RUSSELL'S PARADOX AND HILBERT'S (MUCH FORGOTTEN) VIEW OF SET THEORY

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Abstract. We try to explain the convictions of mathematicians that the axioms of Russell's theory of types and of ZFC are consistent and that mathematics is like a scientific investigation rather than a pure art. We claim that this is an inductive inference (a generalization) based on a mental experiment, namely the experience of a certain physical constructive process occurring in the brains of those who read with understanding the axioms of ZFC. Thus Con(ZFC) is a fairly well established statement or prediction about a physical process. We describe this process by means of concepts introduced by Hilbert, Poincaré and Skolem, and we add a number of remarks which we believe to be in the spirit of Russell's rationalism.

It is a pleasure and honor to be here and to speak on topics which are so important to all of you and to so many others in the world.

1. We all know the message of Russell's famous letter to Frege: The class of all sets which are not members of themselves is not a set.

But why should it be a set? And, since it is not, can we develop a clear view of sets (in our imaginations) which shows in a palpable way that this class is not a set?

These questions have been answered almost one hundred years ago, and the answer is: sets are imagined boxes intended to contain other boxes. Of course we cannot enclose a box into itself. And hence the class of all boxes cannot be put into one box, and the class of Russell is a superbox designed to contain all boxes.

Henri Poincaré said it differently. He said, for example, that a line is not a set of points, although we can construct many points on it. His words potentially infinite set suggest the same view of sets as imagined boxes.

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David Hibert (in 1904 in [3]) used still different words. He wrote that sets are thought-objects which can be imagined prior to the construction of their members.

Georg Cantor used the term definite sets, to stress the difference between sets and proper classes.

But even today, the word set is a bit confusing to some philosophers who are not used to set theory. In fact some are perplexed by Russell's paradox and think that Frege's inconsistent set existence schema is intuitive. It would be better if set theory teachers (and books on set theory) told at the beginning that it is essential to view the universe of sets as a container intended to contain other containers or boxes (intended for other boxes). Of course the set-brackets $\{...\}$ also suggest this view of sets, but this notation should be explained to those who meet it for the first time.

[However, let me defend Frege. When we construct axiomatic theories it is natural to start with axioms as strong and as simple as possible (even if one does not see through very well). Then, if inconsistencies are discovered, of course one has to prune down the axioms. Thus it was natural that Frege started with the simplest and strongest axiom which occured to him. Something similar happened when H. Steinhaus proposed the axiom of determinacy. His original form was too strong and I trimmed it down (then we published it together). But I had also another strong version and D. Scott showed that it is inconsistent, see [6], p. 219. And again when William Reinhardt proposed his axioms of large cardinals K. Kunen showed that some of them were too strong and inconsistent. And recently, some axioms which imply GCH occurred to me and H. Woodin showed that some of them were inconsistent with each other (the weaker versions which still appear to be consistent also imply GCH and seem interesting, see [9]). In this kind of work one has to take some risks.]

This view of sets as containers suggests that there is a constructive process in human imaginations underlying set theory. The Skolem Paradox of existence of countable models of ZFC is no longer a paradox, but a description of this process.

The remainder of this lecture consists of various remarks following from the above view. In particular about the philosophical problem of interpretation (ontology) of pure mathematics, about a relation between Hilbert's ε-symbols and his view of sets (mentioned above), and about a terminology of philosophers which seems quite incongruent with this interpretation of pure mathematics.

2. Let me begin with some events from my early education.

I asked my high-school teacher if mathematics is consistent. He said that he is not sure since there are things called paradoxes in set theory. But he said that some have worried about that question and this stimulated some books. He lent me one, a semiformal exposition of what was then called theoretical arithmetic. It was full of close-knit proofs of obvious facts, the whole thing pedantic and boring, and absolutely inconclusive relative to my question.

Of course I learned later that Hilbert asked the same question, and that Gödel has shown that in a sense we do not and cannot know the answer.

But, inspite of this, I had an experience which I remember vividly: When reading for the first time the formulas expressing the axioms of set theory (in a book by Kuratowski and Mostowski), I formed in my imagination an approximate picture of the universe of sets which was so concrete that I believed immediately that this system of axioms is consistent.

How to describe this experience? What was happaning in my imagination? I believe that I constructed mentally a little finite segment of the Skolem hull of sets around the empty set by means of the operations suggested by the axioms. This construction appeared so regular, in a sense periodic, that I immediately believed that it can be continued for ever. Injecting the infinite set was beautiful, no obstacle whatsoever. See also below, Section **3**. Of course the consistency of ZFC follows from the extendibility of these finite segments of Skolem hulls for as long as we can and wish to do it. (A more detailed mathematical definition of these finite structures arizing in human imaginations and supporting the feeling of consistency and concreteness of mathematical theories is given in [7]).

I do not want to be disrespectfull to those who are believing Platonists, but it seems to me that this experience in mental construction has nothing to do with seeing a Platonic universe of sets. If I had been exposed e.g. to Quine's NF I believe I would not have had such an experience since the Quinian membership relation has a loop $V \in V$ and so the finite segments of Skolem hulls for NF are not that simple. I was relating a personal experience; perhaps all of you had this experience (if not with ZFC, then with PA)?

[A more general observation may be of interest: The fact that we believe the consistency of ZFC is caused by a mental mechanism M which turns into convictions this (and a zillion other) inductive generalizations, based on mental or external experiences. M acts in such a way that it turns into convictions any descriptions or theories which appear to be true (that is to agree with facts) and are the simplest (or a disjunction of the simplest) among those descriptions which we know. Moreover M acts independently of our will. Notice that M is an attribute of good intelligence, and that it counters any general form of scepticism. Let me add that as soon as evolution gave us the power to form freely complex mental models of reality and the language to communicate them, it had to give us also an instinctive preference for the simplest such models, i.e., those which are the easiest to remember and to communicate from scratch. And M is a perfected form of this instinct. I believe that there exists also another reason for that preference: the simplest theories which appear to be true have a better chance to give correct predictions. In the practice of daily life and in science this preference is always obeyed.]

Thus the questions, why mathematics appears to be consistent, why it is so concrete or reconstructible in our minds, are all answered. It is all grounded in the simplicity of the axioms of ZFC and the human ability to carry a mental construction of appropriate finite stuctures when exposed to those axioms. (Although it is possible that some like Zermelo himself may first imagine the structures and then express the axioms.)

Still, I do not think that ZFC has any absolute position. Most extensions of ZFC with axioms of existence of large cardinals even some very strong ones (like Vopenka's axiom or Reinhard's n-extendible cardinal existence axioms), are easily incorporated in the above mental construction which yields in the same way the conviction that they are consistent. But, of course, as the axioms become stronger the conviction becomes weaker. I believe that our minds have some subjective estimates of the probabilities of consistency of those theories; however those values are not easy to measure, they are present only in our subconscious knowledge and they vary with time and experience. (Concerning our evaluating knowledge see below Section **5(e)**.) And, more generally, all our beliefs may have such probabilities in our minds. (Thus the concept of a conviction used above is a simplification. Rather than creating convictions M attaches probabilities of consistency to theories. We are able to observe clearly the presence of these values only when they are close to 1 or close to 0.)

Of course one should remember that the mental constructions supporting the consistency of ZFC and its natural extensions are not full proofs. For example it seems to me that if somebody is not familiar with set theory, and he is exposed to ZFC + AD (where AD is the axiom of determinacy), he may have the same positive experience. AD may appear to him as a natural generalization of a fact on finite games of perfect information (since it requires a longer exploration to find the clash between AD and AC). At least I can tell that if I omit the axiom of choice, that is I consider the theory ZF + AD, then that mental experience happens in my mind. And I believe that whenever we consider a new theory, at the beginning we have only that mental constructive method and no Platonic telescope to see anything.

3. In 1951 the logician J. Slupecki told me: I understand free variables. They are places for the substitution of more concrete terms. But I do not understand quantifiers since often they refer directly to some actually infinite universes which I do not believe to exist. (Like a number of Polish logicians of that time Slupecki called himself a nominalist; so did Kotarbiński and Tarski. They were influenced by K. Twardowski. See also [7].)

But there was an excellent answer to his question which Hilbert gave in 1923 (see [4]) which I did not know in 1951 (it is possible that Slupecki knew it and wanted only to stimulate his student): Hilbert's ε-symbols eliminate quantifiers from the basic formalism of logic. Quantifiers become abbreviations or defined symbols. Namely,

$$\exists y\varphi(x,y) \leftrightarrow \varphi(x,\varepsilon y\varphi(x)), \quad \text{and} \quad \forall y\varphi(x,y) \leftrightarrow \varphi(x,\varepsilon y(\neg\varphi)(x)).$$

In fact, if we add to first-order logic without quantifiers the ε-operator, and Hilbert's axiom schema

$$\varphi(x,y) \to \varphi(x,\varepsilon y\varphi(x)),$$

and the above definitions, then one can prove the usual axiom schemata on quantifiers.

Thus quantifiers are explained via Hilbert's ε-symbols as abbreviations of expressions which do not refer to actually infinite universes. [Concerning the mental role of variables see below Section **5(a)**.]

4. Now let us juxtapose the above mentioned words of Hilbert from 1904 (that sets are actually constructed though-objects whose members need not be actually constructed), with his ε-symbols of 1923 (I do not know if he ever wrote about this connection): It is clear that sets viewed as thought-objects can be denoted by terms involving certain function symbols and the process of creation of those symbols is described by Hilbert's ε-operator which acts upon variables and formulas. It is also clear that the ε-operator is a tool for naming Skolem functions. In this way it explains the mental experience described above, namely the constructive process in our brains which is stimulated by reading expressions with quantifiers such as the axioms of ZFC (see [7] for a mathematical definition of those structures).

5. Finally let me outline some thoughts about the philosophy of mathematics and even philosophy in general. In fact the ontology of mathematics presented above has a more general significance. I believe that it puts in doubt the value of a large part of the philosophical literature. It suggests also that the history of philosophy too often stresses dead ends, and fails to shine light upon the golden threads of knowledge.

(a) First, as much as I know, Hilbert himself in his 1923 paper and later did not write about the above ontological significance of his ε-operator. Perhaps he was not aware of it. This is suggested by the the fact that he proposed a distinction between concrete and abstract objects in mathematics, while the above considerations imply that this distinction is illusory (unless he meant only the distinction between finite and infinite structures in the sense of set theory or between objects with and without direct physical interpretations).

Indeed according to what we said all mathematical objects are imaginary and physical in human brains. There is no objective degree of concreteness of physical objects (I will return to this point in a moment). We have only the objective distinction between those imagined objects which name and are intended to model other physical objects or processes, and those thought-objects (motivated by their formal properties or their role in the structure of theories) for which there is little or no chance of finding physical significance (e.g. a well ordering of the real line). This is also discussed in [10]. But in pure mathematics we write symbols like 2 or $\prec$ (a well ordering of the real line), and I do not know of any clear sense in which 2 is a more concrete thought-object than $\prec$, inspite of the fact that 2 plays an important role in numerous descriptions of the physical reality (e.g. as a property of objects which we view as consisting of two sub-objects) while $\prec$ does not play any such role (and $\prec$ is not individually definable in the language of ZFC prior to its ε-extension). When a mathematician uses $\prec$ in a proof,

$\prec$ is as concrete in his imagination as any other thought-object. And variables are not only places for substitution (as told by Slupecki). When we use them in calculations and in proofs, they get in our minds the status of thought objects, objects which can be called general. Of course terms involving variables are also general objects (but at other times we can think about them as functions). We can only say about the objects of pure mathematics that some are simple and particular while others are general and not completely imagined, and some are so complicated that we can see them only as containers whose intended content cannot be completely imagined.

(b) In his philosophy of mathematics H. Poincaré erred in a different way. With his interpretation of infinite sets by potentially infinite sets (we said boxes or cotainers) he jumped too hastily to the prediction that Cantor's theory of infinite cardinal numbers will never be an interesting or genuine part of mathematics. Thus he thought that one kind of (presumably too abstract) objects can be apriori banned from mathematics. It appears that he believed that good mathematics must belong to natural sciences. But such a requirement is inconsitent with his own thesis that *in mathematics to exist is to be imagined and to be free from any known contradiction.*

(c) Let us reflect upon the question what information or what concrete structure of mathematical objects is so valued by the constructivists or the intuitionists and according to them is lost in classical mathematics? (See e.g. [0].) The above considerations suggest that there is no such thing, that nothing is lost. Indeed, as soon as a normal contribution to mathematics is accomplished, all its objects are equally concrete equally imagined and named in the mind of its author. As already mentioned, the integer 2 is no more concrete in the mind of such an author than a individual well ordering of the real line which he has named and used. But of course, it is only by the mental experiment mentioned above that he is lead to the inductive conclusion (the physical prediction) that his objects are consistent.

[Let me add that intuitionistic logic does not agree with the way we organize theories and descriptions of reality. We observe that in the world at large theories and arguments are organized such that *p or not-p* is accepted as true without necessarily knowing that p is true or knowing that *not-p* is true, and mathematical thinking is not that different from common thinking. (Of course in our imagination p is neither truth nor falsehood, it is a sentential variable, a general object of a certain kind.) Thus human knowledge is organized on the basis of classical logic and not intuitionistic logic; see also [7].]

(d) The philosophers discussing truth or existence in mathematics (see e.g. [1] and [11]) are often unclear. Thus a few words on those concepts are in order. As we know, truth is an agreement between sentences and the reality. [And an excellent mathematical theory of this relation was given by Tarski. This theory inspired model theory, but the latter went far beyond the original epistemological motivation of Tarski.] Thus, since mathematics talks about imaginary objects, the word true in pure mathematics has different meanings than in normal parlance. In fact most often it means provable. Unlike metamathematics, mathematics does not describe directly the structure of mathematical thoughts (this difference is discussed in [8]). We can say that spoken or written mathematics is a kind of reflection but not a description of our finite thought-structures (a description is given in [7]). Also in pure model theory truth or satisfaction do not have any physical meanings. Such meanings appear only when the theory is applied to explain some aspects of brains or texts describing reality. Thus in pure mathematics I do not see any truth (i.e., agreement with reality) other than our reliance on the mental experiments similar to that described in Section **2**.

It follows also that the term exists means something else in philosophy, in science and in everyday language, than in mathematics. In the former the primary role of the concept of existence is to distinguish those thought-objects which are intended to name or correspond to something from those which are purely imaginary that is without such an intention. While in mathematics existence either distinguishes objects which appear consistent from the inconsistent ones or, in other contexts "A exists" means that we were able to prove the theorem that A exists. (Concerning the physical interpretations of mathematical objects, see also [10].)

(e) It is more difficult to explain or to build an interesting theory of intuition (see [11] and references therein). There are striking examples of the importance of intuition: A good chess player uses not only computation but also his intuitive knowledge in order to choose a good move, a mathematician uses intuition to form interesting axioms, definitions, conjectures, and to build proofs. In each case there are too many possible choices and their consequences are too remote to be explored, and one applies one's evaluating knowledge (EK) (see [7]). The EK which is useful in chess and mathematics is fully learned since there was no evolutionary pressure to give us such facilities. In [7] I divided knowledge into EK and descriptive knowledge (DK), and proposed some mathematical definitions and theories of learning of EK and DK. Here I will only add that intuition, the mechanism M and EK are closely related to each other. Indeed the evaluation of the degree of truth of a statement, of the strength of an analogy, or of the simplicity of a theory or a description, are all made on the basis of EK, and they are essential for the action of M. Vice versa M plays an important role in the construction of EK.

Assuming that our mathematical theories of EK and DK (see [7] and references therein) are adequate, the main unsolved problem is to explain how EK and DK influence each other's development and how they collaborate in our minds.

(f) The philosophical significance of proof theory is unclear, since there seems to be no ontological or psychological significance of the proofs of consistency of arithmetic. To me, and perhaps to most of us, like all pure mathematics those proofs are motivated only by the formal value or beauty of their logical and conceptual structures (unlike, say, Mathematical Physics which has also another goal: to describe the physical reality).

(g) Likewise any special philosophical significance of primitive recursive arithmetic

(PRA) is unclear except for those areas which touch upon computer science or brain science. For example, the idea that PRA constitutes the appropriate theory of concrete objects fails, if we agree (see **(a)**) that all thought-objects of pure mathematics are equally concrete in human imaginations.

(h) We do not share Hermann Weyl's concern about the lack of scientific content of mathematics (see e.g. [12]). Weyl did not regard as sufficient the artistic or esthetical motivation of mathematics. As a pure mathematician he did not feel secure and wanted to do applicable mathematics. Unlike him Hilbert felt secure and I think that most of us share Hilbert's feeling. And we are sure of the importance of mathematics since we know that it is a mental art with an extraordinary esthetical quality, that throughout human history art accompanied and often preceded knowledge, and that mathematics is a prerequisite to science. For these reasons the concern of Weyl seems unnatural.

(i) Historians and philosophers divide the philosophy of mathematics into three schools: logicism (Frege, Russell and others), formalism (Hilbert and his school) and intuitionism (Brouwer and his school). As we wrote in [5], and in view of the ontology described in the present paper, this classification seems artificial and confusing. Also philosophers use a number of terms which are derived from the theological disputes in the Middle Ages, and are in conflict with modern natural language (but pleasing to some of them). Let me discuss those points.

First, what is logic? A natural sense of this word is: the ensemble of those methods of correct argument with which nature endowed human beings. (I am aware that postmodernists do not believe in such a thing. They claim that those methods were made by people and not by nature, that they are cultural and not natural. But I do not believe the postmodernists. The agreement between mathematicians and scientists of all races and cultures and the degree to which knowledge is a cumulative construction disproves the postmodernist ideas; see also [13]). If you agree with the above definition of logic, it is reasonable to regard set theory as a part of logic. Indeed all mathematicians know intuitively enough set theoretical constructions and axioms to build the objects and to prove the theorems in their fields of interest. However, many of them do not know those methods and axioms in a conscious way. They have the know-how but not the know-why (Plato pointed out this distinction in a different context). Thus type theory or set theory and some special tools for using them, appear to be ready in our nature. [Some must think that it is only in the nature of gifted people, since they write books and offer courses for the ungifted ones; books with titles like "Learning to Reason", which are hundreds of pages long and have almost no mathematical content. I doubt if they can ever be useful; in fact I believe that they are harmful since they can kill the interest of their readers. The compactness and the expressive power of mathematics are essential qualities which are important in our culture, and these qualities are completely missed in those books. Even their authors seem to know that those texts are training programs, but they seem unaware that they are extremly inefficient as sources of information and are dangerously counterproductive as initiations to science or mathematics.] Thus I believe that the primitive sense of the word logic encompasses set theory (say ZFC, or some system with urelements). Nevertheless, since Gödel's discovery of the Completeness Theorem for first-order logic (FOL), in mathematics, the term logic is often restricted to mean FOL. And of course, after restricting it in this way, one can claim that the program of logicism (of Frege, Russel and Whitehead) of deriving mathematics from logic, is not doable. But, if we return to the old more inclusive sense of the term, then this program is doable and it has been fully completed.

Now, formalism is really the same program, whence the distinction between formalism and logicism rests on the above terminological convention; it vanishes if we adopt the wider sense of the concept of logic. (I am not aware of any significant disagreements between logicists and formalists, unless one takes the Platonism of Frege to be a part of logicism—Russell would have rejected such an inclusion). Thus I think that this distinction is not interesting. Both groups of people tried and (due to the work of Dedekind, Cantor, Zermelo and Skolem) succeeded to construct a simple and formal (i.e., mathematical) account of mathematics.

And the name formalism (apparently invented by Brouwer) is deplorable. First, this term carries a pejorative overtone, second, it is misleading. Indeed the mental experience supporting the consistency of ZFC which was discussed in Section 2 is real and, for example, the distinction between ZFC and NF mentioned there is also real. Thus, between these two theories, only NF could be called formalistic. Also there is a significant difference between sentences and objects or stuctures in that mental experience. Moreover there are many categories of mental objects, and the division into the category of truth valued expressions and object is only the first step in this analysis (see [7]). There exists a rich stucture in our imaginations and memories which is founded on those categories. Therefore the struture of mathematical knowledge is far from arbitrary. Hilbert stressed the role of imagination in mathematics and one cannot attribute to him the view that mathematics is an *arbitrary* game of symbols. Unfortunately as much as I know he did not respond himself to the curious accusation that his philosophy assumes or implies such arbitrariness.

The phenomenon of relatively wide acceptance of biased misnomers like the name formalism for classical mathematics, is not isolated. Time and again we see people who twist language in order to win the popular debate. For example the positive overtones of the word realist must delight Platonists who call themselves realists. (They do not seem to be bothered that this contradicts the common meaning of words according to which only an idealist can think that mathematical objects are real, i.e., exist independently of mankind, while a realist should know that they are ideal, i.e., imagined.)

Let me propose the name *rationalism* or *rational ontology of mathematics* for the idea of Poincaré, Hilbert and Skolem that pure mathematics has no intended meaning, that meanings appear only in its applications. Thus from now on I will use the term rationalism in this sense. In doing so I accept the meaning of rationalism in natural language and I dismiss the meaning of those philosophers who oppose rationalism to empiricism (since in common parlance it is irrational to ignore experiments).

Still on terminology: Poincaré was called a conventionalist. Again I do not know any good justification of this label. It is true that he stressed that the choice of primitive concepts in a mathematical theory is a matter of convenience, a convention. But this is a well recognized fact and not a philosophical position. He may have erred by not realising that his version of special relativity is not only a convention but a real improvement of the Galilean theory (since the concept of ether becomes redundant). This was an oversight, but again not a philosophical position.

(j) I know only one interesting distinction in the philosophy (ontology) of mathematics namely the distinction between Platonists and rationalists, and this motivates the following remarks.

Some Platonists base their conviction on a feeling that PA is true and that every sentence in the language of PA is true or false. What matters is what do they mean by true? After all rationalists have just the same belief: for every sentence in the language of PA either it or its negation expresses a physical law, i.e., a correct prediction about an appropriate physical process. Platonists say also that they are sure that if set theory is done in other planetary systems it is compatible and perhaps very similar to ours. But again rationalists agree since they believe that set theory is founded upon some finite mental experiments and universal physical laws determine the outcomes of those experiments.

Thus the only real difference between rationalists and Platonists is the belief of the latter that there is a non-physical Platonic world of mathematical objects, and there is a human ability to learn that world. Of course rationalism explains mathematics without those assumptions (by accepting the physical reality of thoughts). This Ockhamian economy of ratonalism distinguishes it from Platonism and makes it convincing.

As mentioned in Section **2** we have a psychological mechanism M which acts independently of our will and makes us believe the simplest explanations of things which agree with facts. For this reason rationalism is convincing and Platonism is not. (Notice that although our minds do not tolerate inconsistency, M induces beliefs in some theories which only approximate the reality in some intended degree, a degree which, very often, is not known to the believer. E.g. I believe that each of the three theories of planetary orbits of Copernicus, Kepler and Einstein is true in its intended sense, but I do not know exactly how accurate they are.)

(k) In this state of affairs the existence of Platonists in this day and age is puzzling to us as it was puzzling to Tarski (see [7]) and presumably to Russell. Let me try to answer this question. Of course there were very outstanding Platonists, and among them Frege, Zermelo and Gödel. But notice that Frege and Zermelo did their main works before the invention of Skolem functions or Hilbert's ε-symbols, and hence we can assume that they did not see any way to justify the rationalistic theory. It remains to explain the position of Gödel who knew very well those concepts of Skolem and Hilbert (the philosophy of Gödel was taken very seriously by several authors, see e.g. [11] and [14]). My explanation is that Gödel never thought seriously about the idea that mathematics is a product of the physical brains. It is possible that he had a mystic non-physical concept of mind, since we know that he believed that the mind discovers or sees (rather than constructs) the universe of sets. Of course many people, and many mathematicians, think that way. But I propose that this is a consequence of a gap in their knowledge and a propensity to create a quick fix for that gap, especially that this fix has almost no impact on mathematics. [However it does have influence upon the attitude toward the axioms of set theory, and perhaps on the way they are invented. I cannot believe that the illusion that they tell a truth about some real infinite universe could be better than the knowledge that the only reality inspiring and supporting them is a finite constructive process in human brains.]

But many people think that there are important (humanitarian) reasons to reject rationalism. To explain this let me point out first that rationalism is in agreement with the identity thesis which says that the mind is the functioning brain (that mental life is an electrochemical process in the brain). And that, by its simplicity and agreement with facts, M makes the rationalists believe this thesis independently of their will. But the identity thesis is inconsistent with almost all religious doctrines. Thus many intellectuals feel that the identity thesis is a horrible theory, and that any philosophy which does not lead us away from this belief is objectionable. This hostile reaction of many intellectuals to rationalism may be strengthened by the fact that pseudo-rational political ideologies have tortured humanity, especially in the twentieth century when racism and communism enslaved many minds and gave politicians the power to kill millions. Those disasters led many intellectuals to the belief that rationalism is bad for humanity and that religions are much better. But most rationalists think that those terrible ideologies were gross abuses (and not consequences) of rationalism, and that the rejection of rationalism is not so humanitarian at all. First, it removes a significant intellectual obstacle to persecusions and supression of information or brain-washing in the name of religious faiths. Second, it violates human nature by obstructing or damaging the mechanism M described in Sections **2**, **5(e)** and **(j)**. Of course this negative judgement of the antirational positions is in the spirit of Russell's ethical philosophy. [It is not related to Russell's pacifism, which (I believe) was misguided.]

[Unfortunately we must be aware of the fact that there are also other causes of antirational theories. First, there are those who failed to acquire a minimum of scientific and mathematical education and developed an inferiority complex on this point. Scientific accounts of reality make no sense to them and they like to be told that science is not so important, and that there exists a deeper "spiritual knowledge" which is not based on history, biology and honest introspection (by sane observers). Second, there are those politicians and media people who capitalise on the above mentioned antirational ideologies. Third, there are those whose mechanism M is unhealthy. Such people can acquire strong beliefs in haphazard ways and accept murky ideologies. Of course mental illness or brain washing (which can be induced by others or self-induced) may cause this pathology. Perhaps malfunctions of M can result also from certain kinds of education which spoil or do not exercise M at the appropriate age. People afflicted in this way should be treated gently. And there are cases when error or illusion plays a positive role and must not be destroyed. But one should not tolerate those who profess irrational ideas in order to use or to subjugate others.]

(l) The defense of rationalism offered by philosophers is weak and their influence on scieces almost non-existing. In fact, as the biologist E. O. Wilson and the physicists J. Bricmont and A. Sokal explained, the lack of consilience between philosophy and scieces is striking (and sometimes it reaches comic proportions). Let me mention one example of this lack of contact or awareness. The philosophers D. Armstrong, A. Flew, B. Russell, G. Ryle, L. Wittgenstein and many others do not quote nor use in their theories and historical accounts the work of physiologists such as Charles Sherrington or Santiago Ramon y Cajal, or the mathematician A. Turing. This is perplexing since the discovery of the complexity of the brain and the construction of relatively simple universal Turing machines provided a much higher platform for a rational philosophy of the human being (the identity thesis) than the platform of earlier centuries upon which those philosophers appear to be standing.

I think that philosophy of mathematics (and philosophy in general) becomes relevant and true to its original aims only when it constitutes a contribution to knowledge. And, to propose a sufficiently sharp (perhaps even useful) criterion let me say that *knowledge consists only of the simplest among the fullest descriptions of reality.*

(m) Finally, concerning the future, let me add that I believe that the rational philosophy of mathematics points us toward a better understanding of the structure of mathematical knowledge in human brains, and eventually it will explain how mathematicians invent proofs of conjectures. (I have tried to formulate some preliminary mathematical definitions for such a theory in the lecture [7].)

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