SOBOLEV EMBEDDINGS II

Name: ________________________

First recall from Wednesday:

Fill in the details

Given Banach spaces $\mathcal{B}_1, \mathcal{B}_2$, we say that $\mathcal{B}_1$ is ______________________________________ in $\mathcal{B}_2$ and write $\mathcal{B}_1 \subset \mathcal{B}_2$ if we can choose a value C so that the inequality

$$\|f\|_{\mathcal{B}_2} \qquad \|f\|_{\mathcal{B}_1}$$

holds for each f in ________________________.

Motivated by the FTC, we began looking for inequalities of the form

$$\|u\|_{L^{p^*}(\mathbb{R}^d)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^d)}.$$

We saw that this inequality will certainly fail unless we have

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$$

with $p < d$.
Given p, d we call the value p^* the *Sobolev conjugate*.
Note that while we have determined that the inequality can only hold in this special case, we have not yet proved that this inequality is ever true. A large portion of this project will be devoted to showing that the inequality does indeed hold in this case.

Theorem (Gagliardo-Nirenberg-Sobolev inequality)**.** *Suppose $1 \leq p < d$. There is a constant C depending only on d and p so that is p^* is the Sobolev conjugate then*

$$\|u\|_{L^{p^*}(\mathbb{R}^d)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^d)}$$

for any $u \in C_c^1(\mathbb{R}^d)$.

Before we prove the theorem we examine some consequences.

Corollary. *Suppose $1 \leq p < d$. We have the continuous embedding*

$$W^{1,p}(\mathbb{R}^d) \subset L^{p^*}(\mathbb{R}^d).$$

Exercise

Provide a *brief* proof of the above corollary.

Exercise

Directly from the above embedding, given $1 \le p < d$, for what other values q do we have the embedding $W^{1,p}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$?

Exercise

Considering embeddings for $W^{1,p}(\mathbb{R}^d)$, what additional embeddings can we obtain for $W^{2,p}(\mathbb{R}^d)$?

Exercise

Given $p < q < \infty$, show that for large enough k, we have $W^{k,p}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$.

We are now ready to prove the theorem.

Proof of the Gagliardo-Nirenberg-Sobolev inequality. We will start by trying to prove the theorem for $p = 1$, and see if we can adapt the FTC argument to higher dimension.

Note that we can relate $u_i = u_{x_i}$ and u by the identity

$$u(x) = \int_{-\infty}^{x_i} u_i(x_1, \dots, x_{i-1}, y_i, y_{i+1}, \dots, x_d) dy_i.$$

Hence we have the inequality

$$|u(x)| \le \int_{-\infty}^{\infty} |\nabla u(x_1, \dots, x_{i-1}, y_i, y_{i+1}, \dots, x_d)| dy_i.$$

Exercise

Explain verbally why we can not relate norms of u and ∇u by integrating both sides of the above inequality.

We need to modify the inequality to obtain sufficient decay in the y_i direction on the right hand side.

Exercise

Find a function which has some decay in every direction and bounds $|u(x)|^2$.

Exercise

Consider the case $d = 2$. Integrate $|u(x)|^2$ and the above function in space to obtain the Gagliardo-Nirenberg-Sobolev inequality in this case.

Unfortunately, this trick is not quite enough for dimensions $d \geq 3$. The following two exercises explain why.

Exercise

Suppose we have $u, v \in L^2(\mathbb{R}^2)$. Then for almost every a, The functions $u(a,\cdot)$,$u(\cdot,a)$,$v(a,\cdot)$,$v(\cdot,a)$ are in $L^2(\mathbb{R})$.
Define a function on $\mathbb{R}^3$ given by $w(x, y, z) = u(x, y)v(y, z)$.
For almost every a, b, $w(a, b, \cdot)$ is in which L^p space? What about $w(a, \cdot, b)$ and $w(\cdot, a, b)$?
How does this relate to the above situation?

Exercise

To clarify why this really is a problem, suppose we have some function u on $\mathbb{R}^2$ such that for almost every x, $u(x, \cdot) \in L^q(\mathbb{R})$, and for almost every y, $u(\cdot, y) \in L^r(\mathbb{R})$, with $q \neq r$. Why can we not expect u to live in any L^p space on $\mathbb{R}^2$?

We can balance our dimensions by considering the inequality

$$|u(x)|^d \leq \Pi_i \int_{-\infty}^{\infty} |\nabla u(x_1, \ldots, x_{i-1}, y_i, y_{i+1}, \ldots, x_d)| dy_i.$$

We know that for $p = 1$, $p^* = \frac{d}{d-1}$. so we will modify the equality accordingly.

$$|u(x)|^{d/(d-1)} \leq \Pi_i \Big(\int_{-\infty}^{\infty} |\nabla u(x_1, \ldots, x_{i-1}, y_i, y_{i+1}, \ldots, x_d)| dy_i \Big)^{1/(d-1)}.$$

We will examine what happens when we integrate in one spatial dimension

$$\begin{aligned}
\int_{-\infty}^{\infty} |u(x)|^{d/(d-1)} dx_1 &\leq \int_{-\infty}^{\infty} \Big(\Pi_i \Big(\int_{-\infty}^{\infty} |\nabla u(x_1, \ldots, x_{i-1}, y_i, y_{i+1}, \ldots, x_d)| dy_i \Big)^{1/(d-1)} \Big) dx_1 \\
&= \Big(\int_{-\infty}^{\infty} |\nabla u(y_1, x_2, \ldots, x_d)| dy_1 \Big)^{1/(d-1)} \\
&\quad \cdot \int_{-\infty}^{\infty} \Big(\Pi_{i \neq 1} \Big(\int_{-\infty}^{\infty} |\nabla u(x_1, \ldots, x_{i-1}, y_i, y_{i+1}, \ldots, x_d)| dy_i \Big)^{1/(d-1)} \Big) dx_1 \\
&\leq \Big(\int_{-\infty}^{\infty} |\nabla u(x_1, \ldots, x_d)| dx_1 \Big)^{1/(d-1)} \\
&\quad \cdot \Pi_{i \neq 1} \Big(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\nabla u(x_1, \ldots, x_{i-1}, y_i, y_{i+1}, \ldots, x_d)| dx_1 dy_i \Big)^{1/(d-1)}
\end{aligned}$$

Exercise

Justify the last inequality using Hölder's inequality.

An induction argument gives that

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|u(x)|^{d/(d-1)}dx_1\cdots dx_j$$
$$\leq \Big(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|\nabla u(x_1,\dots,x_d)|dx_1\cdots dx_j\Big)^{j/(d-1)}$$
$$\cdot\,\Pi_{i>j}\bigg(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|\nabla u(x_1,\dots,x_{i-1},y_i,y_{i+1},\dots,x_d)|dx_1\cdots dx_j dy_i\bigg)^{1/(d-1)}.$$

Make sure to convince yourself of the above.

Hence

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|u(x)|^{d/(d-1)}dx_1\cdots dx_d \leq \Big(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|\nabla u(x_1,\dots,x_d)|dx_1\cdots dx_d\Big)^{d/(d-1)}$$

and the proof is complete for $p = 1$.

The proof for general $p < d$ can be obtained by applying the $p = 1$ case to functions of the form $|u|^\gamma$. We will omit the details of this argument.

This completes our examination of the proof.

Classical Derivatives. We now examine further regularity and embeddings. We begin with a redundant exercise.

Exercise

Show that as $p < d$ approaches d, p^* goes to infinity.

We may expect to obtain an L^∞ bound for $p = d$, but it turns out this is not the case. In any dimension $d \geq 2$, the we can obtain a counterexample using log log growth around a point.

Exercise

Skip in class; just for fun. Note that the function $u(x) = \log\lvert\log\lvert x\rvert\rvert$ defined on $B(1)$ is unbounded. Explain why ∇u is in $L^d(\mathbb{R}^d)$. Consider the outward radial derivative $\lvert\partial_r u\rvert = \lvert\nabla u\rvert$

We can obtain a more delicate embedding for $W^{1,d}(\mathbb{R}^d)$ which we will not investigate here. Given what we have already said about properties of embeddings, it should not be a shock to learn that for $p > d$ we do in fact have $W^{1,p}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$. In fact we obtain something much stronger. The following exercise is a special case.

Exercise

Another fun exercise not for in class.
Directly use properties of the L^2 Fourier transform $\mathcal{F}$ to show that if $k > d/2$, then

$$W^{k,2}(\mathbb{R}^d) \subset \mathcal{F}(L^1(\mathbb{R}^d)) \subset C_0(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d).$$

We now present the embedding. We note that there is a well known stronger version involving what are known as Hölder spaces.

Theorem (A weak Morrey's inequality)**.** *If $d < p \leq \infty$ then for any $k \geq 1$, we have the continuous embedding*

$$W^{k,p}(\mathbb{R}^d) \subset C^{k-1}(\mathbb{R}^d).$$

Exercise

Fix $1 \leq p \leq \infty$, $d \geq 1$. Show that for any $l \geq 0$, we can choose k sufficiently large so that the continuous embedding

$$W^{k,p}(\mathbb{R}^d) \subset C^l(\mathbb{R}^d)$$

holds.

Finally, we give some optional homework which applies these ideas to PDE.

Optional Homework: Conservation Laws. In this homework we will derive conservation of mass for an NLS equation which has broad applications to physics. The equation we consider is

$$\begin{cases} i\partial_t u + \Delta u - |u|^2 u = 0 \\ u(0) = u_0 \in W^{2,2}(\mathbb{R}^3) \end{cases}$$

with unknown $u(t,x) : \mathbb{R} \times \mathbb{R}^3$, $\partial_t u$ a time derivative of u, and Δu the spatial Laplacian. After one "classical" exercise, we will not need to worry too much about the interpretation of this equation.

Exercise

Use the equation to show that if we have some $u \in C^2(\mathbb{R} \times \mathbb{R}^3)$, which is also in, say $W^{2,2}(\mathbb{R}^3)$ for each t, then

$$\partial_t \int |u(t,x)|^2 dx = \int \Big[\partial_t (u\bar{u})\Big] dx = 0$$

where you may assume the first equality. Explain why for all $t \geq 0$ we then have

$$\int |u(t,x)|^2 dx = \int |u(0,x)|^2 dx.$$

We will now list some "local" properties of solutions with initial data in $W^{2,2}(\mathbb{R}^3)$ as well as properties of solutions with initial data in $W^{k,2}(\mathbb{R}^3)$ for large k, and ask you to show that all solutions with $W^{2,2}(\mathbb{R}^3)$ initial data also satisfy conservation of mass.

Properties of solutions

Properties of solutions with $W^{2,2}(\mathbb{R}^3)$ initial data:
Suppose we have $u_0 \in W^{2,2}(\mathbb{R}^3)$ Then we have

- (Local existence up to time T around u_0) there is some $T > 0$ and $r > 0$ so that for each $v_0 \in B_{W^{2,2}(\mathbb{R}^3)}(r)$, there is a solution $v(t) : [0,T] \to W^{2,2}(\mathbb{R}^3)$ to the equation, continuous in t with respect to the $W^{2,2}(\mathbb{R}^3)$ norm, such that $v(0) = v_0$.
- (Uniqueness) For each $v_0 \in B_{W^{2,2}(\mathbb{R}^3)}(r)$, the solution v as described above is the unique function mapping $[0,T] \to W^{2,2}(\mathbb{R}^3)$, continuous in time and such that $v(0) = v_0$.
- (continuous dependence) Furthermore, the map $v_0 \to v(t)$ on $B_{W^{2,2}(\mathbb{R}^3)}(r)$ is continuous in $W^{2,2}(\mathbb{R}^3)$ for each t.

Properties of solutions with $W^{k,2}(\mathbb{R}^3)$ initial data:
There is some large enough k so that for $u_0 \in W^{k,2}(\mathbb{R}^3) = W^{k,2}(\mathbb{R}^3) \cap W^{2,2}(\mathbb{R}^3)$, if u is any solution as described above, then we additionally have $u \in C^2(\mathbb{R} \times \mathbb{R}^3)$.

These properties are enough to establish conservation of mass.

Exercise

Show that for any $w_0 \in W^{2,2}(\mathbb{R}^3)$, if $w : [0,T) \to W^{2,2}(\mathbb{R}^3)$ is any solution which is continuous in time with respect to $W^{2,2}(\mathbb{R}^3)$ norm and $w(0,x) = w_0(x)$, then

$$\int |w_0(x)|^2 dx = \int |w(t,x)|^2 dx.$$