems to be more natural than **OT'**. (But a reader inclined to a minimalism sharper than mine but less radical than Fredkin's, may still prefer **OT'**. For his convenience we will present in Section **6** a discussion of the realm of computability and point out some of its surprising properties and related unsolved problems.)

The reader may feel that such sweeping statements as **OT** do not belong to science and perhaps not even to knowledge. Of course I disagree and I believe that getting rid of such general claims or convictions would leave us with a disjointed view of reality. General remarks about epistemology, examples of other "sweeping statements", and the reasons which make them convincing are discussed in Section **5**.

The third topic, discussed in Section **7**, are the practical consequences of our theory. Indeed **OT** forces us to reexamine the value of pure mathematics, and to define in a clear way the goals of mathematical and scientific education. We feel that this analysis shows the necessity of significant changes of current educational philosophies. In particular it suggests that many textbooks are missing the point and stupefy the students. Our main point is that **OT** illustrates some of the differences between understanding and being able to cope, or between knowledge and know-how, while those books confuse those things, and try to impart knowledge through know-how. Indeed, **OT** shows the necessity of using mathematical objects which are neither algorithms nor directly interpretable in order to understand algorithms and continuous functions.

Finally, in Section **8** we will repeat some of our general conclusions and add some critical remarks.

[In view of my separation of the set of physically interpretable objects within the larger class of all mathematical objects, I ought to explain my ontological position toward the latter. In fact I have already tried to explain this in the papers [$M_1$, $M_2$], so let me give here only a brief summary. I view the objects of pure mathematics (as long as they are not intended to represent physical objects or processes) as physical phenomena which occur in human brains. Of course I do not attribute to them any descriptive role, and they do not represent anything independent of humanity. On the other hand the laws of physics and the structure of human brains (and computers) are such that pure mathematics is not arbitrary and it is more reproducible than most other thoughts or mental constructions. In fact it crystallizes in our brains (when these are exposed to the appropriate sensory inputs) into very precise forms described by classical logic and set theory. Hence, however I do not believe in the existence of any Platonic reality, I reject formalism, i.e., the view that mathematics is an arbitrary game of symbols. I believe that mathematics is an exploration of such theories which natural intelligence (natural logic) can create, and they happen to be far from arbitrary. I believe also in a certain unicity of mathematics which stems from a canonical set theory, see [DM] and [$M_7$].[1] Notice that our philosophy differs from that of L. Kronecker who accepted the reality of integers but not the reality of real numbers, while it agrees with the philosophy of the French geometer R. Thom, who expressed the view that the continuum of real numbers has a fundamental and irreducible nature.]

I am indebted to Jerome Malitz for many long conversations on the topic of computable real functions which influenced Section **6**, and to Andrzej Schinzel and the referee for suggesting many improvements.

**2. Preliminaries on coding, physical significance, and computability.** In mathematics functions are almost always understood to be sets of ordered pairs, and the notation $f : X \to Y$ means that $f \subseteq X \times Y$ and that for every $x \in X$ there exists a unique $y \in Y$ such that $(x, y) \in f$ (this $y$ is denoted by $f(x)$). But in this paper functions are the primitive concepts and the sets $X$ and $f[X] = \{f(x) : x \in X\}$ are defined in terms of $f$. Thus, as we shall see, $f$ may have intended physical meanings while those sets $X, Y$, and $f[X]$ have no such meanings. (However, in this paper, the elements of $X$ and $Y$ will always have potential physical meanings.)

In spite of the above we will define our realm of meaningful functions by means of some meaningless sets. Probably, using axiomatic category theory, it would be possible to avoid this. But I feel that such an approach would be quite cumbersome. In fact I believe that our brains evolved in such a way as to manipulate sets and objects defined from sets with relative ease, but they are not as good with concepts of category theory. For example consider the function

$$f(x) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{2^{m+n}} \sin \frac{1}{|x| - \frac{m}{n}}.$$

It is easy to program a computer such that, given the decimal development of an irrational number $x$, it will compute $f(x)$ with arbitrary desired accuracy. (For other examples see Section **6**; $f$ is computable in the sense defined there.) Thus $f$ can represent the law of functioning of a real mechanism. But we cannot imagine $f$ without imagining its domain, which is the set of irrational numbers. Indeed notice that the function $f$ is not defined at any rational number $x$ and has a positive oscillation at each of them. Hence the set of irrational numbers or the property of irrationality is needed to explain $f$ to a human being. But this set has no direct physical reality since it and its complement are everywhere dense in the real line. Thus $f$ has a direct physical meaning, but its set-theoretic analysis does not! Nevertheless this analysis is essential for our understanding of $f$. Thus we feel free to introduce the following auxiliary concepts and notations, some of which will be without any physical interpretations.

$\mathbb{N}$ denotes the set of positive integers, $\mathbb{Q}$ the set of all real rational numbers and $\mathbb{R}$ the set of all real numbers. Let $q_1, q_2, \ldots$ be a fixed recursive enumeration of $\mathbb{Q}$ and $Q_1, Q_2, \ldots$ a fixed recursive enumeration of all quadruples of the form $(p, m, q, n)$, where $p, q \in \mathbb{Q}$ and $m, n \in \mathbb{N}$. Every irrational real number $y$ of the open interval (0, 1) has a unique development into an infinite continued fraction

$$y = 1/(n_1 + 1/(n_2 + 1(n_3 + \cdots))), \tag{1}$$

where $n_i \in \mathbb{N}$, and hence $y$ codes a sequence $n_1, n_2, \ldots$. Thus $y$ can be interpreted as a code of the sequence of the corresponding quadruples $Q_{n_1}, Q_{n_2}, \ldots$. If $Q$ is in this sequence we will say that $Q$ *is in* $y$.

A real number $t \in \mathbb{R}$ is called *computable* iff there exists a recursive function $\varphi : \mathbb{N} \to \mathbb{Q}$ such that

$$|t - \varphi(n)| < \frac{1}{n}$$

for all $n \in \mathbb{N}$. One can check that $t$ is computable iff the decimal development of $t$ is recursive.

$\mathbb{T}$ denotes the set of computable real numbers. As well known $\mathbb{T}$ is a real closed field and $\mathbb{T}$ contains all the constants of mathematical analysis.

Notice that all rational numbers are computable and that an irrational number $y$ in $(0,\ 1)$ is computable iff the sequence $n_1, n_2, \dots$ defined by (1) is recursive.

A function $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}$ is called a *real function.*

**3. The ontological thesis.** From a model theoretic point of view, the main structure considered in this paper is the following model:

$$\mathbf{M} = \langle \mathbb{R},\ U,\ r \rangle_{r \in \mathbb{R}},$$

where $U$ is a binary partial function which is defined as follows. The equation

$$U(x, y) = z$$

holds iff the triple $x, y, z \in \mathbb{R}$ has the following two properties:

(i) $y$ is an irrational number in the interval $(0,\ 1)$ and for every $n$ there exists a quadruple $(p, m, q, n)$ in $y$ such that

$$|x - p| < \frac{1}{m}; \tag{1}$$

(ii) for every $(p, m, q, n)$ in $y$, if $x$ satisfies (1) then

$$|z - q| < \frac{1}{n}.$$

It is clear from this definition that $U$ is continuous (and even computable, see Section **6**). Indeed, in order to evaluate $U(x, y)$ at a point $(x, y)$ in the domain of $U$ with accuracy $\frac{1}{n}$ it is enough to evaluate $x$ and $y$ with sufficient accuracy to find $(p, m, q, n)$ in $y$ such that (1) holds. Then $q$ is the desired approximation of $U(x, y)$.

Still $U$ has the following universality property which is easy to prove.

*If $f : D \to \mathbb{R}$ is a continuous real function, then there exists a $y_0$ such that*

$$f(x) = U(x, y_0),$$

*for all $x \in D$.*

As we will show in the next section this proposition is true in the stronger sense that if $D$ is a $\mathbf{G}_\delta$ (i.e., a countable intersection of open sets) then $y_0$ can be such that the domain of $U(\cdot, y_0)$ is exactly $D$. We will show also that the domain $D_U$ of $U$ is a subset of the plane $\mathbb{R}^2$ of the class $\mathbf{G}_\delta$.

By the above universality of $U$ we can interpret every real function, functional, operator, etc., which is continuous in an appropriate sense, as a real number (or *vice versa* some real numbers can be interpreted as such objects). Indeed: Let $\mathcal{T}$ (called the set of *types*) be the least set such that $1 \in \mathcal{T}$ and if $\alpha, \beta \in \mathcal{T}$, then the ordered pair $(\alpha, \beta) \in \mathcal{T}$. We intend 1 to be the type of real numbers and $(\alpha, \beta)$ to be the type of functions from objects of type $\alpha$ to objects of type $\beta$. Namely, we define recursively

$$x^1 = x,$$

for all $x \in \mathbb{R}$, and

$$y^{(\alpha,\beta)}(x^\alpha) = (U(x, y))^\beta$$

for all $x, y \in \mathbb{R}$ such that, for all $z \in \mathbb{R}$, if $z^\alpha = x^\alpha$ then $(U(z, y))^\beta = (U(x, y))^\beta$. Of course for many pairs $\alpha \in \mathcal{T}$ and $x \in \mathbb{R}$ the symbol $x^\alpha$ represents merely the empty function. We will not develop here any theory of these representations; this would lead us too far afield.

[By the above remarks the first-order theory of the model **M** is very rich. It contains the theory of real-closed fields and even arithmetic. Hence, of course, it is not effectively axiomatisable. This theory will not be explicitely developed in this paper.]

Of course every finite or infinite sequence of reals can be represented by a function from $\mathbb{N}$ to $\mathbb{R}$. Thus every such sequence is of the form $x^{(1,1)}$. Likewise every finite or countable discrete structure can be represented in such a form. For this reason, we do not have to talk explicitly about the $n$-dimensional space $\mathbb{R}^n$ or about the space of infinite sequences of reals, or finite or countable models. [And the function $U$ itself is an object of type ((1,1),1).]

Now we can state our main thesis **OT**:

**Ontological Thesis.** *An object $A$ of pure mathematics has a potential for a direct physical interpretations iff $A$ can be presented in the form $x^\alpha$, where $x \in \mathbb{R}$ and $\alpha \in \mathcal{T}$.*

The class of objects of the form $x^\alpha$ will be called *the realm of continuity.* The main argument supporting **OT** will be given in Section **5**. But let me address immediately five possible concerns.

First, the term "presented" is not a priori clear. For example I claim that any finite of countable structure and every continuous mapping from a Polish space to a Polish space can be presented in that way. Several examples will follow and should clarify this isue. **OT** is similar to the familiar thesis going back to Cantor, Dedekind and Zermelo that every mathematical object can be presented as a set (or a class of sets). **OT** claims that the realm of physically interpretable mathematical objects is interpretable in the model **M** in the sense in which the realm of all mathematical objects is interpretable in the model consisting of the universe of sets and the membership relation.

Second, **OT** is motivated by the dichotomy between thoughts (or imagined forms) having an intended physical meaning and those without such intentions. (We do not forget that this intention may be lacking and still the thought may be interesting and useful.) Some readers may fail to appreciate the significance of this dichotomy, or find it uninteresting (this was the case of one referee of this paper). For them this paper will make little sense, and we can only refer them back to the examples mentioned at the beginning of this paper.

Third, the class of concepts which can have direct physical interpretations may appear so vague that **OT** can be easily contested. In fact nature does not let us measure its constants very well; only computable reals can (in principle) be evaluated with arbitrary accuracy. Thus **OT** rests upon the assertion (which is true in a certain measure and is extremly important in the present state of scientific knowledge) that every real number has potential direct physical interpretations. Once this is accepted **OT** follows since then (a) if nature gives us a real number $x$ with certain properties then $x^\alpha$ does represent a possible physical computing mechanism (even though in practice such mechanisms are not given in the form $x^\alpha$, and the $\alpha$'s which appear in practice are not too complicated), and (b) there appears to be enough real numbers to represent all physical quantities.

Fourth, there could be too many real numbers. This problem will be discussed in Section **6** where we consider the possibility of replacing the real line $\mathbb{R}$ by the set of computable reals $\mathbb{T}$.

Fifth, it could appear that there are not enough objects of the form $x^\alpha$ in order to represent all models of physical objects which are used in science. But I claim that, all these models can be represented in that form [and this representation is no more unvieldy than, say, set-theoretic representations of rational numbers (by means of equivalence classes of ordered pairs), of real numbers (by means of certain sets of rationals (see below *Remark 1*), or by equivalence classes of Cauchy sequences of rationals), or of functions (by means of sets of ordered pairs)]. Some examples of representations in the form $x^\alpha$, in particular finite and countable models, were mentioned prior to the statement of **OT**. Let me give five additional ones, and hope that they will convince the reader that our realm of continuity is rich enough. More examples will be given in later sections.

1. Certain sets are represented as domains or as sets of values of continuous functions. [But those are not representations of the form $x^\alpha$, and hence, in general, sets are not objects of the continuous realm. For example, the distinction between two intervals with the same ends, one closed and the other open, is not physically meaningful (in spite of the big difference between the classes of continuous functions with such intervals as their domains). We return to this issue in Section **4**, *Remarks 4, 5*.]

2. A topological (or differentiable) manifold can be presented in the form $x^\alpha$, since it can be represented by a sequence of projection functions (i.e., coordinates on charts) and a system of continuous (differentiable) functions relating those charts.

3. The equivalence class of a measurable function $f : [0,1] \to \mathbb{R}$ defined up to a set of measure 0 can be also presented in that way. First, as mentioned above, a sequence of reals can be interpreted as a function from $\mathbb{N}$ to $\mathbb{R}$. Then using the isomorphism between the Hilbert spaces $L^2[0,1]$ and $l^2$ we can represent equivalence classes of square-integrable function by single square-summable sequences of reals. Then $f$ can be represented by the sequence of functions obtained by truncating the range of $f$ to the intervals $[-n, n]$, since these truncations belong to $L^2[0,1]$. [There are other ways to do the same. E.g. one can represent an absolute square summable

sequence $(a_0, a_1, a_2, ...)$ of real (or complex) numbers by the entire function $g(z) = \Sigma a_n z^n/n!$. (Representation of this kind are used in various calculations.)]

4. Measures and distributions in the Sobolev-Schwartz sense can be interpreted as continuous functionals over appropriate spaces of continuous functions.

5. Unpredictability or unstability of physical processes does not require objects outside of the continuous realm. We observe that dynamical systems which are unstable, that is the trajectories of the system display an extreme sensitivity to the initial conditions, can nevertheless be continuous. That is, the trajectories can depend in a continuous way on the initial positions (when the topology in the space of trajectories is that of almost uniform convergence, see e.g. [AP]). Examples of computable dynamical systems given by simple formulas but unstable at every initial position were given by I. Kan [K]. (Such must be in particular dynamical systems which model the trains of thoughts and emotions in the brain. Indeed it is natural to interpret free will as unpredictability, i.e., unstability of the system, see [$\mathrm{M}_3$, $\mathrm{M}_4$, MS].)

**Problem 1.** Are all the domains of the functions, functionals, operators, etc. of the form $x^\alpha$, viewed as sets of real number codes of their elements, of the class $\mathbf{G}_\delta$ in the topology of $\mathbb{R}$? (A Polish space is a topological space which admitts a complete separable metrization. As well known (see e.g. [Ku]) a subset of a Polish space is a $\mathbf{G}_\delta$ iff it is a Polish space. A positive solution of Probem 0 would show an intersting regularity of the realm of continuity.)

Again, this presumed regularity of domains and continuity of the functions of the form $x^\alpha$ may induce doubts about **OT**. To disperse these doubts let me add the following. The Axiom of Choice simplifies and strengthens set theory in a beautiful way, but it creates many objects unrelated to physical reality. However, in all applications of mathematics a weaker principle, the Axiom of Dependent Choices suffices. Moreover, we have a model $L[\mathbb{R}]$ (the class of sets constructible in the sense of Gödel with real parameters) which is certainly comprehensive enough to interpret is a natural way all of applicable mathematics. And, $L[\mathbb{R}]$ satisfies the axioms of set theory with the Axiom of Choice replaced by the Axiom of Dependent Choices, plus a very strong additional axiom, the Axiom of Determinacy[2]. Then (see e.g. [M$_5$]) those axioms imply the following theorem:

*For every function $f: M \to \mathbb{R}$, where $M$ is a Polish space, there exists a set $D \subseteq M$ which is a $\mathbf{G}_\delta$ and is everywhere dense in $M$ such that $f$ restricted to $D$, i.e., $f \cap (D \times \mathbb{R})$, is a continuous function, and, if $M = \mathbb{R}^n$, there exists a sequence of continuous functions $f_1, f_2, \dots$ which are restrictions of $f$ to closed sets $F_1, F_2, \dots,$ such that the complement of the union of those sets $F_n$ is of Lebesgue measure $0$.*

Since this sentence is true in $L[\mathbb{R}]$, and sequences of continuous functions are in the continuous realm, this suggests that there are enough functions of the form $x^\alpha$ for all applications of mathematics.

*Remark 1.* We should stress again that our definition of the universe of interpretable objects of mathematics is based on the universe $\mathbb{R}$ of real numbers. Let us recall a version of the well known definition of $\mathbb{R}$ due to Eudoxus and Dedekind. $r \in \mathbb{R}$ iff $r$ has the following four properties: $r \subseteq \mathbb{Q}$; $\emptyset \neq r \neq \mathbb{Q}$; if $p \in \mathbb{Q}$, $q \in r$ and $p < q$, then $p \in r$; and there is no maximal number in $r$. (In other words $r$ is a left open ray in $\mathbb{Q}$.) Thus it follows that for every set $X \subseteq \mathbb{R}$, if $X$ is bounded from above, i.e., $\bigcup(X) \neq \mathbb{Q}$, then $\bigcup(X) \in \mathbb{R}$. (In other words $\mathbb{R}$ is connected.) It would be erroneous to infer that this definition yields a fully saturated real line. In fact the meaning of this definition depends on the axioms of the ambient set theory: the stronger the set theory the more left open rays in $\mathbb{Q}$ can be proved to exist. And, Gödel's Second Incompleteness Theorem implies that this process of saturation of $\mathbb{R}$ is open ended.

**4. Technical remarks and theorems.**

*Remark 2.* I chose $\mathbb{R}$ for the universe of the basic model **M** in order to get simple definitions of all the objects which are important in applications of analysis and geometry. (A recursion theorist may prefer the Baire space $\mathbb{N}^{\mathbb{N}}$, and a set theorist may prefer the Cantor set $\{0,1\}^{\mathbb{N}}$. But in each case the representations of real functions would have been more complicated.)

*Remark 3.* It may be instructive to show how to define $y$ such that $U(\cdot, y)$ becomes the constant function 1 over the closed interval $[0, \infty)$ or the open interval $(0, \infty)$. (Theorem 2 below will generalize these constructions.)

In the first case let $y$ code of the set of all quadruples $(p, m, q, n)$ such that $q = 1$, $m = n$ and $p \geq 0$. Then for every $n$ and $x \in [0, \infty)$, the computation of $U(x, y)$ yields 1 (since there are quadruples $(p, n, 1, n)$ in $y$ such that $|x - p| < \frac{1}{n}$). But if $x < 0$ and $n$ is such that $x + \frac{1}{n} < 0$, the search for a quadruple $(p, n, 1, n)$ in $y$ such that $|x - p| < \frac{1}{n}$ never ends, so $x$ is not in the domain of $U(\cdot, y)$.

In the second case let $y$ code the set of all quadruples $(p, m, 1, n)$, where $p \geq \frac{1}{m}$. Then if $x \in (0, \infty)$, for every $n$ we can find a quadruple $(p, m, 1, n)$ in $y$ such that $|x - p| < \frac{1}{m}$ (since we have such quadruples with $\frac{1}{m} < x$). But if $x \leq 0$ there is no pair $(p, m)$ such that $|x - p| < \frac{1}{m}$ and $p \geq \frac{1}{m}$ in the quadruples of $y$. Hence the search for the required quadruple never ends and so $x$ is not in the domain of $U(\cdot, y)$.

*Remark 4.* Remark 3 and Theorem 2 below show that the distinction between intervals with the same ends but either open or closed, or half-closed, and even between any two different analytic sets of real numbers (i.e., continuous images of the set of irrational numbers), has an indirect physical significance in as much as it corresponds to distinct behavior of certain computing devices which can exist in nature. Still, assuming **OT**, no sets of reals are directly interpretable in the physical world. Sets are only imagined objects very helpful for the human understanding and the mathematical analysis of real objects. In particular as mentioned earlier, it seems that the best way to imagine a function is to view it as a set of ordered pairs and to imagine (to some extent) its domain and range.

*Remark 5.* Sets which are not analytic (recall that every Borel set is analytic but the complements of some analytic sets are not analytic) seem more remote from physical reality. This corroborates the view of L. E. J. Brouwer and of N. Lusin about the abstract character of complementation of sets and of negation. [However, this does not go far enough to give a sufficient motivation for intuitionistic mathematics.[3]]

Now we will prove the theorems announced in Section **3**.

**Theorem 1.** *The domain $D_U$ of $U$ is a $\mathbf{G}_\delta$ subset of the plane $\mathbb{R}^2$.*

*Proof.* By the definition of $U$, $(x, y) \in D_U$ iff

$$0 < y < 1 \;\&\; y \notin \mathbb{Q} \;\&\; \forall\, n(\exists\,(p, m, q, n) \text{ in } y)\left[|x - p| < \frac{1}{m}\right], \tag{1}$$

and

$$\exists\, z(\forall\,(p, m, q, n) \text{ in } y)\left[|x - p| < \frac{1}{m} \to |z - q| < \frac{1}{n}\right]. \tag{2}$$

It is clear that (1) defines a $\mathbf{G}_\delta$ set of pairs $(x, y)$ in $\mathbb{R}^2$. To prove the same about (2) we show first that (2) is equivalent to the following condition:

$$(\forall\, k \in \mathbb{N})(\forall\,(p_1, m_1, q_1, n_1), \ldots, (p_k, m_k, q_k, n_k) \in (\mathbb{Q} \times \mathbb{N})^2) \tag{3}$$

$$\left[\forall\, i \le k \left[(p_i, m_i, q_i, n_i) \text{ is in } y \text{ and } |x - p_i| < \frac{1}{m_i}\right]\right.$$

$$\to (\exists\,(p, m, q, n) \text{ in } y)\left[|x - p| < \frac{1}{m} \;\&\; \left[q - \frac{1}{n}, q + \frac{1}{n}\right]\right.$$

$$\subseteq \bigcap_{i=1}^{k} \left(q_i - \frac{1}{n_i}, q_i + \frac{1}{n_i}\right)\Bigg]\Bigg],$$

here $[a,b]$ and $(a,b)$ denote the closed and open interval from $a$ to $b$ respectively. It is clear that $(2) \to (3)$. To prove the converse consider the intersection

$$A_{x,y} = \bigcap \left[q - \frac{1}{n}, q + \frac{1}{n}\right],$$

where the pairs $(q,n)$ are picked from all $(p,m,q,n)$ in $y$ satisfying $|x-p| < \frac{1}{m}$. Then, by the local compactness of $\mathbb{R}$, (3) implies that $A_{x,y}$ is not empty and contains exactly one point. Let $z$ be that point and it is easy to check that $z$ is the number whose existence is claimed in (2). Thus $(3) \to (2)$ is also proved.

Now, it is clear that the set of pairs $(x,y)$ defined by (3) is also a $\mathbf{G}_\delta$. Hence $D_U$ being the intersection of the sets defined by (1) and (3) is a $\mathbf{G}_\delta$. $\square$

It is easy to extend every continuous real function $f$ to such a continuous function $f^*$ whose domain is everywhere dense in $\mathbb{R}$ and such that this extension is maximal. Theorem 1 and the next Theorem 2 imply that the domain of such a maximal $f^*$ is always a $\mathbf{G}_\delta$.

**Theorem 2** (The universality of $U$)**.** *A real function $f : D \to \mathbb{R}$ is continuous and $D \in \mathbf{G}_\delta$ iff there exists a $y_0$ such that $f$ is of the form $U(\cdot, y_0)$.*

*Proof.* By Theorem 1 the implication from right to left is true. To prove the converse implication let $D = \bigcap_{i=1}^{\infty} A_i$, where $A_i$ are open sets such that

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots \tag{4}$$

Let $y_0$ code the set of quadruples $(p,m,q,n)$ such that

$$(\exists x \in D)\left[|x-p| < \frac{1}{m} \;\&\; m \geq n \;\&\; \left(p - \frac{1}{m}, p + \frac{1}{m}\right) \subseteq A_n \;\&\; |f(x) - q| < \frac{1}{n}\right].$$

Then, by (4) and the definition of $U$, it is clear that $f = U(\cdot, y_0)$. $\square$

**5. Argument supporting the Ontological Thesis.** Some arguments were already given in Section **3**, and Theorem 2 completed one of them. Formally speaking **OT** is an equivalence between two concepts (one rather vague and the other more precise). Now, we will give an argument of a much more general kind, an argument which applies to many other statements of a similar type, any equivalence of concepts, equality between sets or classes, and even between quantities. Notice that such relations simplify and add power to our knowledge. But, for the relation to be non-trivial, the two sides of the equivalence must suggest sufficiently different points of view or different methods of classification or of measurement. Let me list three other examples and stress their common nature.

**CT**. (Church's Thesis) A function $f : \mathbb{N} \to \mathbb{N}$ is computable iff there exists a Turing machine for evaluating $f$.

[When A. Church proposed this thesis he expressed it by means of different technical concepts and his claim was far from obvious. But after Turing defined his machines, and proved that all the former definitions were equivalent, **CT** became obvious. (Still some people were contesting **CT**. I believe that they were diehards uwilling to appreciate Turing's achievement.) Of course there is a salient analogy between **OT** and **CT**.]

**IT**. (The Identity Thesis) The mind is the living brain.

[The living brain has some mathematical models, e.g. a functioning universal Turing machine or certain adaptive dynamical systems. These models are still very abstract (see [$M_2$, $M_3$, $M_4$, MS]), but Turing's analogies

(Turing machine)/(input-output tape) = (organism)/(its environment),

(universal Turing machine)/(its program) = (brain)/(its knowledge),

demistify the situation to a great extent. Many people are contesting the thesis **IT**, but it is difficult to understand their complicated arguments. I think that these critics have failed to form a sufficiently abstract concept for which both names *mind* and *living brain* appear adequate and to realise that this concept is of fundamental importance for psychology, brain science, and medicine. (We remark that they may have ulterior motives for resisting this idea: They do not appreciate the strength of the constructive drives in human nature and the possibility to build over them. In consequence they think that, if it were widely accepted, **IT** would have a disastrous effect upon the human society.) We note that, like **IT**, any interesting equivalence of two notions can be rejected if one refuses to form an appropriate abstract concept and to realize its importance.]

**DT**. (Darwin's Thesis) The mechanism which constructed the living species is the mechanism of evolution by natural selection.

[Comments very similar to those which I made about **IT** apply also to **DT**.]

Today **OT** and **CT**, unlike their analogs **IT** and **DT**, do not contradict any living prejudice or any accepted theory. And it seems to me that these analogies are so strong that if somebody can reject **IT** or **DT** he cannot really accept **CT** or **OT**. In fact rejections of the former suggest a kind of general scepticism about any generalization or inductive inferences. It signifies an incomplete acceptance of any science in favor of a kind of scepticism. In other words, an escape (if only a partial escape) from any conviction or knowledge into the realm of know-how in lieu of knowledge. [In order to avoid the accusation of naiveté, let me add that our convictions, or degrees of convictions, are essential factors of our ability to use our knowledge and to communicate it to others.]

I think that such an escape violates human nature. For it is natural to believe inductive generalizations, and to accept any simplification of one's knowledge which appears to be true. And, as mentioned above, every equivalence (or equation) simplifies knowledge. In practice, to be convinced about an identity of two sufficiently clear concepts $A$ and $B$ it suffices to observe the possibility of this identity! In other words it is enough to check that we (and presumably nobody else) can find a bona fide counterexample to $A = B$. Thus, as soon as this lack of counterexamples is observed, an intelligent person who thinks that $A = B$ is interesting enough, will believe that $A = B$ without any action of his will (and he is incapable of loosing this belief unless he forgets it or he learns a clear counterexample).

[But one has to remember that normal convictions are not stuborn convictions. First, the necessary vagueness of the relation of truth or physical significance must be taken into account. Second, every generalization must be regarded as a gamble, which may fail. This wisdom does not mean that we accept ideas of postmodernists who think that the nature of science is the same as the nature of myths or religious beliefs. They make the error of identifying science with a system of "stubborn convictions", discounting the importance of such concepts as *truth in some intended measure* or *approximation*. They make also the error of viewing every new law as a refutation rather than a refinement of former knowledge (if only a realization of its limits). In this way they are missing the cumulative character of knowledge which is so obvious to scientists. In fact their writings often suggest that they have a chip on the shoulder on the subject of science and mathematics; for deeper criticism see [SB, W] and references therein.]

Since there is always some risk in the acceptance of any inductive inference, we can ask what makes us believe such generalizations. In particular, why **OT** is convincing? We propose a very general answer: *As soon as evolution gave us the ability to imagine complicated models and descriptions of reality, it had to give us also an instinctive preference for the simplest of them (which appear to be true in some intended measure), and a mechanism $M$ which turns them into convictions without any action of will.* [This implies that scepticism is unnatural since it is countered by the action of $M$. And, I have often the impression that, people declaring general scepticism do not express an opinion but only adopt a pose. This implies also that if we are not aware of certain facts then $M$ can induce involuntary and false beliefs. Such convictions are honest, but when beliefs are supported by will, i.e., by active avoidance of information, and then are communicated to others as truths, they are malicious.]

The utility of such an instinctive preference for simplicity and of such a mechanism $M$ inducing convictions is clear since the simpler theories are *easier to remember*, *easier to communicate from scratch*, and *more generally applicable*. [Let me add (in response to those philosophers who have expressed doubts about the significance of the concept of simplicity in epistemology) that in practice the two qualities *easier to remember* and *easier to communicate* are very important. Human intelligence instinctively evaluates these things (in a quantitative way), and they constitute our natural measures of simplicity or complexity of descriptions. Thus we accept it as a fact (and regard it as a great mystery) that generalizations and mathematical models so often yield predictions which are sufficiently accurate.]

Thus, since every intelligent person $P$ has a well functioning mechanism $M$, all identities between concepts (like **OT**) and all equations which rest merely upon the lack of any known counterexamples are convincing to $P$. On the other hand, as mentioned above, acceptance of such identities demands the formation of abstract concepts which to some extent are new, and this is perhaps most important role of these identities.

**6. The realm of computability.** A function, functional, operator, etc. will be called *computable* iff it is of the form $t^\alpha$ where $t \in \mathbb{T}$ and $\alpha \in \mathcal{T}$, and the set of objects representable in this form will be called *the realm of computability.* As mentioned in the Introduction, we can obtain a new thesis **OT'** substituting in **OT**

the realm of continuity by the realm of computability. But now we will list some reasons to think that **OT'** is not as natural as **OT**.

Our first reason is that there is no evidence of any kind that all physical constants (such as the speed of light, the mass of a proton, Planck's constant, etc.), are computable. But this is a weak argument since we know only approximate values of these constants and hence for some unknown reason they could be computable. Of course since there are only countably many computable reals the computability of these constant may seem improbable. But this is not a good reason since even set theory has countable models (the so called Skolem paradox), and, since set theory interprets all of mathematics, physical reality can be represented just as well by countable models.

But we have reasons which may be a little stronger for the need of non-computable reals in such models. Namely the realm of computability is not closed under some operations which appear to make physical sense, and the theory of the realm of continuity seems simpler than that of the realm of computability. One purpose of this section is to describe more fully this situation. Theorems 3 and 4 suggest that the computable realm could be large enough, but Example 1, *Remarks 9* and *10*, Example 3, and *Remark 13* suggest that it is too small. We will also point out some unsolved Problems 2 – 6.

We can also define for realm of computability models similar to $\mathbf{M}$, namely $\langle \mathbb{T}, U_0, t\rangle_{t\in\mathbb{T}}$ where $U_0$ is the restriction of $U$ to $\mathbb{T}^2$, or $\langle \mathbb{N}, u, n\rangle_{n\in\mathbb{N}}$, where $u : \mathbb{N}^2 \to \mathbb{N}$ is defined by a universal Turing machine. But, like in the case of $\mathbf{M}$, the first-order theories of these models will not be developed here.

For other studies of computable functionals and operators, see [PR, Po] and references therein. Our basic definitions are simpler and more general, and the main difference between [PR] and this paper is that we do not restrict the domains of computable functions to be intervals in $\mathbb{R}$. Instead we consider the domain of $t^\alpha$ which is defined by $t$ (and is a subset of $\mathbb{R}$ and not of $\mathbb{T}$).

We will focus upon computable functions, i.e., the objects $t^{(1,1)}$. We have the following simple definition of these functions:

A function $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}$, is computable iff there exists a Turing machine $M$ such that

$$f(x) = y$$

iff whenever we start $M$ on an infinite tape on which the full decimal developments of the real $x \in \mathbb{R}$ and of an integer $n \in \mathbb{N}$ are printed, then, after a finite number of steps, $M$ will stop and yield an output $q \in \mathbb{Q}$ such that

$$|y - q| < \frac{1}{n}.$$

It is easy to check that, this definition is equivalent to $f = t^{(1,1)}$, for some $t \in \mathbb{T}$. Notice two facts which will be used later: if $f : D \to \mathbb{R}$ is computable then $f[D \cap \mathbb{T}] \subseteq \mathbb{T}$; and, the composition of computable functions is again computable.

Of course all computable real functions are continuous on their domains but those domains can be complicated $\mathbf{G}_\delta$ sets. Two examples show such complications:

$$f_1(x) = \sum_{q_n < x} \frac{1}{2^n}, \qquad f_2(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sin \frac{1}{x - q_n},$$

where $q_1, q_2, \ldots$ is a recursive enumeration of $\mathbb{Q}$, and $x \in \mathbb{R} - \mathbb{Q}$. It is easy to check that both $f_1$ and $f_2$ are computable, $f_1$ is strictly increasing, and they have no continuous extensions to any point of $\mathbb{Q}$.

*Remark 8.* Given Turing machines defining two computable reals $t_0$ and $t_1$, the relation $t_0 = t_1$ is undecidable. Likewise for any $\alpha \in \mathcal{T}$, the relation $t_0^\alpha = t_1^\alpha$ is undecidable. Hence our form $t^\alpha$ of individual objects of the computable universe (or more generally the form $x^\alpha$ for $x \in \mathbb{R}$) a priori does not clarify those objects in any interesting way. It serves only to define the totality of computable (or physically interpretable) objects.

Whenever we write $f(x)$ we assume that $x$ is in the domain of $f$.

**Theorem 3.** (The Computable Intermediate Value Theorem). *If $f : D \to \mathbb{R}$ is computable, $\mathbb{T} \subseteq D$, $f(a) < f(b)$, then $f$ assumes all computable values of the interval $[f(a), f(b)]$, in other words,*

$$[f(a), f(b)] \cap \mathbb{T} \subseteq f[[a, b] \cap \mathbb{T}].$$

*Proof.* (The idea of this proof if known, see [PR], p. 41].) We will assume without loss of generality that $a, b \in \mathbb{T}$. Suppose to the contrary that $t \in [f(a), f(b)] \cap \mathbb{T}$ but $f(s) \neq t$ for all $s \in [a, b] \cap \mathbb{T}$. Then let us evaluate $f\left(\frac{a+b}{2}\right)$ and $t$ with enough accuracy to determine which of the following inequalities is true

$$f\left(\frac{a+b}{2}\right) > t, \qquad f\left(\frac{a+b}{2}\right) < t$$

(by the assumption of our proof equality cannot hold). If the first inequality holds we will replace the interval $[a, b]$ by $\left[a, \frac{a+b}{2}\right]$, if the second is true we replace it by $\left[\frac{a+b}{2}, b\right]$. Repeating this procedure we get a nested sequence of intervals

$$[a, b] \supset [a_1, b_1] \supset [a_1, b_2] \supset \cdots,$$

where $|b_n - a_n| = 2^{-n}|b - a|$ and $a_n, b_n \in \mathbb{T}$ for all $n$. Hence $f(a_n) \leq t \leq f(b_n)$ for all $n \in \mathbb{N}$, there exists a limit $s = \lim a_n = \lim b_n$ and it is easy to check that $s \in \mathbb{T}$. Then $f(s)$ exists and $f$ is continuous at $s$. Hence $\lim f(a_n) = \lim f(b_n) = t$, and

$$f(s) = f(\lim a_n) = \lim f(a_n) = t,$$

contrary to our assumption. This contradiction concludes the proof of Theorem 3. $\square$

**Theorem 4.** (The Computable Maximum Value Theorem). *If $f : [0,1] \to \mathbb{R}$ is a computable function, and $f(x) < f(c)$ for all $x \neq c$, then $c \in \mathbb{T}$.*

*Proof.* (This theorem is stated without proof in [PR].) Let $f = U(\cdot, t)$, where $t \in \mathbb{T}$. Since $[0,1]$ is compact, for every $n$ we can find a finite system $(p_i, m_i, q_i, n)$, $i = 1, \ldots, k(n)$ of quadruples in $t$ such that

$$\bigcup_{i=1}^{k(n)} \left(p_i - \frac{1}{m_i}, p_i + \frac{1}{m_i}\right) \supseteq [0,1].$$

We can also assume without loss of generality that $p_i \neq p_j$ for $i \neq j$. Then we define

$$g_n(p_i) = q_i,$$

and we extend $g_n$ in a linear way to all other points of $[0,1]$. Thus $g_n$ is piecewise linear and

$$|f(x) - g_n(x)| < \frac{1}{n}$$

for all $x \in [0,1]$. Let now $[a_n, b_n]$ be the shortest interval including the set

$$A_n = \left\{x : g_n(x) \geq \max_{0 \leq y \leq 1} g_n(y) - \frac{2}{n}\right\}.$$

It follows that $a_n \leq c \leq b_n$ for all $n$. By the unicity of the maximum (which is assumed in Theorem 4) it follows that the diameter of $A_n$ converges to 0, i.e., $|b_n - a_n| \to 0$. Since the points $(p_i, q_i)$ are rational it follows that the numbers $a_i, b_i$ are also rational and effectively computable. Also for every $m$ we can find an integer $n(m) \in \mathbb{N}$ such that

$$|c - a_{n(m)}| \leq |b_{n(m)} - a_{n(m)}| < \frac{1}{m}.$$

Hence $c \in \mathbb{T}$. $\square$

Theorem 4 immediately implies the following

**Corollary.** *If $f : [0,1] \to \mathbb{R}$ is computable, $f(c) = \max_{0 \le x \le 1} f(x)$ and $c \notin \mathbb{T}$, then $c$ is an accumulation point of the set $\{x : f(x) = f(c)\}$.*

**Problem 2.** Can the conclusion of the Corollary be strengthened to claim that there exists a perfect (i.e., closed and dense in itself) set $P$ such that $c \in P$ and $f(x) = f(c)$ for all $x \in P$?

The example which follows (and will be constructed later) yields non-trivial examples of functions satisfying the assumptions of the above Corollary.

**Example 1.** There exists a computable function $f : [0,1] \to [0,1]$ such that $\min_{0 \le x \le 1} f(x) = 0$ but $f(t) > 0$ for all $t \in [0,1] \cap \mathbb{T}$.

*Remark 9.* This example may suggest that some non-computable real numbers have direct physical interpretations. If so, we cannot replace in **OT** the Eudoxus–Dedekind real line $\mathbb{R}$ by the set $\mathbb{T}$ of Turing computable numbers (see also Example 3 below).

*Remark 10.* Example 1 shows also how fragile computable real functions can be. Indeed that function $f$ consists of a perfect set in the line $y = 0$ and of countably many arcs whose interiors are above that line. Now, if we flip any non-empty proper subset of those arcs below the line $y = 0$ (by changing the sign of the appropriate part of $f$) the resulting function is no longer computable. [This follows immediately from Theorem 3.]

Results related to Example 1 are due to E. Specker, G. Kreisel and D. Lacombe, see [PR] and references therein. We will use this example again in order to build Example 2. Thus, for convenience of the reader, I include the following construction of Example 1 (it is folklore, I learned it long ago from A. Ehrenfeucht and again recently from R. Dougherty).

Let $A$ and $B$ be two recursively enumerable, recursively inseparable subsets of $\mathbb{N}$. (E.g. $A$ and $B$ could be the sets of Gödel numbers of theorems and of negations of theorems of Peano's Arithmetic.) Let $A = \{\alpha(n) : n \in \mathbb{N}\}$ and $B = \{\beta(n) : n \in \mathbb{B}\}$, where $\alpha$ and $\beta$ are recursive and injective functions. A binary tree is a set $T$ of finite sequences of 0's and 1's such that any initial segment of any sequence in $T$ is in $T$. By an *infinite path* in $T$ we mean an infinite sequence $(\varepsilon_1, \varepsilon_2, \dots)$ such that all its finite initial segments are in $T$.

**Lemma.** *There exists a recursive infinite tree without any recursive infinite path.*

*Proof.* $T$ is the set of all finite sequences $(\varepsilon_1, \dots, \varepsilon_n)$ such that for all $m \leq n$ if $\alpha(m) \leq n$ then $\varepsilon_{\alpha(m)} = 1$ and if $\beta(m) \leq n$ then $\varepsilon_{\beta(m)} = 0$.

It is clear that $T$ cannot have any recursive infinite path since such a path would define a recursive set separating $A$ from $B$. $\square$

Now using $T$, we can define the function $f$ of Example 1. First let $x$ belong to the ternary Cantor set, i.e., $x$ is of the form

$$x = \sum_{n=1}^{\infty} \frac{2\varepsilon_n}{3^n},$$

where $\varepsilon_n \in \{0, 1\}$ for all $n$. Let $T$ be a tree satisfying the Lemma. Then, for such $x$, we define

$$f(x) = \begin{cases} 0, \text{ if } (\varepsilon_1, \varepsilon_2, \dots) \text{ is an infinite path in } T; \\ \dfrac{1}{n+1}, \text{ if } (\varepsilon_1, \dots, \varepsilon_n) \text{ is the longest initial} \\ \text{segment of } (\varepsilon_1, \varepsilon_2, \dots) \text{ which belongs to } T. \end{cases}$$

Then we extend $f$ linearly over all segments of the complement of the Cantor set. By the Lemma it is clear that $f$ has all the properties required in Example 1.

**Example 2.** There exists a bounded computable function $h : D \to \mathbb{R}$ such that $\mathbb{T} \subseteq D$ but $h$ has no continuous extension to all of $\mathbb{R}$. [To show this let $f$ be as in Example 1. The function $g(x) = 1/(e^x + 1)$ injects $\mathbb{R}$ onto the interval (0,1) and $g$ is computable. Thus $f(g(x))$ is bounded (since $0 \leq f(x) \leq 1$), computable, its minimum value is 0 and $f(g(t)) > 0$ for all $t \in \mathbb{T}$. Let $h(x) = \sin(1/f(g(x)))$. Hence $h$ has no continuous extension to any $x$ such that $f(g(x)) = 0$, but this happens only for some non-computable $x$. Thus $h$ has all the required properties.]

*Remark 11.* The above example satisfies the assumption of the Computable Intermediate Value Theorem (Theorem 3). This shows a similarity between Theorem 3 and the theorem of Darboux that the intermediate value property is true for derivatives of everywhere differentiable functions. Indeed those derivatives can also have many discontinuities.

*Remark 12.* Given a continuous function $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}$, we denote by $S(f)$ the set of points where $f$ has a positive oscillation, i.e., the set of all $x \in \mathbb{R}$ such that

$$(\exists\, \varepsilon > 0)(\forall\, \delta > 0)(\exists\, u, v \in D)[|x - u| < \delta \;\&\; |x - v| < \delta \;\&\; |f(u) - f(v)| > \varepsilon].$$

It follows easily from this definition that $\mathbb{R} - S(f)$ is always a $\mathbf{G}_\delta$ everywhere dense in $\mathbb{R}$. It is easy to see that every continuous $f$ can be extended to a continuous $f^* : (\mathbb{R} - S(f)) \to \mathbb{R}$. Thus $f^*$ is a maximal continuous extension of $f$.

**Problem 3.** Let $f : D \to \mathbb{R}$ be computable, $\mathbb{T} \subseteq D$ and $S(f)$ be non-empty. Must there exist a non-empty perfect set $P \subseteq S(f)$?

In spite of Remark 12, $f$ can be computable but such that $f^*$ cannot be computable. Indeed the next example (mentioned without proof in [PR]) shows that removing one point from the graph of a certain continuous but noncomputable function $f : \mathbb{R} \to \mathbb{R}$ can turn it into a computable one. This point will be the top of an ultrasharp peak of the graph of $f$.

**Example 3.** There exists a bounded, even, and continuous function $f : \mathbb{R} \to \mathbb{R}$ such that the value $f(0)$ is not in $\mathbb{T}$, but $f$ restricted to the domain $\mathbb{R} - \{0\}$ is computable. Moreover $f(0) > f(x)$ for all $x \in \mathbb{R} - \{0\}$. [Indeed let $\gamma : \mathbb{N} \to \mathbb{N}$

be a recursive function which is injective but such that the image set $\gamma[\mathbb{N}]$ is not recursive. Then the real number

$$\xi = \sum_{k=1}^{\infty} \frac{1}{2^{\gamma(k)}}$$

is not in $\mathbb{T}$. Let

$$f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \sum_{k=1}^{n} \frac{1}{2^{\gamma(k)}},$$

and extend $f$ such that $f(x) = f(1)$ for $x \leq -1$ and $x \geq 1$, $f$ is linear in each interval $\left[-\frac{1}{n}, -\frac{1}{n+1}\right]$ and $\left[\frac{1}{n+1}, \frac{1}{n}\right]$, and $f(0) = \xi$. It is clear that $f$ has all the required properties.]

*Remark 13 (On derivatives and integrals).* The derivative of a computable function need not be computable even when this derivative is continuous. To show this notice first that if $g = t^{(1,1)}$, $g : [a,b] \to \mathbb{R}$, where $a, b, t \in \mathbb{T}$, then for every $N \in \mathbb{N}$ there exists a finite set of quadruples $(p_i, m_i, q_i, n_i)$ in $t$ such that $n_i > N$ for all $i$, and the open intervals $(p_i - 1/m_i,\ p_i + 1/m_i)$ cover $[a, b]$. Of course given $N$ such a set of quadruples can be found effectively. It follows that if $f : \mathbb{R} - S \to \mathbb{R}$ is a bounded computable function, where $S$ is a finite set, then the Riemann integral

$$F(x) = \int_0^x f(u)du,$$

is a computable function with domain $\mathbb{R}$. Let now $f$ be the function of Example 3, then $F$ is computable but its derivative is $f$ and it is not computable. (For another example and related results see [PR], Ch. 1.)

Finally we wish to show that the domains of computable functions can be intervals with non-computable ends. (Closed and open computable ends are possible, see *Remark 3.*)

**Example 4.** There exist constant computable functions with the open domain $(-\infty, \xi)$ and with the closed domain $[\xi, \infty)$, where $\xi$ is the noncomputable real number defined in Example 3. [Indeed let $y_0$ code in a recursive way the set of quadruples $(p, m, 0, n)$ such that

$$p + \frac{1}{m} < \sum_{k=1}^{m} \frac{1}{2^{\gamma(k)}},$$

then $y_0 \in \mathbb{T}$, the domain of $U(x, y_0)$ is $(-\infty, \xi)$ and its value is 0 for all $x < \xi$. And if $y_0$ codes the set of quadruples $(p, m, 0, n)$ such that

$$p - \frac{1}{m} > \sum_{k=0}^{n} \frac{1}{2^{\gamma(k)}},$$

then again $y_0 \in \mathbb{T}$ and the domain becomes $[\xi, \infty)$.]

**Problem 4.** Is it true that the intervals $(-\infty, \xi]$ and $(\xi, \infty)$, where $\xi$ is as above, cannot be domains of computable functions? And, if

$$\eta = \sum_{k=1}^{\infty} \frac{1}{2^{\alpha(k)}} - \sum_{k=1}^{\infty} \frac{1}{2^{\beta(k)}},$$

where $\alpha$ and $\beta$ are as in Example 1, then none of the four intervals $(-\infty, \eta)$, $(-\infty, \eta]$, $(\eta, \infty)$ and $[\eta, \infty)$ can be the domain of a computable function?

As well known there exist sets of the class $\mathbf{G}_\delta$ which are everywhere dense in $\mathbb{R}$ but have measure 0.

**Problem 5.** Does there exist a computable real function whose domain (range) includes $\mathbb{T}$ and is of measure 0?

**Problem 6.** Let $f$ and $g$ be two real computable functions such that the union $f \cup g$ is a continuous function. Must $f \cup g$ be computable? (I suspect that the answer is negative.)

*Remark 14.* Concerning certain connections between computability theory, measure theory, and randomness, see [V] and references therein.

**7. Practical consequences.** As every mathematical definition **OT** is a sharp tool, and since it pertains to the significance of mathematics it can be easily abused.

On the other hand I hope that, by shedding a clear light upon the concept of direct physical meaning, **OT** can prevent certain errors, and discourage some bad practices. The purpose of this section is to discuss those issues which seem to me to be quite important.

First recall the simplistic view that only applied mathematics matters. According to this view it is natural that there should be distinct departments of pure and applied mathematics and that, when funds are lacking, the first can be disbanded without any harm to the society at large. At first sight **OT** can be thought to corroborate this view, sugesting that there is a clear conceptual difference between pure and applicable mathematics, and only those mathematicians who work in the finite, countable, and continuous realms, are important. But I think that **OT** tells us almost the opposite. It shows that the human mind cannot imagine directly the physical reality, that it can imagine it only by means of sets. This conclusion follows alredy from several examples given in this paper. Now I will develop this thought a little more.

Let the view that all mathematics is worthwhile be called *the Greek attitude*, and the view that only those parts of mathematics which are intended to talk about physical reality deserve our efforts be called *the Roman attitude*. The Greek attitude is suggested by the "declaration of independence" of G. Cantor, D. Hilbert and H. Poincaré: *in mathematics to exist is to be imagined and to be free of any known contradiction*. The Roman attitude is suggested by such ideas as the rejection of infinite cardinal numbers by L. Kronecker and H. Poincaré, by the belief of H. Weyl that one should do only such mathematics which is relevant to natural sciences, and by the claims of L. E. J. Brouwer and his successors (especially E. Bishop) that only mathematics in their style (intuitionism) is worthwhile.

We are strongly in favor of the Greek attitude. Our main argument supporting it is psychological. We claim that human brain has a certain system or framework $F$ for building language and forming descriptions and $F$ *leads us beyond the realm of continuity.* Let me try to prove this claim. The system $F$ is described in mathematical logic by means of the concept of a definitionally closed first-order set theory with a Hilbert $\varepsilon$-operator for the formation of names of objects. Of course, in order to become functional in the human mind, $F$ has to be activated by proper training or education (at an appropriate age), but the theory of $F$, that is logic and set theory, does not have to be explicitely known or taught. Hence, I believe that $F$ is natural and not an artifact of culture. [There are mathematical simplifications in the standard system of axioms ZFC of set theory such as the full Axiom of Extensionality (which excludes urelements) or the Axiom of Regularity, which are not natural. These do not seem to be parts of $F$ since they are useful only in mathematics. (E.g. natural logic accepts urelements.) So let me forget here those those mathematically useful but artificial simplifications appearing in ZFC.] The natural character of the logic and set theory of $F$ is visible from the fact that most mathematicians can use it correctly without being conscious its axioms and rules. Moreover, it is almost impossible to do mathematics in a clear way, without the set-theoretic framework. Confusing old expositions of mathematical theories become completely clear when they are translated into the set theoretic language (which is today the standard language of mathematics). Even in everyday life we use sets and classes in order to describe the world. In fact we think about the physical reality by means of a fairly deep hierarchy of sets. Natural languages freely create names of new objects (properties) and then analyze them as sets, or collect them into sets. I repeat that the human brain has evolved to think in this way. But there comes the surprise: this system $F$ given by nature eventually induces concepts (especially in mathematics) which do not make any direct physical sense!

[But in view of this there exists a threat from the demagogues or sectarian philosophers. Today I have in mind those who have a chip on the shoulder concerning their ignorance of mathematics. They would be happy to introduce a censorship rejecting much of pure mathematics. Let me relate some old personal experience. In my Ph.D. thesis, I proved the existence of set $A$ in $\mathbb{R}^3$ such every set obtained from $A$ by adding and removing from it any countable sets is geometrically congruent to $A$, and of a set $B$ on the spherical surface $S^2$, such that, for every cardinal number $\alpha$ such that $3 \leq \alpha \leq 2^{\aleph_0}$, the sphere $S^2$ can be split into $\alpha$ disjoint sets each of them rotation-congruent to $B$. Of course such sets $A$ and $B$ are purely imaginary. This was happening in Poland in 1957, and there was a real danger that communist activists will make trouble to my thesis advisor prof. S. Hartman. But we were lucky and nothing happened. A related episode happened in 1963. In a letter which I got from A. N. Kolmogoroff from Moscow, he wrote that the problems of determinacy of Borel and projective sets (then unsolved) are very interesting. But, as I learned later, he told in his entourage that the Axiom of Determinacy is a very bad idea. Certainly he was worried that AD could become a communist dogma and the Axiom of Choice be banned. Fortunately this did not happen. See also below, *Remarks 16 – 20.*]

I believe that a more primitive form of $F$ exists also in all higher animals. They do not classify reality in the form of sets or classes of various ranks as fully as we do, but they percieve it in the form of a structure of objects and processes. While humans have evolved to build also these set theoretic hierarchical descripions or explanations, and we have an inherited algorithm to explain things in this framework $F$. Thus, as I wrote above, set theory it is not only of theoretical value, it has a deep psychological motivation.

*Remark 16.* It follows from the above observations that the tendency to reduce mathematics to its applications damages education. Unfortunately this tendency is quite common. This is not to suggests that interpretations of abstract objects in physical reality should not be taught (they are important for applications and they stimulate intelligence). What I wish to criticize is the *algorithmic view* of mathematics which assumes that it has a core consisting of a system of algorithms which should be tought directly. The *classical view*, which I believe to be more natural, is that mathematics is a system of interesting imagined structures which have to be described just like real objects, while algorithms are only consequences of our familiarity with these structures. In their well known writings and reforms of the British system of exams Hardy and Littlewood eloquently criticised the algorithmic view.[4] But contemporary educational theories and the practice of teaching mathematics (prior to graduate studies) are still partial to this view. It leads to the production of books which attempt to train students in the performance of algorithms rather than to inform them about mathematical structures.[5] By presenting mathematics as a repetitive activity those books misrepresent the mathematical method. Being essentially training manuals, they fail to take advantage of natural intelligence or to stimulate it. Know-how cannot replace knowledge. Imagined structures challenge analytic imagination and inspire the mind to invent other interesting structures. The aphorism that the more you know the less you have to think is only partly true, strictly speaking *if you know more you can think more efficiently.* Of course algorithms are tools rather than descriptions, and their importance rests upon repetitive performance. Hence they can be appreciated only in real action.[6] We must take into account that every intelligent child knows that school offers only a preparation for real action, that the activity of learning in school is not the same as real work. [It follows that effective teaching of algorithms or even of the ability to read and write requires a good motivation of the children. If do not reap quick benefits by reading things that interest them or by writing to others, they have a hard time to acquire those abilities. And the same is true for mathematical algorithms.]

*Remark 17.* Concerning the contraposition of the knowledge of mathematical structures with algorithmic skills we add: Knowledge is so easy to communicate, that we think of it as interpersonal, while skills are more personal since they can be developed only by individual training. Many animals are very skillful but they are not knowleadgeable. It is easier to give new knowledge than to give new skills.

*Remark 18.* Returning to set theory (the system ZFC) we add that it has the following surprising property. It suggests its own extensions which appear to be consistent. Namely, new axioms postulating the existence of large cardinals such as inaccessible, measurable, strongly compact, etc. In this way one reaches axioms which yield important theorems (e.g. such as the truth of AD in the model $L[\mathbb{R}]$, see Section **3**). Still the universes of those stron set theories seem so regular (or repetitive), that we believe that they are consistent. It seems that no theory other than set theory has such a self-propelling quality. This property is another important motivation the set-theoretic foundations of mathematics.

*Remark 19.* The above considerations suggest that logic and set theoretic foundations, are also essential for the project of building an artificial intelligence which would be similar to human intelligence in its ability to prove mathematical conjectures.

**8. Conclusions and miscellaneous remarks.** 1. Our definition of potentially physically interpretable mathematical objects as objects of the form $x^\alpha$ ($x \in \mathbb{R}$, $\alpha \in \mathcal{T}$), which we have called the continuous realm, implies that a large part of pure mathematics is a study of that realm. Namely, most of number theory, analysis, geometry, and those parts of algebra which are significant in these theories, are studies of the continuous realm. But for mathematics this fact is of little consequence since the human mind has evolved in such a way as to use logic and set theory, and this lead us directly into the realm of subsets of $\mathbb{R}$, and hence outside of the continuous realm. Thus in mathematics any strictures of constructivism appear artificial and are counterproductive. Moreover, the beauty of Cantor's universe of sets and of its substructures makes us unwilling to accept any such limitations.

2. Our main thesis **OT** suggests a definition of applied (or applicable) mathematics. Namely, *mathematical results are applicable iff if they pertain to the continuous realm*. But this definition and the elementary nature of the continuous realm may be seductive and induce errors. This seems to explain the insular opinions of some analysts who think that they have the authority to define what is good mathematics. E.g. it explains the error of H. Poincaré and other mathematicians who believed (and still believe) that infinite cardinal numbers, logic, and set theory, are unimportant just because they have no *direct* relevance for science. To avoid such errors we should keep in mind three things: (a) Human brains are organized in such a way that it is impossible to understand the continuous realm without introducing sets which do not belong to it. (b) Logic and set theory belong to applied mathematics in as much as they explain the structure of mathematics and the functioning of intelligence (for related considerations and references see [$\mathrm{M}_2$]). (c) Not only set theory but even number theory and combinatorics, when it studies very large finite sets (e.g. Ramsey theory, see [GRS]), constitute an abstract art, that is, they do not contribute to our knowledge of any observed physical phenomena (unless we interpret them as a studies of the possibilities of human imagination).

3. The ontological theory of this paper (see Section **3**) would loose its significance if we accepted a more liberal sense of the words *direct physical interpretation*. Namely, if we accepted the interpretation suggested by Gödel's Completeness The-

orem plus any physical interpretation of the structure of positive integers with the operations $+$ and $\cdot$. Indeed for every recursively enumerable theory T which we believe to be consistent we can form the theory $\mathrm{T}^* = \mathrm{PA} + Con(\mathrm{T})$, that is, the extension of Peano's first-order arithmetic PA with an additional axiom $Con(\mathrm{T})$ claiming the consistency of T. Then, by Gödel's theorem, we can define in $\mathrm{T}^*$ a model $M$ of T, and prove in $\mathrm{T}^*$ that T is true in that model. Thus, since the elements of $M$ are integers, T has a physical interpretation (since integers and any finite sequences of symbols can be interpreted as physical objects).[7] Of course, this interpretation is different from the usual physical interpretations of mathematical objects which are appear in science.

## Endnotes

[1]When writing the paper [$\mathrm{M}_7$] I was unable to decide if the axiom *all sets are ordinal-definable* should be included in the canonical system of axioms of set theory. Later I reached the conviction that it should, and the reason is explained in [DM]. But let me add that A. Enayat (see [E], Section 2) has proved theorems which suggest that in the future the situation may change, that real numbers which are not ordinal-definable may gain some importance in mathematics.

[2]The fact that the standard system ZFC of axioms of set theory imply that $L[\mathbb{R}]$ satisfies the Principle of Dependent Choices is easy, see e.g. [Ka]. The fact that $L[\mathbb{R}]$ satisfies also the Axiom of Determinacy is a deep theorem of D. A. Martin, J. R. Steel and W. H. Woodin. It is based on ZFC plus some (relatively weak) large cardinal axioms, see [$\mathrm{N}_1$, $\mathrm{N}_2$] and references therein.

[3]Our critical view of intuitionism is expressed in the papers [MM] and [$\mathrm{M}_1$].

[4]I do not have in mind Hardy's later Apology where he claimed that good mathematics is neither applicable nor inspired by applications. This book expressed with great clarity a number of interesting thoughts about mathematics, but that

claim is not true.

[5]Unfortunately it is good business to sell big and expensive books where short ones would do. Hence for the textbook industry the Roman attitude with its insistence on know-how (that is procedures) is profitable. And indeed authors of textbooks on science and mathematics, at all the levels prior to graduate university texts, are asked to make them easy, comprehensive and to stress their utility. These three demands constitute a recipe for inflation and boredom. Indeed, most of those books are written by committees and they are very expensive, several hundreds of pages long, richly illustrated with irrelevant pictures. The information which they contain is diluted by haphazard examples and repetitive exercises, they lack any focus. They are written as if their goal was not to inform the students but to train them. In America those texts and the style of teaching which they promote are disastrous. They hide the accessibility and the synthesis of knowledge offered by the mathematical method and they kill the natural interest of the youth in mathematics and science. [When a child or a student is given a book in which most concepts are new to him, he can and read it (and carry it) only if the book is reasonably short. I believe that no young person $n$ years old should be required to read a book on mathematics or science of more than $10n$ pages. Longer books ought to be available for the interested but should not be followed in classes.] A basic ability to discern good from bad textbooks is missing in the committees chosing the textbooks (see [F], p. 291 - 302). In this unfortunate state of affairs I can imagine only one remedy: a sound philosophical education making people more familiar with the structure of natural intelligence and a more critical attitude toward textbooks, especially these on mathematics and sciences.

[6]In spite of these psychological obstacles, today the majority of physicists think that algorithms are the only way to describe some fundamental phenomena. Indeed,

since about 1915 physicists have adopted the philosophy that in quantum mechanics one cannot build adequate models, that in this area the only possible knowledge consists of algorithms for predictions (or statistical predictions). In fact *physicists accepted algorithms as definitive descriptions of reality.* But, as R. P. Feynman clearly expressed in his lectures, this cannot be called explaining (see [FLS], Vol. III, Ch. 1–4). (Personally I have not lost hope that it will be possible to understand quantum mechanics, that is to explain it by means of models, but everybody tells me that this is foolish.)

[7] There exist other finitistic interpretations of first order theories which do not depend on Gödel's method, see [$M_6$] and [P].

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