Foundations of Mathematics in the Twentieth Century

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Abstract

We outline the growth and the present state of logic and foundations of mathematics with a special emphasis on the developments that occurred in the 20th century. The paper is divided into three sections: Set Theory, Logic with Model Theory, and Computability Theory. The most important concepts and some representative theorems of each of these three areas are explained.

Foreword

A shorter version of this paper appeared in the Polish magazine *Delta*, issues 9 and 11 of 1999, and 1 and 3 of 2000. Another shortened version is to appear in the May 2001 issue of *American Mathematical Monthly*. The present version includes additional philosophical remarks, and fuller definitions of some mathematical concepts, some proofs including Gödel Completeness theorem, and a brief discussion of real-computable functions. The authors are indebted to Abe Shenitzer who translated the *Delta* version into English and encouraged us to expand it to its present form.

1 Introduction and early developments

Logic and foundations constitute a domain of mathematics which can be divided into three problem areas: basic mathematical structures (in terms of which one can define all other mathematical structures), the correctness and significance of mathematical reasoning, and the effectivity of mathematical computations. In the twentieth century, these areas have crystallized into three large chapters of mathematics, called *set theory, mathematical logic* *(including model theory)* and *computability theory*, which are intertwined in very significant ways. In this paper we will try to describe the evolution and *present state* of each of them. In the modern times the study of logic and foundations has attracted such eminent mathematicians and philosophers as Cantor, Cohen, Frege, Gödel, Hilbert, Kleene, Martin, von Neumann, Russell, Solovay, Shelah, Skolem, Tarski, Turing, Zermelo and others, and has given rise to a large body of knowledge. Although our paper is only a brief sketch of this development, we will discuss such essential results as Gödel's theorems on the completeness of first-order logic and on the incompleteness of most mathematical theories, some independence theorems in set theory, the role of axioms of existence of large cardinal numbers, Turing's work on computability, and some more recent developments.

Some mathematicians believe that, since Logic and Foundations have accomplished their goal of building a precise and clear basis for mathematics, they do not yield today any exciting or desirable new theorems. Jean Dieudonné, one of the influential members of the Bourbaki group was a proponent of this view. We believe that this is an expression of ignorance or lack of interest in in a number of exciting fields of thought such as (1) philosophy (ontology of mathematics), (2) brain science (how the brain thinks), (3) computability theory (the theory of complexity of calculations) (4) extensions of mathematics beyond of the beaten path of set-theoretic tools (like the problem of measurability of projective sets, the Generalized Continuum Hypothesis, combinatorial problems related to large cardinals, etc.). Of course we hope that this paper may be interesting to those who have such interests and that it may help dissipate the above mentioned intolerant views. Pure Mathematics is certainly an art rather rather than a science and intolerance or a bad intellectual climate can easily force it into stagnation (recall its sad fate in the Roman Empire).

There are many interesting unsolved problems in logic and foundations. For example they *do not* explain what is "good" mathematics. We know that mathematics has a very precise structure: axioms, definitions, theorems, and proofs. Thus we know what is correct mathematics but not why the works of certain mathematicians delight us while others strike us as downright boring. Nor do foundations tell us *how* mathematicians construct proofs of their conjectures. Since we have no good theoretical model of this process, we are far from having truly effective computer programs for proving theorems, although isolated spectacular successes have been achieved in this area. Other unsolved problems are mentioned at the end of this paper.

We begin with a brief sketch of the history of logic and foundations prior to 1900. The ancient Greeks were interested in the problems of what are correct arguments and what are the real numbers. As partial solutions of these problems they created the theory of syllogisms and a theory of commensurable and incommensurable magnitudes. These problems resurfaced in the 18th century. Their reappearance was due to the development of analysis and to the lack of sufficiently clear concepts of sets, functions, continuity, convergence, etc. In a series of papers (1878–1897) Georg Cantor created set theory. In 1879 Gottlob Frege[1]

[1] *Begriffschrift, eine der arithmetischen nachgebildete Formelssprache des reinen Denkens*, Halle. Translated in Jean van Heijenoort editor, *From Frege to Gödel, A Source Book in Mathematical Logic, 1879-1931*, Harvard University Press, Cambridge, MA and London, 1967 (third printing in 1977).

described a formal system of logic that sufficed for explaining precisely the logical structure of all mathematical proofs. In 1858 Richard Dedekind[2] gave a definitive solution to the question of what are the real numbers by defining them in terms of sets of rational numbers, and proved the axiom of continuity of the real line, an axiom accepted hitherto (beginning with the Greeks) without proof.

We will say a little more about some of the points of this historical account. Aristotle was interested in the question of characterizing correct arguments or proofs. This led him to create the theory of syllogisms that codifies certain forms of proofs. In terms of modern logic, these were rules pertaining to unary relations. A relevant example is the syllogism: *if Socrates is a Greek and if all Greeks are mortal then Socrates is mortal.* Written in modern logical notation this syllogism takes the form

$$[Greek(Socrates) \wedge \forall x(Greek(x) \Rightarrow mortal(x))] \Rightarrow mortal(Socrates).$$

In this formula $Greek(\cdot)$ and $mortal(\cdot)$ are symbols of unary relations. We used the universal quantifier $\forall x \ldots$ which means *for all x we have* ... Similarly, one introduces the existential quantifier $\exists x \ldots$ which means *there exists x such that* ... For two thousand years the theory of syllogisms was the core of logic. In the 17th century G.W. Leibniz hoped to create a "characteristica universalis", a language that would make it possible to express all mathematical sentences, and a "calculus ratiocinator" that would reduce all reasoning to computation. In the middle of the 19th century G. Boole, guided by the algebra of numbers, introduced an algebra of sentences. We owe further evolution of these ideas to A. de Morgan, C.S. Pierce, G. Frege, and other mathematicians and philosophers. In particular, it became clear that the logic used in mathematics goes beyond syllogisms. Indeed, mathematics involves relations between more than one object. For example, "less than" relates two numbers and is thus a *binary* and not a *unary* relation.

Cantor made the most important contributions to the determination of fundamental mathematical structures. He showed that (almost) all objects used by mathematicians can be thought of as sets. Moreover, it turned out that such an interpretation removed all ambiguities previously encountered in mathematics. For example, using the concept of set it was easy to define the concept of a natural number. In turn, natural numbers could be used to define integers and rational numbers. B. Bolzano, R. Dedekind, and Cantor showed independently how to use sets of rational numbers to construct the real numbers and to prove their fundamental properties. Then functions were defined as sets consisting of ordered pairs.[3] This new definition was simpler and more general than the earlier approach, which treated a function f as an algorithm for computing the value $f(x)$ for a given element x in the domain of f. In the 1880s and 1890s Cantor used this general concept of function to prove many theorems of set theory. He extended the concept of a cardinal number to infinite sets, analyzed notions such as linear order, and introduced the concept of well-ordering and the

[2] *Essays on the Theory of Numbers by R. Dedekind*, W.W. Benjamin, Chicago and London, 1901 (English translation).

[3] Somewhat later, ordered pairs were defined in terms of sets, by N. Wiener (in 1914) and K. Kuratowski (in 1921), and ordinal and cardinal numbers were defined in terms of sets by J. von Neumann in 1923.

related concept of an ordinal number. He also used transfinite induction in definitions and proofs. This was a new method that went beyond induction over the natural numbers used already in Euclid's *Elements*.

There was no theory of computability at the end of the 19th century. The original concept of functions as algorithms was developed only later, beginning with the work of Turing in 1935. However, issues of effectiveness have concerned mathematicians and philosophers since ancient times. There were many examples of algorithms, such as Euclid's algorithm for finding greatest common divisors, Newton's algorithm for finding zeros of differentiable functions, and Gauss' algorithm involving the arithmetic-geometric mean. Some computing devices already existed in antiquity.[4] In many cultures there were types of abaci, some of which are still in use today. There appeared mechanical computing devices for addition and multiplication needed for censuses. Science, artillery, and finance required many accurate computations. The demand for computing devices for astronomy, engineering, taxation bureaus, banks, etc. grew steadily. In the first half of the 19th century C. Babbage proposed the construction of a machine similar to a modern computer using stored programs, but he was not able to complete this project. And in mathematics many great analysts, up to the middle of the 19th century, did not accept existential statements unsupported by algorithms. For instance, to validate the claim that the initial value problem $y' = -y, \quad y(1.3) = 2.4$ has a solution, one had to know how to construct such a solution, i.e., one had to give a general prescription such that given any real number x one could compute the value $y(x)$ of the solution function y. Eventually, in the 20th century, the mathematical concept of a computable function was created by A. Church, K. Gödel, E. Post, and in a more definitive way by A. Turing.

An interesting prelude to the development of logic and foundations in the 20th century occurred in 1900. At the International Congress of Mathematicians in Paris, David Hilbert presented a lecture in which he stated 23 unsolved problems.[5] Three of them, Nos. 1, 2, and 10, deal with logic and foundations. Problem 1 belongs to set theory and was originally posed by Cantor. It is called the *continuum hypothesis* and asks: *are there infinite sets of real numbers whose cardinality is different from the cardinality of the set of integers as well as from the cardinality of the set of all reals?* Problem 2 belongs to logic and asks: *are the axioms of analysis consistent?*[6] Problem 10 belongs to the theory of computability and asks: *is there an algorithm for deciding the existence of integral solutions of a polynomial equation with integral coefficients and several unknowns*? Thus the three areas of foundations of mathematics already appeared in Hilbert's lecture. In the 20th century those three problems were solved in very surprising ways: K. Gödel and P.J. Cohen have shown that the continuum hypothesis is consistent and independent of the axioms of set theory which are generally

[4]For a description of a complex clockwork that calculates astronomical phenomena from the first century B.C. on see D.J. Solla Price, *Scientific American*, June 1959

[5]See e.g. *Mathematical Developments Arising from Hilbert's Problems*, F.E. Browder editor, Proc. Sympos. Pure Math. vol. XXVIII, parts 1,2, AMS, Providence, R.I., 1976.

[6]In his lecture, Hilbert asked about the consistency of arithmetic. But by arithmetic he meant what is currently called analysis, i.e., arithmetic of the integers augmented by set variables and a full comprehension scheme. (See Section 2.)

accepted until now.[7] Gödel showed that Problem 2 is unsolvable, moreover, no consistent theory containing enough combinatorics, whose axioms are explicitly given, can prove its own consistency. M. Davis, Yu. Matiyasevich, H. Putnam, and J. Robinson have shown that Problem 10 has a negative answer, i.e., that no such algorithm exists.

Before continuing our presentation of logic and foundations let us add some remarks about the relation of this paper to the accounts of other historians and philosophers.

Some authors divide the study of logic and foundations into three orientations: intuitionism, logicism and formalism. There are reasons to think that this division is misleading, first because these three concepts are not of the same kind and second because logicism and formalism are terminological oxymorons. Specifically, intuitionism is not an attempt to explain what mathematics is but rather a proposal for an alternative mathematics. On the other hand, logicism and formalism are such attempts, but, as we shall explain below, the difference between them is rather insignificant. Since these terms turn up so frequently in the literature, it makes sense to discuss them in greater detail.

Intuitionism was invented in 1908 by L.E.J. Brouwer as a reaction to the freedom of imagination proposed by Cantor, Hilbert and Poincaré (*In mathematics, to exist is to be imagined and to be free of any known contradiction*), and to the program of formalizing (that is, defining in mathematical terms) the rules of logic and the axioms of mathematics. Brouwer thought that these two things (freedom of imagination and formalization) are incompatible; he also thought that many objects of classical mathematics are not sufficiently constructive. Developing these ideas, he rejected the existence of least upper bounds of nonempty bounded sets of reals (that is the continuity of the real line) and many other nonconstructive statements. He proposed an alternative mathematics, called intuitionism, which was intended to be more meaningful than classical mathematics, and to have a more meaningful logic (for example, the general theorem of classical logic *p or not p* was rejected; *p or not p* would be proved only by proving *p* or proving *not p*), and this logic would undergo perpetual development. This program attracted some outstanding mathematicians, such as H. Poincaré and H. Weyl (although in their practice both were far from accepting the strictures of intuitionism) and E. Bishop (who accepted them fully). However, intuitionism proved to be rather unsuccessful. It had two unpleasant technical features: its weak logic made it hard to prove theorems, and many theorems of classical mathematics were rejected outright; and it has an unwieldy proliferation of concepts that are equivalent in classical mathematics but not under the weak logic of intuitionism. But Brouwer raised an issue that, if valid, would prevail over these technical difficulties. He claimed that many concepts and objects of classical mathematics are meaningless. For example, he accepted the integers but rejected well-orderings of the real line. (Even today many mathematicians who are not familiar with logic and foundations would say that this sounds reasonable.) But in 1904 Hilbert observed that, like any large integers, the infinite sets of pure mathematics are actually imagined in the sense that they are like containers whose intended content has not yet been constructed (imagined). The same view of sets was also held by Poincaré.[8] This observation removed

[7] See Table I for a list of those axioms. Recently, the second author proposed some new axioms which imply the continuum hypothesis.

[8] So the words *set*, *class*, *family* or *collection*, when they are used in pure mathematics, may be confusing.

the existential (ontological) problem of infinite sets. Moreover, in 1923 Hilbert proposed an extension of the logic of Frege in which quantifiers are no longer primitive concepts (they become abbreviations for certain quantifier-free expressions). In this way the apparent reference to infinite universes in pure mathematics, suggested by quantifiers, becomes a metaphor for a finite constructive process actually occurring in human imaginations (see Section 3). Thus the critique of classical mathematics raised by Brouwer and his followers collapsed. The authors of this article know of no clear motivation for intuitionism. The integers and their algebra (if we include some very large integers such as $10^{10^{10}}$) are no more constructive and finitary than any other mathematical objects and their "algebras". All we can say today in defense of Brouwer's idea is that some concepts and objects of mathematics are useful in science and others are not, and must be regarded as pure art.[9] But both kinds of objects are equally solid constructions in human imaginations and they become more permanent when expressed in writing.

Next we turn to logicism and formalism. Unlike intuitionism, they are attempts to explain what is mathematics. The first is attributed to G. Frege, B. Russell and A.N. Whitehead, and its spirit is close to the philosophers of the Vienna circle. The second is attributed to D. Hilbert and his collaborators and to Th. Skolem, and is close to the Polish school of logic. The dividing line between logicism and formalism is not very significant because it depends merely on a terminological difference. Specifically, it depends on whether set theory, or some of its variants (such as the theory of types), is or is not included in logic. The term *logic* is used since a long time in the narrower sense (often called *first-order logic*), which does not include set theory (the growth of set theory and of model theory has motivated the modern terminology). But we must add that there is an important fact that explains the former, more inclusive, sense of logic (of Frege et al.): the natural character of the axioms of set theory (see Table I) which makes them so intuitive that they belong to the "mental logic" of almost all people who work in pure mathematics or use mathematics in science. Another reason why the term logicism is not used anymore is that its program — to derive mathematics from logic in the wider sense — has been fully accomplished. This fundamental achievement is due mainly to Cantor and Dedekind. We will say more about it below.

To complicate matters, the term formalism has a number of meanings. First, it is associated with the problem proposed by Hilbert in 1900 of proving the consistency of stronger (more expressive) theories in weaker theories, because this requires formalization of the stronger theory in the weaker one. Although in 1931 Gödel proved that such proofs do not exist, the idea of Hilbert gave rise to the problem of classification of mathematical theories according to their strength: We say that S is *stronger* than T if T is interpretable in S (e.g., analysis in set theory) but not vice versa. The work of Gödel implies that S is stronger than T if the consistency of T can be proved in S. Work on the ensuing classification, which is sometimes called the "Hilbert program", is still in progress. Second, formalism is used as a name for the philosophy of Hilbert that we discussed previously in connection with

Indeed, they indicate thought-objects with the role of containers rather than completed collections.

[9] For example there is no physical significance to the distinction between a closed and open interval of the real line, or to a well-ordering of the real line.

intuitionism. Brouwer called Hilbert a formalist, meaning that Hilbert professed that pure mathematics is just a formal game of symbols (Stalin levelled the same accusation against western art, which he called bourgeois art). This was unfair since Hilbert was clear about the fact that mathematics is, first and foremost, a structure of thought-objects in our minds rather than of symbols on paper, and, if the word *game* can be applied to it, then the adjectives *interesting*, *beautiful* and *often applicable* would have to be added as well. Thus, contrary to Brouwer's misnomer for Hilbert's philosophy, the role that mathematics plays as an interface between science and art is essential in Hilbert's philosophy.[10]

2 Set Theory

In the first decade of the 20th century mathematicians clarified the concept of a set. A naive approach to this concept, proposed by G. Frege and B. Russell, can be expressed in the form "every class one can think of is a set", in symbols $\exists x \forall y[y \in x \iff \varphi(y)]$, where φ is any property expressed in the language of set theory. Russell himself found that this led to a contradiction when φ is $y \notin y$. Indeed, suppose that R is the collection $\{y : y \text{ is a set and } y \notin y\}$. It follows immediately from this definition that if R is a set then

$$R \in R \iff R \notin R.$$

This is a contradiction, hence R cannot be a set. It became clear that the concept of set had to be refined so as to avoid such contradictions. This is not to imply that the mathematical community, in particular Cantor, ever accepted the full comprehension scheme of Frege and Russell. Cantor knew that this scheme was too liberal and he had other examples to prove it. Collections (defined by arbitrary formulas φ of the language of set theory) are called *classes*. Cantor found that if the class of ordinal numbers were a set, then there would be an ordinal number not in that class — a contradiction; and if the universal set (i.e., the class of all sets) were a set, then its power set would have a cardinal number greater than itself — again a contradiction. Something had to be done to make the concept of set clear and useful. This problem was solved by Russell and Whitehead, who constructed a weak set theory called the theory of types, and in a more satisfying way by E. Zermelo. In 1908 Zermelo proposed a system of axioms describing sets and methods for constructing them. These axioms do not imply that Russell's R, or Cantor's other classes, are sets. Zermelo's system, with certain improvements introduced by Th. Skolem and A. Fraenkel, became the generally accepted axiomatization of set theory. The theory based on them is called Zermelo-Fraenkel set theory, or ZFC (See Table I). Almost all of mathematics can be developed within this theory.

[10] We need to warn the reader that several such misnomers are used by philosophers. We wish that they were avoided. Thus the plain rationalism of Hilbert and Poincaré concerning infinite sets in mathematics is called formalism in the case of Hilbert and conventionalism in the case of Poincaré, the term rationalism is meant as an opposition to empiricism, and Platonism is called realism. These meanings do not agree with the usual meanings of the words and they carry misleading overtones. Indeed, in normal parlance it is bad to be a formalist but it is good to be rational and to pay attention to experiments. And a Platonist is an idealist rather than a realist since he seems to have trouble distinguishing thoughts with a reference from thoughts without reference.

Table I. The Axioms of Set Theory ZFC
(Cantor, Zermelo, Skolem)
(Primitive symbol $\in$)

A1 Extensionality (Sets containing the same elements are equal):

$$\forall z[z \in x \leftrightarrow z \in y] \to x = y;$$

A2 Union (The union of all elements z of a set x is a set; $u = \bigcup(x)$):

$$\exists u \forall y[y \in u \leftrightarrow (\exists z \in x)[y \in z]];$$

A3 Powerset (The collection of all subsets y of a set x is a set; $p = \mathcal{P}(x)$):

$$\exists p \forall y[y \in p \leftrightarrow (\forall z \in y)[z \in x]];$$

A4 Replacement (The image r of a domain d by a mapping φ exists):

$$\forall x \exists z \forall y[\varphi(x,y) \to y = z] \to \forall d \exists r \forall y[y \in r \leftrightarrow (\exists x \in d)\varphi(x,y)];$$

(The axiom A4 is a scheme, for each formula φ we get a separate axiom. The formula φ may contain free variables other than x and y but not the variables d, r or z)

A5 Infinity (There exists an infinite set s):

$$\exists s[\exists x[x \in s] \wedge (\forall x \in s)(\exists y \in s)\forall z[z \in y \leftrightarrow (z \in x \vee z = x)]];$$

A6 Regularity (The relation $\in$ is well-founded):

$$\exists y[y \in x] \to (\exists y \in x)(\forall z \in y)[z \notin x];$$

A7 Choice (Every set x of non-empty, pairwise-disjoint sets y possesses a selector s):

$$\begin{aligned}(\forall y \in x)\exists z[z \in y] \quad \wedge \quad & \forall yzt[y \in x \wedge z \in x \wedge t \in y \wedge t \in z \to y = z] \\ \to \quad & \exists s(\forall y \in x)\exists t \forall u[(u \in y \wedge u \in s) \leftrightarrow u = t)].\end{aligned}$$

These axioms allow us to build almost all sets which are needed in mathematics and they describe a hierarchy of sets. First, by axiom A4 (for any formula φ such that φ is always false), we get a set without any elements. By the axiom A1 that set is unique, we call it the *empty set* and will denote it by $\emptyset$ or 0. By axiom A3, we have a set $\mathcal{P}(\emptyset) = \{\emptyset\}$ (also denoted by 1), which has the single element $\emptyset$. Then we have $\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$ (which is also denoted by 2). Then given any two sets u and v we can build the set $\{u, v\}$ by using the axiom A4 with $d = \{\emptyset, \{\emptyset\}\}$ and

$$\varphi(x, y) := [(x = \emptyset \to y = u) \wedge (x = \{\emptyset\} \to y = v)].$$

We denote $\bigcup(\{u, v\})$ by $u \cup v$ and define $x \subseteq y \leftrightarrow x \cup y = y$. We form the ordered pair $(x, y) = \{\{x\}, \{x, y\}\}$ and we define the cartesian product $a \times b = \{(x, y) : x \in a \wedge y \in b\}$.

Another important definition (due to von Neumann) is the class *Ord* of ordinal numbers. These are sets α (like $0, 1$, and 2 above) with the property:

$$\forall x, y[x \in y \in \alpha \to x \subseteq y \subseteq \alpha].$$

The class *Ord* of ordinal numbers is not a set. One proves that the class *Ord* is the smallest class containing $\emptyset$, closed under the operation $x + 1 = x \cup \{x\}$ and under arbitrary unions of sets of its elements. An ordinal λ is called limit, if $\bigcup(\lambda) = \lambda$. ω or N denote the least infinite ordinal number, thus $\omega = N = \{0, 1, 2, \ldots\}$.

It turns out that one can assign to every set x a unique ordinal number $rank(x)$. We define it as follows: $rank(\emptyset) = 0$ and $rank(x) = \bigcup(\{rank(y) + 1 : y \in x\})$ otherwise. Then the set of all sets of rank $\leq \alpha$ exists. We denote that set by V_α. The following is an alternative inductive definition of V_α (where α is any ordinal):

$$V_0 = \emptyset, \quad V_{\alpha+1} = \mathcal{P}(V_\alpha), \quad V_\lambda = \bigcup_{\xi<\lambda} V_\xi$$

for limit λ (see Fig. 1). The class of all sets is denoted by V.

It is a surprising and important fact (established by Cantor, Dedekind, von Neumann and others) that all mathematical objects can be interpreted in a natural way as sets in V. The simplest but important step of this interpretation was the construction (by Wiener and Kuratowski) of ordered pairs (x, y) given above (see footnote [3]). More recently D. Scott constructed cardinal numbers without using the axiom of choice.[11].

The axiom of choice, A7, troubled some mathematicians. This axiom asserts that for any set x of nonempty and pairwise disjoint sets there is a set that has just one element in common with each element of x. This is a generalization of an obvious property of finite collections of sets, but it leads to sets that are not individually definable in terms of the membership relation. By now mathematicians freely use the axiom of choice in many of its equivalent forms, such as Zorn's lemma or the well-ordering principle. Obviously, acceptance of the axiom of choice implies the acceptance of its consequences, for instance the paradoxical decomposition of a ball.[12].

[11] The cardinal number of x is a set c whose elements are all sets y of the least possible rank such that y can be bijectively mapped onto x.

[12] In 1924 S. Banach and A. Tarski, improving on an earlier theorem of F. Hausdorff, showed that it is possible to divide a ball into a finitely many (five) disjoint sets which can be moved by isometries to form two balls of the same size as the initial ball. Of course, this contradicts physical experience. Thus one can say that the mathematical concept of a set is far more liberal than the concept of a physical body. In general, the consequences of the axiom of choice (for uncountable families of sets) have less to do with applications of mathematics than theorems whose proofs do not involve this axiom.

There is a weaker form of the axiom of choice (called *axiom of dependent choices* or DC) which does not have such odd consequences. DC is the following statement: If $A_0, A_1, \ldots$ is a sequence of nonempty sets and $R_i \subseteq A_i \times A_{i+1}$ are relations such that

$$(\forall x \in A_i)(\exists y \in A_{i+1})[(x, y) \in R_i],$$

then there exists a sequence $a_0, a_1, a_2, \ldots$ such that

$$\forall i[(a_i, a_{i+1}) \in R_i].$$

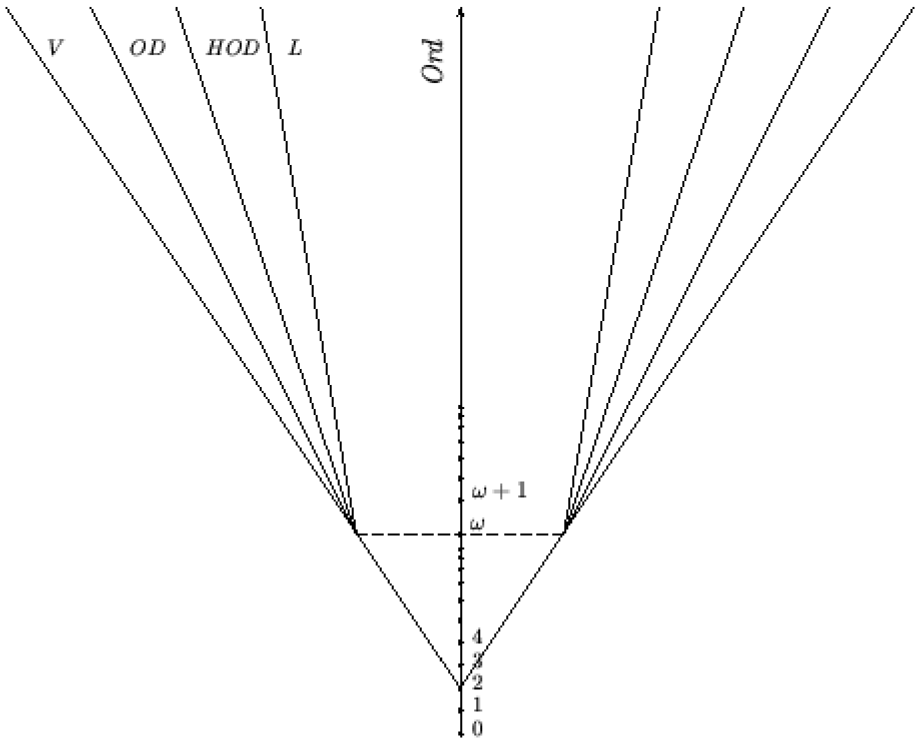
A set x is *ordinal-definable* if, for some ordinal α and some formula $\varphi(\cdot)$, x is the set of elements y in V_α that satisfy the formula $\varphi(y)$ in the structure $\langle V_\alpha, \in\rangle$. The class of ordinal-definable sets is denoted by OD. Let HOD denote the universe of *hereditarily ordinal-definable sets* i.e. sets such that they, their elements, the elements of their elements, and so on, are ordinal definable. Gödel showed, without using the axiom of choice, that HOD satisfies all the axioms of ZFC. Thus, if ZF (i.e. ZFC without A7) is consistent then ZFC is also consistent. The assumption $V = OD$ implies $V = HOD$ and the axiom of choice.

As mentioned above, the continuum hypothesis asserts that the cardinal number of any infinite set of real numbers is either equal to the cardinal number of the set of integers (denoted by $\aleph_0$) or to the cardinal number of the set of all real numbers (denoted by $\mathfrak{c}$). ZFC implies that this is equivalent to the assertion that $\mathfrak{c} = \aleph_1$, where $\aleph_1$ is the cardinality of the shortest well-ordered uncountable set. Attempts to decide whether $\mathfrak{c} = \aleph_1$ have led to many interesting results. In the late 1930s Kurt Gödel showed that the continuum hypothesis is consistent with the axioms of ZFC and in 1964 Paul J. Cohen showed that the axiom of choice does not follow from ZF and the continuum hypothesis does not follow from ZFC.

Gödel's technique amounts to reversing the objections to the axiom of choice. Instead of asking how to define choice sets (selectors), he limits all sets to those constructible by means of the following transfinite process. Working in ZF we inductively define "constructible levels" such that sets belonging to a particular level are definable in the structure consisting of the objects of the previous levels, i.e., we define $L_0 = \emptyset$, $L_{\alpha+1} = Def(L_\alpha)$, and $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ for limit λ. Here $Def(L_\alpha)$ is the operation that adds to the set L_α new elements representing all subsets of L_α definable in the relational system $\langle L_\alpha, \in\rangle$ in terms of elements of L_α. Thus

$$\begin{aligned} L_{\alpha+1} \quad = \quad & \{\{x \in L_\alpha : \varphi(x, a_1, \ldots a_k) \text{ is true in } \langle L_\alpha, \in\rangle\} : \varphi \text{ any formula} \\ & \text{of } n+1 \text{ variables, } n = 1, 2, \ldots, \text{ and } a_i \in L_\alpha, \text{ for } i = 1, \ldots, n\} \end{aligned}$$

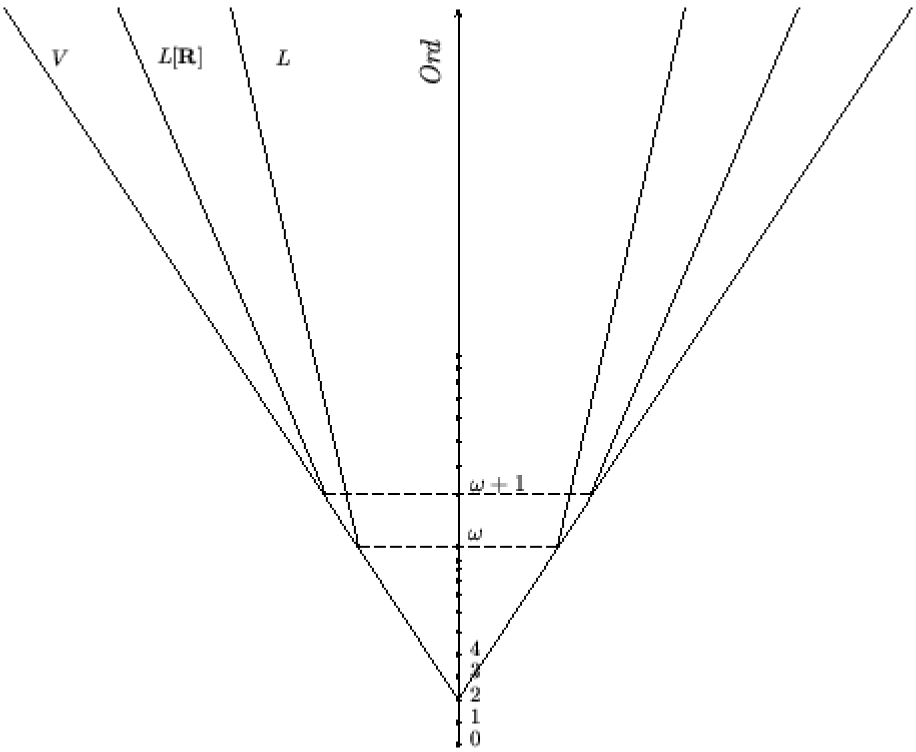
This definition of L_α can be formalized in ZF because definitions are formulas, and formulas can be encoded by natural numbers. In fact, we can even construct a well-ordering of the universe L of constructible sets. This is done by induction. It is enough to define the order of a given level L_α. First we order "k-tuples" of objects from the previous levels. Then we order the given level by comparing pairs consisting of a code of a formula and a sequence of parameters (from the previous levels). The ordering constructed in this way allows us to prove that the axiom of choice holds in the universe L of constructible sets, and hence that it is consistent with ZF. To obtain the continuum hypothesis in the universe L, Gödel proved a deeper theorem: all elementary substructures of L_α are themselves isomorphic to the levels L_β with $\beta \leq \alpha$. This suffices to show that all constructible real numbers are already constructed in levels L_α with indices α that are denumerable in L. This readily implies the continuum hypothesis in L. What remains to be shown is that all the axioms of ZFC hold in the universe L. We saw above how the axiom of choice is shown to hold in L. Showing the validity of other axioms of ZF in L is easy. Hence the axioms of ZFC remain consistent after the addition of the continuum hypothesis. Even the *generalized* continuum hypothesis — which asserts that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every ordinal α — holds in the universe L of constructible sets. Gödel showed also that all constructible sets (and only those) are constructible in L. Thus all the consequences of ZFC, and of the sentence asserting that all sets are constructible (denoted $V = L$), are true in L.



$Ord \subseteq L \subseteq HOD \subseteq OD \subseteq V$;
ZFC is true in V, in HOD, and in L;
ZFC + GCH + V = L is true in L;
It is consistent with ZFC that not all real numbers are in OD;
$Ord:\ 0 = \emptyset,\ \ \xi + 1 = \xi \cup \{\xi\},\ \ \lambda = \bigcup_{\xi<\lambda} \xi,\ \ \lambda$ limit;
$V:\ V_0 = \emptyset,\ \ V_{\xi+1} = \mathcal{P}(V_\xi),\ \ V_\lambda = \bigcup_{\xi<\lambda} V_\xi,\ \ \lambda$ limit;

Figure 1: Ordinal-definable, hereditarily ordinal-definable and constructible sets in the universe of Set Theory

Although it is consistent with the axioms of ZFC to assume $V = L$, it is more natural to assume that $V \neq L$ and even that there are real numbers that are not in L and not even in OD. Then the the class L can be enlarged by adding sets from $V \setminus L$ and using them in an inductive definition of a larger universe. For example, if L_0 is replaced by $V_{\omega+1}$ in the above construction, then we get an interesting universe denoted $L[\mathbf{R}]$, which will be discussed later in this paper (see Fig.2).



$Ord \subseteq L \subseteq L[\mathbf{R}] \subseteq V$;

ZFC is true in V and in L, ZF is true in $L[\mathbf{R}]$;

Under suitable large cardinal axioms ZF + $\forall X[AD(X)]$ + DC is true in $L[\mathbf{R}]$;

Figure 2: Constructible and constructible from reals sets in the universe of Set Theory

For a long time mathematicians were unable to prove the independence of the the axiom of choice or the continuum hypothesis. As mentioned above, this was established by Cohen in 1964. Cohen's argument is more complicated than Gödel's. It can be explained as the construction of a suitable Boolean model — a class of functions whose values belong to the Boolean algebra of open-closed subsets of a topological space and suitable definitions of the membership and equality relations between such functions. This construction assigns the Boolean value **1** to all the axioms of ZFC. For a suitable choice of that topological space the value of the formula describing the continuum hypothesis in this universe is **0**. But it can be shown that whatever is provable from formulas that take on the value **1** in this model likewise has the value **1**. Hence the continuum hypothesis is an independent sentence — it is not provable from ZFC. In a similar way Cohen proved the independence of A7 (the axiom of choice) from ZF. His method is called *forcing*. Forcing was used later to obtain other independence results, including many well-known problems in set theory, real analysis, and algebra.

While these results show the independence of various sentences in ZFC, the truth or falsehood of most of these sentences can be established in the two natural subuniverses L and $L[\mathbf{R}]$. However, the proofs of these theorems in $L[\mathbf{R}]$ require some extensions of ZFC. Specifically, they require cardinal numbers so large that their existence cannot be proved in ZFC.

The role of different universes, i.e., of different models of ZF, becomes particularly clear if we look at the *axiom of determinacy* introduced by J. Mycielski and H. Steinhaus in 1962. Let X be a set of infinite sequences of 0s and 1s. Consider the following game between two players I and II. Player I chooses 0 or 1, then player II chooses 0 or 1, then I again chooses 0 or 1, etc. When making their choices, both players know X and they know the previous choices. Player I wins if the sequence of 0's and 1's belongs to X, while player II wins if that sequence does not belong to X. The axiom of determinacy for X, $AD(X)$, states that one of the players has a winning strategy. Using the axiom of choice, it is easy to show that there are sets X for which $AD(X)$ is false, i.e., for which neither of the players has a winning strategy. But if a model of ZF satisfies $AD(X)$ for all X, then this model is closer to physical reality than any model of ZFC. For example, the Banach-Tarski paradoxical decomposition of a ball (see footnote (14)) is impossible in that model. In 1989 D.A. Martin, J.R. Steel, and H. Woodin showed that if one assumes ZFC plus appropriate large cardinal axioms,[13] then the model $L[\mathbf{R}]$ satisfies $AD(X)$ for all X, as well as the axiom DC of dependent choices. Whether $AD(X)$ holds for all $X \in OD$ remains an open problem (see the problems listed at the end of this paper).

The universes HOD, L and $L[\mathbf{R}]$ are very different. For example, in the models HOD and L there are definable non-Lebesgue measurable sets of reals, while (under the assumption of the existence of a suitable large cardinal) in the model $L[\mathbf{R}]$ all sets of reals are Lebesgue measurable and $|\mathbf{R} \cap HOD| = \aleph_0$.

For these reasons, and because of their intrinsic interest, work on large cardinal numbers has become an important topic in set theory. There are many classes of such large cardinals. Here we will introduce one of them, called *measurable cardinals*. We call a cardinal number κ *measurable* if there is a measure with values 0 and 1, defined on the family of all subsets of a set X of cardinality κ, such that the measure of every one-element set is 0, the measure of the whole set X is 1, and the measure of the union of fewer than κ sets of measure 0 is still 0. $\aleph_0$ is the smallest measurable cardinal. It is not possible to prove the existence of measurable cardinals greater than $\aleph_0$ in ZFC. In fact, D. Scott showed that if $V = L$ then there is no measurable cardinal $\kappa > \aleph_0$. Thus we see that $V \neq L$ is a more natural assumption than $V = L$ since it does not rule out the natural idea of measurable cardinals. The assumption of the existence of measurable cardinals (and of other large cardinals) has many interesting and deep consequences in real analysis, in the combinatorics of infinite sets, and even in the

[13]Some simple large cardinal axioms will be explained below.

combinatorics of finite sets.

Practicing mathematicians, whether algebraists, geometers, or analysts, almost never use axioms beyond ZFC. This is surprising since set theory was invented by Cantor more than a hundred years ago and ZFC expresses only Cantor's assumptions. It is even more surprising, since mathematics is constantly enriched by the natural sciences, by definitions and theorems suggested by physical reality. Nonetheless, such famous problems as the four-color conjecture, Fermat's last theorem, Bieberbach's conjecture (and many others) were solved in ZFC, in fact in weak subtheories of ZFC. It may be of some interest that axioms of existence of large cardinals (e.g. those used by Martin et al.) are not inspired by physical experience but rather by concepts arising in the human imagination for which there are no ready physical interpretations outside of human brains. On the other hand, the axiom of determinacy was motivated by H. Steinhaus' wish to build a system of mathematics closer to physical experience than ZFC.

We note that the consistency and independence of a sentence in set theory tell us nothing about its future, since it is possible that future developments of mathematics will decide this sentence. All they tell us is that the sentence is false in some model of ZFC and true in another. If we limit ourselves to the universe of constructible sets, then we obtain theorems about sets of real numbers that are very different from those true in the universe of sets with a measurable cardinal.

Today set theory plays a role similar to that played by Euclidean geometry for over 15 centuries (up to the time of the construction of mathematical analysis by Newton and Leibniz). Namely, it is a universal axiomatic theory for modern mathematics. It bears repeating that its axiomatization is not complete and that sometimes new fundamental principles about sets are discovered or invented. Relevant examples are large cardinal axioms and the axiom of determinacy for the class $L[\mathbf{R}]$. The evolution of mathematics decides which axioms become generally accepted.

There are axiomatic theories weaker than ZFC that are still very strong in the sense that large chapters of mathematics can be developed in them. The simplest is PA (*Peano arithmetic*, see Table II). PA is so powerful that it allows us to prove all known theorems of classical number theory. However, there are theorems of finite combinatorics (which can be expressed somewhat artificially as number-theoretic statements) that are provable in ZFC but not in PA. We will meet such theorems in the next two sections.

Table II. The Axioms of Peano's Arithmetic PA

(Dedekind, Peano)

(Primitive symbols 1, $+$, $\cdot$)

PA1: $x + 1 \neq 1$;

PA2: $x \neq y \to x + 1 \neq y + 1$;

PA3: $x + (y + 1) = (x + y) + 1$;

PA4: $x \cdot 1 = x$;

PA5: $x \cdot (y+1) = (x \cdot y) + x$;

PA6: $\varphi(1) \wedge \forall x[\varphi(x) \to \varphi(x+1)] \to \forall x \varphi(x)$,
for every formula $\varphi(x)$ (which may contain free variables other than x).

The theory PA has a natural extension called Z_2 or *second order arithmetic.* Z_2 is obtained from PA by adding a second sort of variables $X, Y, \ldots$ called set variables, a binary symbol $\in$, the axiom of extensionality, the comprehension scheme, and the set-theoretic form of the axiom of induction.

Table III. The Axioms of Second Order Arithmetic Z_2
(Hilbert)
(Primitive symbols 1, $+$, $\cdot$ and $\in$)

Axioms $Z_2 1$–$Z_2 5$ are the axioms PA1–PA5 of table II

$Z_2 6$: $\forall x[x \in X \leftrightarrow x \in Y] \to [X = Y]$

$Z_2 7$: $\exists X \forall y[y \in X \leftrightarrow \psi(y)]$, where ψ is any formula of the extended language without the free variable X;

$Z_2 8$: $1 \in X \wedge \forall x[x \in X \to x+1 \in X] \to \forall x[x \in X]$.

This theory Z_2 is a natural framework for the development of mathematical analysis (however is much weaker than ZFC)[14].

3 Logic and Model Theory

We now turn to a discussion of mathematical logic and model theory (and to Hilbert's second problem). While Frege discovered the formalism of logic as early as 1879, the introduction of a good system of notation took a considerable time. In their *Principia Mathematica*, the first volume of which appeared in 1910, B. Russell and A.N. Whitehead popularized the modern notation for the Fregean syntax of logic. However, they also included a part of set theory in their system of logic. The first truly modern textbook of mathematical logic, written by D. Hilbert and W. Ackermann, was published in 1928. Then, in 1930, Gödel showed in his doctoral dissertation that Frege's rules of proof are complete in the sense that a sentence φ is provable by those rules from a set of axioms A if and only if φ is true in all models of the system A. We call this result the *completeness theorem for first-order logic.* It motivated the modern terminology according to which logic is the complete system of axioms and rules of Frege.

[14] Z_2 and other weak theories adequate for various fragments of analysis, were studied in S.G. Simpson, *Subsystems of Second Order Arithmetic*, Springer, Berlin, 1999.

In 1924 Hilbert extended Frege's system by adding an operator ϵ for creating functions. Thus, if $\varphi(\bar{x}, y)$ is any formula (here $\bar{x}$ denotes any finite, possibly empty, string of free variables), then Hilbert's operator allows us to introduce a function symbol $\epsilon y\varphi$ of the variables $\bar{x}$ (see axiom L6 of Table IV). This operator is a formal counterpart of the mathematical practice of naming new objects and functions with desired properties. If there is no y such that $\varphi(\bar{x}, y)$ is true, then the value $(\epsilon y\varphi)(\bar{x})$ does not matter and L6 will be automatically true. Similarly, for any formula $\varphi(\bar{x})$ we can introduce a relation symbol R_φ with the variables $\bar{x}$ of φ, which abbreviates the formula φ (see axiom L7 of Table IV). Hilbert's extension has three advantages: (a) it simplifies the rules of logic, (b) the axiom of choice (A7 in Table I) can be derived from the remaining axioms of ZFC,[15] and, most importantly, (c) it allows us to solve the ontological mystery of quantification over infinite universes (see below).

Let us generalize the languages of ZFC, whose primitive concepts are two relation symbols $\in$ and $=$, and of PA, whose primitive concepts are the constants 1, two function symbols $+$ and $\cdot$, and one relation symbol $=$, to the case of an arbitrary system of relation symbols R_i $(i \in I)$ and an arbitrary system of function symbols f_j $(j \in J)$. Each R_i and f_j has a prescribed finite number of arguments $m(i)$ and $n(j)$. Thus our generalized language allows us to express properties of structures (called *models* of the language), of the form:

$$M = \langle A, \tilde{R}_i, \tilde{f}_j \rangle_{i \in I,\ j \in J},$$

where A is a nonempty set, $\tilde{R}_i \subseteq A^{m(i)}$, and $\tilde{f}_j : A^{n(j)} \to A$ when $n(j) > 0$, and $\tilde{f}_j \in A$ if $n(j) = 0$. Given any model M of the language and any sentence σ in the language (for example, an axiom of ZFC or PA) we can talk about the *truth* of σ in M. M is called a *model of a theory* T if all the axioms of T are true in M.

In Table IV we list all the rules and axioms of logic. Notice that L1–L7, L11 and L12 are rules since φ, ψ and χ are arbitrary formulas and R and f are arbitrary relation or function symbols. Similarly, A4 in Table I and PA6 in Table II were also rules; the other statements in Table I and II are single statements (axioms). The system presented in Table I and Table IV formalizes almost all of mathematics.

Table IV. The Axioms and Rules of Logic

(Essentially after G. Frege and D. Hilbert)

Primitive symbols: $\neg$ negation; $\to$ implication; $=$ equality symbol; we use strings of variables $\bar{x} = (x_1, \ldots, x_n)$ and $\bar{y} = (y_1, \ldots, y_n)$, $(n \geq 0)$; ϵ an operator such that if $\varphi(\bar{x}, y)$ is any formula then $\epsilon y\varphi$ is a function symbol of n arguments (or a constant symbol if $n = 0$).

L1: $(\varphi \to \psi) \to [(\psi \to \chi) \to (\varphi \to \chi)]$;

L2: $\varphi \to (\neg\varphi \to \psi)$;

[15] This is visible if we substitute the equation $y = (\epsilon z(z \in x))(x)$ for $\varphi(x, y)$ in A4.

L3: $(\neg\varphi \to \varphi) \to \varphi$;

L4: If φ and $\varphi \to \psi$ have been proven then ψ is proven;

L5: If $\varphi(x)$ is proven, then $\varphi(f(\bar{y}))$ is proven, where $f(\bar{y})$ is any term;

L6: $\varphi(\bar{x}, y) \to \varphi(\bar{x}, (\epsilon y\varphi)(\bar{x})))$;

L7: $R_\varphi(\bar{x}) \leftrightarrow \varphi(\bar{x})$;

L8: $x = x$;

L9: $x = y \to y = x$;

L10: $x = y \to (y = z \to x = z)$;

L11: $[x_1 = y_1 \wedge \ldots \wedge x_n = y_n] \to [R(\bar{x}) \leftrightarrow R(\bar{y})]$,
where R is any relation symbol of n variables;

L12: $[x_1 = y_1 \wedge \ldots \wedge x_n = y_n] \to [f(\bar{x}) = f(\bar{y})]$,
where f is any term of n variables.

It is natural to add to Hilbert's system various abbreviations. The following are particularly important and some of them were already used above.

$(\varphi \vee \psi) := (\neg\varphi \to \psi)$;

$(\varphi \wedge \psi) := \neg(\varphi \to \neg\psi)$;

$(\varphi \leftrightarrow \psi) := (\varphi \to \psi) \wedge (\psi \to \varphi)$;

$(x \neq y) := \neg(x = y)$;

$(x \notin y) := \neg(x \in y)$;

$\{y : y \in z \wedge \varphi(\bar{x}, y)\} := \epsilon t \forall y[y \in t \leftrightarrow y \in z \wedge \varphi(\bar{x}, y)](\bar{x})$ (in the case of ZFC);

$\exists y\varphi(\bar{x}, y) := \varphi(\bar{x}, (\epsilon y\varphi)(\bar{x}))$;

$\forall y\varphi(\bar{x}, y) := \varphi(\bar{x}, (\epsilon y(\neg\varphi))(\bar{x}))$.

The importance of this formalization (using the ϵ-operator) was already mentioned in the introduction. Namely, since it avoids any reference to infinite universes, it defuses Brouwer's criticism of classical mathematics. Indeed, variables are merely places for substituting constant terms, and the quantifiers $\exists$ and $\forall$ are defined by the last two formulas.

We will show a natural application of this extended formalism, namely, we will prove Gödel's completeness theorem (stated at the beginning of this Section). But first we introduce some fundamental concepts and make some observations.

A *sentence* is a formula without free variables. In fact, if we eliminate quantifiers (by means of the ϵ-operator), then such formulas contain no variables at all. A *theory* T is a set of sentences closed under derivations (that is, if φ is any sentence in the language of T and φ can be derived from T by the rules L1–L12 then φ belongs to T). A theory T is said to be *consistent* iff it does not contain a sentence of the form $\alpha \wedge \neg\alpha$ (if it did, it would contain all sentences). A theory T is said to be *complete* iff, for every sentence α, T contains either α or $\neg\alpha$.

It is easy to prove that a theory T does not contain a sentence α if and only if $T \cup \{\neg\alpha\}$ is consistent. Thus to prove Gödel's completeness theorem it suffices to show that every consistent theory T has a model $\langle A, \tilde{R}_i, \tilde{f}_j\rangle_{i\in I,\ j\in J}$, where $\tilde{R}_i$ are relations over A corresponding to the relational symbols R_i of T and $\tilde{f}_j$ are functions on A corresponding to function symbols of T. Such a model can be constructed as follows. Using Zorn's lemma we extend the theory T to a consistent and complete theory $T^\star$. Let C be the set of constant terms of T (that is, expressions constructed by the composition of function symbols and constants). Note that all the function symbols generated by the ϵ-operator occur in $T^\star$ and in C. Define the equivalence relation $\equiv$ over C by

$$c_1 \equiv c_2 \text{ iff } c_1 = c_2 \text{ is a theorem of } T^\star.$$

Let A be the set of equivalence classes of $\equiv$, that is, $A = (C/\equiv)$. We have to define the relations $\tilde{R}_i$ and the functions $\tilde{f}_j$. Let

$$\tilde{R}_i(c_1/\equiv, \ldots, c_m/\equiv) \text{ iff } R_i(c_1, \ldots, c_m) \text{ is a theorem of } T^\star$$

and

$$\tilde{f}_j(c_1/\equiv, \ldots, c_n/\equiv) = (c_0/\equiv) \text{ iff } f_j(c_1, \ldots, c_n) = c_0 \text{ is a theorem of } T^\star.$$

It is easy to check that these are correct definitions of relations and functions in A (that is, they do not depend on the choice of the representatives c_i in the equivalence classes $c_i/\equiv$). It is clear that the structure $\langle A, \tilde{R}_i, \tilde{f}_j\rangle$ is a model of $T^\star$ and thus also of T.

In 1950 Alfred Tarski initiated a systematic study of first-order theories and of the classes of their models. Model theory has applications in algebra, algebraic geometry, functional analysis, and in other areas of mathematics. One such application is *nonstandard analysis.* While 17th and 18th-century mathematicians used infinitesimals without defining them in a clear way, and while Cauchy introduced the ϵ–δ definition of continuity that allowed mathematicians to avoid their use, in the 1960s Abraham Robinson used model theory to show that it is possible to introduce infinitesimals by clear and precise definitions and to make good use of them. However, nonstandard analysis is conservative in the sense that if a theorem is stated by means of standard concepts and can be proved using nonstandard concepts, then it can also be proved without them.

In mathematical practice we almost always use many-sorted logic, that is, first-order language with various sorts of variables restricted to appropriate kinds of objects. See for example second order arithmetic Z_2, defined at the end of Section 2. Likewise when we analyze a ring, then, as a rule, we discuss not only the set of its elements but also the set of its ideals, that is, a family of its subsets. And in geometry we may use several sorts of objects, points, lines, planes etc. However, one can reduce many-sorted logics to first-order logic presented above (by means of additional unary relation symbols for each sort of variables). In turn, first-order logic can be reduced to the simpler two-sorted equational logic of G. Birkhoff, which is studied in a chapter of mathematics called universal algebra.

Recall that Hilbert's second problem asked for a proof of the consistency of the theory Z_2 (see Section 2). Hilbert recognized that as soon as a theory T is axiomatized and its rules of proof are clearly spelled out, the problem of consistency of T becomes a combinatorial problem. It is easy to see that one can interpret hereditarily finite sets (i.e. V_ω) as numbers, and so PA is adequate for expressing all problems of finite combinatorics. Hence Hilbert's proposal to prove the consistency of Z_2 in PA. [PA (Table II) appears to be consistent. For instance the induction scheme (PA6) is corroborated by the following experiment: if one knocks down the first of a row of dominoes, and if every falling piece knocks down the next, then, no matter how many pieces there are, they all fall down.]

Generalizing, we can ask for a proof of the consistency of ZFC, i.e., of most of mathematics, in PA. Hilbert thought that PA is complete and so he believed that such proofs can be found. But in 1931 Gödel proved that PA is not complete and moreover, that one cannot prove the consistency of PA *within* PA. Hence it is impossible to prove the consistency of Z_2 or of ZFC in PA.

These theorems of Gödel are called the *first* and *second incompleteness theorems*. We will state them in a more explicit way. Note that we can effectively enumerate the formulas of the language of PA. Indeed, a formula is a string of symbols of the (finite) alphabet of the language of PA, and so we first enumerate the symbols of the language and then look at finite sequences of symbols. We code the symbols $\neg, \rightarrow$, $\forall, \exists$, 1, $+, \cdot$, $($, $)$, x_n by the numbers $1, \ldots, 9, n+10$ respectively. Then we associate with the sequence of positive integers $\langle a_1, \ldots, a_k\rangle$ the number $2^{a_1} \cdot 3^{a_2} \cdot \ \ldots \ \cdot p_k^{a_k}$ (here p_k is the k-th prime). Some numbers code correctly constructed formulas and others do not. The codes of formulas are called their *Gödel numbers*. It is easy to tell in the language of PA when a number m is the Gödel number of a formula of the language of PA. It is tedious but relatively easy to tell when a number is a Gödel number of an axiom of PA. Similarly, we can code sequences of formulas by numbers and, again, it is easy to tell when a number is a code of a correct proof. Thus we can state in the language of PA a sentence Con(PA) that tells us that PA is consistent; Con(PA) says "There is no proof of the formula $1 \neq 1$ based on the axioms of PA." And Gödel showed that if Peano's arithmetic is consistent, then there is no proof of Con(PA) in PA! More generally, he showed that, for every consistent theory T containing PA in which the consistency of T can be expressed by a sentence $\mathrm{Con}(T)$ as above, T does not prove $\mathrm{Con}(T)$. Of course, we can prove Con(PA) in ZFC. We can also show in ZFC that the negation, $\neg$Con(PA) has no proof in PA (because ZFC proves that PA has a model). Hence PA is not a complete theory and no consistent extension of PA by a set of axioms whose set of codes is definable in the language of PA is complete. Thus Gödel proved in a very strong sense that Hilbert's project of proving consistency cannot be accomplished.

This brings us to an important hierarchy of theories

$$\mathrm{RCF} < \mathrm{PA} < \ldots < \mathrm{A}_i < \ldots < \mathrm{ZFC} < \ldots < \mathrm{ZFC} + \mathrm{LC}_i < \ldots$$

Here RCF (called the theory of real closed fields) is the complete theory of the field of real numbers $\langle \mathbf{R}, +, \cdot, < \rangle$ (its axioms are those of ordered fields plus axioms saying that every positive element is a square and that every polynomial of odd degree has a root; there is a separate axiom for each odd number n since this theory does not have the concept of the set of integers[16].); PA was defined in Table II; A_i are various axiomatic systems of analysis (Z_2 among them); ZFC was defined in Table I; LC_i are various axioms of existence of large cardinal axioms (some were discussed in Section 2); $S < T$ means that $\mathrm{Con}(S)$ is provable in T.

Some interesting combinatorial theorems of ZFC that can be stated in the language of PA have been found to be independent of PA. This is the case for some strong forms of Ramsey's theorem on the coloring of graphs. In fact, it turned out that some such sentences are intimately related to various axioms of large cardinals. H. Friedman, L. Harrington, R. Laver, J. Paris, and R. Solovay have proved several such results.

Using methods related to those of Gödel, Tarski proved in 1932 an important theorem about the concept of truth. If M is any model in which the model $\langle N, +, \cdot, 1 \rangle$ is definable (where $N = \{0, 1, 2, \ldots\}$), then the set of Gödel numbers of sentences that are true in M is not definable in M. In other words, there is no formula φ in the language of M such that $\varphi(n)$ is true in M iff n is a Gödel number of a sentence true in M (here $n = \underbrace{1 + \cdots + 1}_{n}$). This theorem has the following philosophical significance: to talk about the truth of sentences of any language L containing enough arithmetic we must go over to a larger language, since the concept of truth for sentences of L is not expressible in L itself.

4 Computability Theory

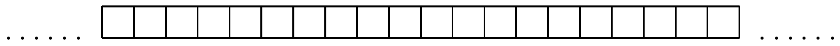
Now we turn to problems of computability (and to Hilbert's 10th problem). While proving the incompleteness theorems discussed in Section 3, Gödel considered a class of functions now called computable or recursive (later Emil Post, and Alonzo Church, gave other definitions of this class). A function $f : N \to N$ is *computable* iff there exists a formula $\varphi(x, y)$ of the language of PA such that: $\varphi(n, m)$ is a theorem of PA if $f(n) = m$, and $\neg\varphi(n, m)$ is a theorem of PA if $f(n) \neq m$. Here m and n denote any terms of the form $1 + \cdots + 1$ and $f(n)$ is also represented in this way. This definition of computability is robust in the sense that if we replace PA by a stronger theory, e.g. by ZFC, we still get the same class of computable functions. Call a set of natural numbers *computable* iff its characteristic function is computable, and call it *recursively enumerable* (RE) if it is empty, or it is the set of values of a computable function. Gödel showed that the set of codes of the theorems of PA is RE but not computable. All consistent extensions of PA such that the set of Gödel numbers of

[16]The theorem that this is an axiomatization of the complete theory RCF is due to A. Tarski (1939).

their axioms is RE have the same property. We owe to Alan Turing a better understanding of the concept of computability. In 1935 he introduced the concept of a machine, now called a *Turing machine*, and proved the existence of a *universal machine*, i.e., a machine such that, given an appropriate code, it is capable of simulating the work of any other Turing machine.

We now give Turing's definition of computability. This definition is equivalent to the former but it does not depend on the notion of proof and it leads to many interesting concepts, theorems and still unsolved problems.

A *Turing machine* M is a map $M : A \times S \to A \times S \times C$ where A is a finite set called an *alphabet* (in our paper we will assume without loss of generality that A consists of only three letters, $0, 1$ and $\Box$ where $\Box$ is the blank symbol), S is a finite set called the *set of states* of M or the *memory* of M, and C consists of three symbols: R, L, Q called *right*, *left* and *quit*, which we call *commands*. The letter $\Box$ and a certain state s_0 in S play special roles. Such a machine M acts upon an infinite tape



whose squares are filled with single letters of A. In other words, at time 0 we have a doubly infinite string of letters $\ldots a_{-1}^0\ a_0^0\ a_1^0 \ldots$. The machine looks at a_0^0 and it evaluates $M(a_0^0, s_0) = (a_0^1, s_1, c)$ (it always starts at the location 0 and in the state s_0). It erases a_0^0, and prints a_0^1 in its place. Then it moves left or right or quits (i.e., halts) depending on the value of the command c. If it quits, it declares that its job is finished. If it moves right, it looks at the current content of the cell with the index 1, and if it moves left, it looks at the current content of the cell with the index -1. Then, if this content is a, it evaluates $M(a, s_1)$, and acts again in the same way. M works in this way until (if ever) the third component c becomes Q, in which case it stops. Notice that only a single cell on the tape is affected at each time step. Thus the machine creates the sequence of consecutive tapes $\ldots a_{-1}^n\ a_0^n\ a_1^n \ldots$, where $n = 0, 1, \ldots$ is the time parameter.

The initial state of the tape $\ldots a_{-1}^0\ a_0^0\ a_1^0 \ldots$ is called the *input*, and the final state of the tape (if M reaches Q) is called the *output*. From now on we assume that the input is of the form $\ldots \Box\Box w \Box\Box \ldots$ where w is a finite string of letters from the alphabet A and we will denote it briefly by w. Also we assume that at time 0 the first letter of w is at the location 0.

It is easy to code a machine M by means of a finite sequence of 0s and 1s of length at most $K|S| \log |S|$, where K is an absolute constant. We will denote the code of the machine M by $code(M)$.

A machine M_0 is called *universal* if given the input $w_1 \Box w_2$, where w_1 is a code of a Turing machine M_1 and w_2 is any sequence of 0s and 1s, M_0 will work like M_1 on the input w_2. Thus M_0 stops on the input $w_1 \Box w_2$ iff M_1 stops on the input w_2, and it yields the same output. An important theorem of Turing tells us that there exist universal Turing machines.

The concept of a Turing machine has an important philosophical significance because of the following suggestive analogies:

$$\frac{\text{Turing machine}}{\text{Its tape}} = \frac{\text{Organism}}{\text{Its environment}}$$

$$\frac{\text{Universal Turing machine}}{\text{The code part of its input}} = \frac{\text{Human being}}{\text{Its knowledge}}.$$

These analogies are supported by the fact that for every real-world computational procedure (algorithm) there exists a Turing machine that can perform that procedure.

A universal Turing machine M_0 is a theoretical model for a programmable computer. Now, Turing machines lead to a more general concept of a computable function. A partial function $f : D \to N$, where $D \subseteq N$ is called *computable* iff there exists a Turing machine M such that for any input consisting of a binary representation of n, $n \in D$, the output of M is the binary representation of $f(n)$, and for $n \notin D$ the machine M does not halt (that is it does not reach Q). Turing proved that if $D = N$ then this definition is equivalent to the Gödel's definition stated at the beginning of this section (the main difficulty was to show that there exists a Turing machine that enumerates all the codes (Gödel numbers) for theorems of PA and only such numbers).

An important concept of computability theory, due (independently) to Solomonoff, Kolmogorov, and Chaitin, is that of an incompressible sequence. Let $|w|$ denote the length of the string w. A finite string v of 0s and 1s is called *incompressible* relative to a universal Turing machine M_0 iff, for every code w such that M_0 with the input $w\Box 0$ yields the output v, we have $|w| \geq |v|$. Since there are 2^n sequences of the length n and $2^n - 1$ sequences of lengths $< n$ there exists an incompressible sequence of any length n.

Chaitin has shown that for every recursively enumerable theory T and every universal machine M_0 there are only finitely many sequences s such that T can prove that s is incompressible relative to M_0. (This is another interesting concrete form of the Gödel incompleteness theorem).

Let us present Chaitin's interesting proof. Let M_T be a Turing machine that enumerates the theorems of T. We upgrade M_T to a Turing machine M_T^c ($c \in N$) which, given the input 0, searches for a sequence s with $|s| \geq c$ that T proves to be incompressible relative to M_0, and if it finds such a sequence s it prints s and stops. It is easy to define M_T^c so that the code of M_T^c relative to M_0 is no longer than

$$|code(M_T)| + K_1 \log_2 c + K_2,$$

where K_1, K_2 are constants independent of c. Now choose c such that

$$|code(M_T)| + K_1 \log_2 c + K_2 < c.$$

Then M_0 applied to the input $code(M_T^c)\Box 0$ does not halt (otherwise its output s would be compressible, contrary to the definition of M_T^c). Therefore T has no theorem of the form "s is incompressible", where $|s| \geq c$.

This leads to a class of problems of the following kind: find the smallest Turing machine adequate for a given task. Indeed, let $M_0^\star$ be a Turing machine such that $M_0^\star$ applied to the input $w_1\Box w_2$ works like M_0 on the input $w_1\Box 0$ yielding an output v and then it works like M_0 applied to the input $v\Box w_2$. It is clear that $M_0^\star$ is again a universal Turing machine. We will say that M is *minimal* if its code relative to $M_0^\star$ is incompressible. Thus, by the previous theorem, we see that for every recursively enumerable theory T there are only finitely many Turing machines M for which T can prove that they are minimal. So the problem of finding minimal machines will always remain an art (it can not be mechanized).

There are other important concepts of computability theory which we will define here. Let $\{0,1\}^\star$ denote the set of finite sequences of 0s and 1s. Let $L \subseteq \{0,1\}^\star$. We say that L is *computable* if there a Turing machine M such that given an input $w \in \{0,1\}^\star$, M yields the output 1 if $w \in L$ and the output 0 otherwise. We say that L is of *polynomial time*, in symbols $L \in$ PTIME, iff there is a Turing machine M and a polynomial $p(x)$ (polynomials of the form $x^k + a$ where a and k are positive integers suffice) such that given any input $w \in \{0,1\}^\star$, M yields the output 1 if $w \in L$ and the output 0 if $w \notin L$ and if in both cases M does not take more than $p(|w|)$ time steps.

We define the class PSPACE (*polynomial space*), by a definition like the one above, except that now we require that M does not visit more than $p(|w|)$ cells of the tape, while the number of time steps does not matter.

Finally, we say that $L \in$ NPTIME (*nondeterministic polynomial time*) iff there exists a Turing machine M and a polynomial $p(x)$ such that for all inputs $w \in \{0,1\}^\star$, $w \in L$ iff there exists a $v \in \{0,1\}^\star$, called a *guess*, such that given the input $w\Box v$, M stops and makes no more than $p(|w|)$ time steps.

The significance of the classes PTIME and PSPACE is clear. We illustrate the significance of NPTIME and its importance for logic and foundations by three examples.

A positive integer n will be identified with its binary representation, thus it belongs to $\{0,1\}^\star$. Then it is easy to convince oneself that the set L_0 of composite integers belongs to NPTIME. (If $w \in L_0$, the guess v should code two factors of w, both bigger than 1.) It is not known if $L_0 \in$ PTIME.

We consider finite graphs G with sets of vertices of the form $\{1,\ldots,n\}$, that is G is a binary symmetric relation on $\{1,\ldots,n\}$. We identify G with its characteristic function that is a symmetric $n \times n$-matrix of 0s and 1s. We code such a matrix by a single string of 0s and 1s of length n^2. A set $C \subseteq \{1,\ldots,n\}$ is called a *clique* iff $C^2 \subseteq G$. Let L_1 be the set of codes of graphs G such that G possesses a clique C with $|C| \geq \frac{n}{2}$, $(n = 1,2,\ldots)$. Then again $L_1 \in$ NPTIME and it is not known if $L_1 \in$ PTIME.

A theory T is *decidable* if the set of Gödel numbers of theorems of T is computable. As stated above, Gödel proved that no consistent recursively enumerable theory containing PA is decidable. Let L_{PA} be the set of strings of the form

$$w0\underbrace{11\ldots1}_{n},$$

where $w \in \{0,1\}^\star$ is a code (Gödel number) of a theorem τ of PA such that τ has a proof (in PA) that has a code (in $\{0,1\}^\star$) of length $\leq n$. Then $L_{\mathrm{PA}} \in$ NPTIME. Indeed,

$w0\underbrace{11\ldots1}_{n} \in L_{\mathrm{PA}}$ iff there exists a $v \in \{0,1\}^\star$, $|v| \leq n$ such that v codes a proof of τ. Now, the machine can check if v is such a code using no more than $p(n)$ time steps. Hence $L_{\mathrm{PA}} \in$ NPTIME. Again we do not know if $L_{\mathrm{PA}} \in$ PTIME. This is a very important problem. Indeed, if the answer is yes, then we would have a machine M and a polynomial $p(x)$ such that given a conjecture τ and an integer n, M could find out in time $\leq p(n)$ if τ has a proof of length $\leq n$ (Gödel posed this question in a letter to von Neumann, S. Cook defined the class NPTIME and proved some important theorems about it). Most mathematicians believe that the answer is negative, that the problem is harder than that.

It is easily seen that

$$\text{PTIME} \subseteq \text{NPTIME} \subseteq \text{PSPACE}.$$

An important theorem due to S. Cook tells us, for many interesting sets $L \subseteq \{0,1\}^\star$, that if $L \in$ PTIME then

$$\text{PTIME} = \text{NPTIME}.$$

L_1 and L_{PA} are among those sets. Whenever T is a theory that does not prove the formula $x = y$ then L_T (defined in a similar way) is among those sets. And there are many other such sets which are important in combinatorics, in computer science, and in many applications of mathematics (they are called NP-hard).

In the late 1930ies Banach and Mazur extended the concept of computable functions to functions whose arguments and values are real numbers. Using the concept of a Turing machine we can define those real-computable functions as follows. $f : D \longrightarrow \mathbf{R}$, where $D \subseteq \mathbf{R}$, will be called *computable* iff there exists a Turing machine M such that for all $x \in D$ if the input tape contains the code of a positive integer n and codes of a a sequence of rational numbers $q_1, q_2, \ldots$, such that $|x - q_k| < \frac{1}{k}$ for all k (the codes begin at the 0^{th} cell of the tape and extend to the right, each q_k being represented by by a pair of binary codes of integers r_k, s_k such that $q_k = r_k/s_k$), then after a finite number of steps M halts and yields (to the left of the above-defined input) codes of integers r and s such that $|f(x) - \frac{r}{s}| < \frac{1}{n}$. It follows easily from this definition that every real computable function is continuous on its domain[17].

In this way the old concept of an effectively computable function is precisely defined. We can even attempt to answer the general question: Which mathematical objects have physical interpretations?

In order to answer this question first we have to extend the concept of real-computable functions to functions of two variables. The definition is quite similar to the former, x and q_k are vectors in $\mathbf{R}^2$, and the inequality $|x - q_k| < \frac{1}{k}$ is replaced by $\|x - q_k\| < \frac{1}{k}$ where $\|\cdot\|$ is the usual norm on $\mathbf{R}^2$. Then it is easy to prove that there exists a computable function

[17] But such domains can be quite complicated. For instance, the function

$$f(x) = \sum_{i=0}^{\infty} \frac{1}{2^i} \sin \frac{1}{x - q_i},$$

where $q_0, q_1, \ldots$ is an effective enumeration of all rational numbers, is computable but its domain is the set of irrational numbers and f cannot be continuously extended to any rational number.

f_0 of two variables such that for every continuous (not necessarily computable!) function $f : D_f \longrightarrow \mathbf{R}$ there exists a constant c such that

$$f(x) = f_0(x, c) \qquad \text{for all } x \in D_f.$$

Thus c can be interpreted as a code for f. Then functions from codes to reals or from codes to codes can be interpreted as functionals or as operators, however all of them will be continuous over their domains of codes. Now we have the following answer to the previous question. Assuming that each real number has a physical interpretation it follows that each function, functional and operator which is continuous in the above sense has a physical interpretation (to avoid redundant objects we assume that the domains of definition are maximal[18]). We think that, conversly, every mathematical theory which has a physical interpretation can be interpreted in such continuous functional analysis. For example let $k : \mathbf{R} \longrightarrow \mathbf{R}$ be integrable, i.e. $k \in L^1(\mathbf{R})$. Ik k is not continuous then k has no direct physical interpretation, but consider the functional

$$F(f) = \int_{D_f} f(x)k(x)dx$$

and the operator

$$M_f(x) = \int_{D_f} f(y)k(x-y)dy,$$

where D_f is the domain of f, and both F and M are viewed as functions on the set of real numbers coding f, whose values are reals or codes. Then F and M are continuous on their domains[19]. Consequently, F and M have a direct physical significance.

The theory of Banach and Mazur supplied the missing part of the answer to the 19th-century question "What are functions?" Indeed, first we have a precise definition of a general function (mentioned above in our discussion of set theory), second we have a precise definition of a computable function (from a set of real numbers to a set of real numbers); only the latter involves the concept of computation.

The theory of computability has a large component, independent of computer science, that investigates the complexity of infinite sequences of 0s and 1s. For two such sequences x and y we say that $x \leq_T y$ if there exists a Turing machine M such that, if M is allowed to read y, then it can enumerate the consecutive terms of x. This preorder $\leq_T$ defines the so-called Turing degrees of sequences and there are many deep and beautiful theorems about these degrees.

Let us return to Hilbert's 10th problem. After years of efforts by many mathematicians, M. Davis, Y. Matiyasevich, H. Putnam, and J. Robinson proved that there is no algorithm for deciding if a polynomial equation of several variables, with integer coefficients, has a solution in integers. Moreover, they showed that for every recursively enumerable set $X \subseteq N$ there

[18] i.e. the function in question has no continuous proper extension.

[19] This follows form Lusin's theorem that every measurable function k can be restricted to a closed set K such that $k \restriction K$ is continuous and the complement of K is of arbitrarily small measure.

exists a polynomial $p(x, x_1, \ldots, x_n)$ with integer coefficients such that $x \in X$ if and only if there exist integers $y_1, \ldots, y_n$ such that $p(x, y_1, \ldots, y_n) = 0$. Thus they proved that even elementary number theory is a profound subject, so profound that there is no uniform method for solving its most natural problems.

For lack of space (and competence) we will not discuss many other connections between computer science and foundations of mathematics, for example, the connections between category theory and object-oriented programming (OOP), or Scott's applications of lattice-theoretic investigations to the semantics of programming languages.

5 Concluding Remarks

What is mathematics? Mathematical practice and logic and foundations indicate that mathematics is the art of constructing deductive theories. For example, while the Goldbach conjecture, which asserts that every even number greater than 2 is a sum of two primes, is a well established empirical fact, nevertheless mathematicians do not include it as an axiom because it is not a basis for an interesting theory (and also because it may turn out to be provable on the basis of PA or ZFC). Thus pure mathematics is the art of deduction rather than a list of facts. On the other hand, the study of nature often suggests new mathematical conjectures and sometimes even their proofs, and from time to time new fundamental axioms that yield interesting theories. In particular, logic is a study of ways in which nature has prepared us to describe reality, a study of natural intelligence. Even the concept of infinite sets is suggested by some apparently unending processes, and by the physical space-time continuum. Thus pure mathematics is not based on human imagination alone. Mankind in contact with nature generates the art of mathematics.

The main achievements of logic and foundations in the 20th century are the following:

(A) A precise language for mathematics has been constructed, so that we know what are mathematical sentences and what is a proof of a sentence based on a set of axioms and definitions. Gödel's completeness theorem for first-order logic shows that this concept of proof matches the set-theoretic semantics developed later by Tarski. When proving theorems, mathematicians have always appealed to logical intuition. In the 20th century this intuition has been fully and precisely described (see Table IV).

(B) An axiomatic set theory ZFC whose only primitive notions are those of logic plus the membership relation, has been constructed. Modern mathematics is based on this theory. Its axioms are so simple that they can be presented on a single page (see Table I). On the other hand, as shown by Gödel in 1931, and later by Cohen and others, many questions in finite combinatorics, in the arithmetic of natural numbers, in analysis, and in set theory cannot be decided in ZFC. The work of extending ZFC by new axioms with interesting consequences continues.

(C) A theory of computable functions and algorithms has been constructed. We have a clear distinction between the general classes of functions with integer arguments and values and with real arguments and values, and their subclasses of computable functions, defined by means of Turing machines. More concrete classes concerned with the time and space requirements of computations, such as PTIME, NPTIME, and PSPACE, are also being investigated.

Thus the fundamental questions posed in the 19th century have been answered. Those answers are surprising and beautiful. One can ask whether mathematics and computer science in the 21st century will be founded on the same concepts as in the 20th century. We conjecture that set theory will remain the most useful and inspiring universal theory on which all of mathematics can be based.[20]

Finally, let us formulate three open problems in logic and foundations which seem to us of special importance.

1. *To develop an effective automatic method for constructing proofs of mathematical conjectures, when these conjectures have simple proofs!* Interesting methods of this kind already exist but, thus far, "automated theorem proving procedures" are not dynamic in the sense that they do not use large lists of axioms, definitions, theorems and lemmas which mathematicians could provide to the computer. Also, the existing methods are not yet powerful enough to construct most proofs regarded as simple by mathematicians, and conversely, the proofs constructed by these methods do not appear simple to mathematicians.[21]

2. *Are there natural large cardinal existence axioms* LC *such that* ZFC + LC *implies that all OD sets X of infinite sequences of* 0*s and* 1*s satisfy the axiom of determinacy* $AD(X)$*?* This question is similar to the continuum hypothesis in the sense that it is independent of ZFC plus all large cardinal axioms proposed thus far.

3. *Is it true that* PTIME $\neq$ NPTIME, or at least, that PTIME $\neq$ PSPACE*?* An affirmative answer to the first of these questions would tell us that the problem of constructing proofs of mathematical conjectures in given axiomatic theories (and many other combinatorial problems) cannot be fully mechanized in a certain sense.[22]

[20] Some mathematicians, who study *category theory*, prefer another (more complicated) basis which was not described in this paper.

[21] However, we should point out that in an ongoing work S. Fajtlowicz constructed a program generating interesting mathematical problems that inspired several mathematical papers.

[22] In May 2000 the Clay Mathematics Institute announced a prize of one million dollars for a proof or disproof of the conjecture PTIME $\neq$ NPTIME. See `http://www.claymath.org/prize_problems/index.htm` for details.

Suggested further reading

1. General Information

    - J. Barwise, editor, *Handbook of Mathematical Logic*, North Holland, Amsterdam and New York, 1977.
    A comprehensive presentation of all three areas of logic and foundations at the beginning of the fourth quarter of the 20th century.
    - Y.L. Ershov, S.S. Goncharov, A. Nerode and J.B. Remmel, editors, *Handbook of Recursive Mathematics*, Elsevier, Amsterdam, New York, 1998, Vol. I and II.
    An extensive and comprehensive presentation of effective mathematics.
    - D. Gabbay and F. Guenther, editors, *Handbook of Philosophical Logic*, Reidel, Dordrecht and Boston, 1985, vols. I–IV.
    A very extensive presentation of aspects of logic and foundations as studied by philosophers (often inconsistent with the views expressed in our paper).
    - J. van Leeuven, editor, *Handbook of Theoretical Computer Science*, Elsevier, Amsterdam and New York, and MIT Press, Cambridge, MA, 1990, vol. I and II.
    A very extensive presentation of syntactic and semantic issues of modern theoretical computer science.

2. Set Theory

    - P.J. Cohen, *Set Theory*, W.A. Benjamin, New York and Amsterdam, 1966.
    An excellent introduction to axiomatic set theory and independence proofs.
    - A.A. Fraenkel, Y. Bar-Hillel and A. Levy, *Foundations of Set Theory*,North Holland, Amsterdam, 1976.
    An interesting treatment of the history of axiomatic set theory.
    - T. Jech, *Set Theory*, Academic Press, New York, 1977.
    A comprehensive course on classical set theory.
    - K. Kunen, *Set Theory, An introduction to independence proofs*, North Holland, Amsterdam and New York, 1980.
    A very useful course on set theory and on Cohen's method of forcing.
    - K. Kuratowski and A. Mostowski, *Set Theory: with an introduction to descriptive set theory*, North Holland, Amsterdam, 1976 (third edition, revised).
    An advanced course on axiomatic set theory.

3. Model Theory

    - C.C.Chang and H.J. Keisler, *Model Theory*, North Holland, Amsterdam and New York, 1990.
    A very beautiful exposition of model theory.

- W. Hodges, *Model Theory*, Cambridge University Press, Cambridge and New York, 1993.
  A comprehensive monograph on model theory with a large bibliography.

4. Computability Theory

- P. Odifreddi, *Classical Recursion Theory*, North Holland, Amsterdam and New York, 1989-1999, vol. I and II.
  An extensive presentation of classical recursion theory.
- H. Rogers, Jr., *Theory of Recursive Functions and Effective Computability*, MIT Press, Cambridge, MA 1987.
  A very beautiful presentation of computable function theory up to 1967.
- R.I. Soare, *Recursively Enumerable Sets and Degrees: a Study of Computable Functions and of Computably Generated Sets*, Springer-Verlag, Berlin and New York, 1987.
  A comprehensive study of degrees of computability.

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