INEQUALITIES, LIMITS, AND APPROXIMATION

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1. Inequalities

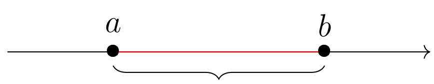
Suppose we have a sequence $\{a_n\}_{n=1}^{\infty}$ and a number L such that

$$\lim_{n\to\infty} a_n = L.$$

One rough way to interpret this limit statement is to say "For very large values of n, the value a_n is a very good approximation of L."
If we had a computer printing out the terms of the sequence $\{a_n\}_{n=1}^{\infty}$, one at a time, I could get as good an approximation of L as I wanted, as long as I was willing to wait long enough.

1.1. **Error.**

How do we make the notion of "good approximation" precise? When we are dealing with numbers, we have a good notion of how far apart two numbers are! Given two numbers a, and b, we will use the notation $d(a, b)$ to denote **the distance between** a and b. We use the standard definition $d(a, b) = |a - b|$.



$$d(a, b) = |b - a|$$

Recall our sequence $\{a_n\}_{n=1}^{\infty}$, for which

$$\lim_{n\to\infty} a_n = L.$$

Here we can think of $d(a_n, L)$ as how bad of an approximation the sequence is at every step. Intuitively, we should have

$$\lim_{n\to\infty} d(a_n, L) = \lim_{n\to\infty} |a_n - L| = 0.$$

It can be shown that $\lim_{n\to\infty} d(a_n, L) = 0$ **if and only if** $\lim_{n\to\infty} a_n = L$, that is, these statements mean the exact same thing.

Question. *Recall that we have a notion of distance for points on the plane as well! Given points (x_1, y_1) and (x_2, y_2), we define the distance in the usual way*

$$d\Big[(x_1, y_1), (x_2, y_2)\Big] = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Consider the list of points

$$\{(x_n, y_n)\}_{n=1}^{\infty} = \Big((\frac{1}{2}, \frac{1}{1}), (\frac{1}{2^2}, \frac{1}{2}), (\frac{1}{2^3}, \frac{1}{3}), \ldots, (\frac{1}{2^n}, \frac{1}{n}), \ldots \Big).$$

Without doing any computations, how do we think that the value $d\Big[(x_n, y_n), (0, 0)\Big]$ will behave when we take n to be very large? [If you feel you must do computations, the algebraic limit theorems are sufficient] Think about what is happening graphically as we move down the list of points. What do you think we can take away from this?

Note that these ideas also apply to functions. As an analogue to one of the above facts, the statements $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} d(f(x), L) = 0$ are also equivalent. Clearly, the idea of distance, and hence the idea of "error of approximation" provides a nice framework from which to consider limits.

1.2. **Inequalities.** In the context of error, inequalities are very useful. Often times if we are approximating a value, we are not asking "exactly how far away am I?", but rather "am I off by a small enough amount?".

Question. *Suppose there is a value L we are trying to approximate, and we have a single value a we are trying to use to approximate L. For each of the following scenarios, draw a number line, and mark L as a point on the line. Then draw the points that a could be.*

(1) $|a - L| < 3$
(2) $|a - L| \leq 3$
(3) $|a - L| \geq 3$
(4) $|a - L| < 4$
(5) $|a - L + 2| < 2$
(6) $|a - L - 1| < 2$

Which scenarios are clearly better than others for approximation? Now draw the picture for (4) on the same line you used for (1). Which one of these scenarios gives us more information about a?

Question. *Recall the previous question, and the scenarios $|a - L| < 3$, and $|a - L| < 4$. If we have that $|a - L| < 4$ can we also say $|a - L| < 3$? If we have that $|a - L| < 3$, can we also say that $|a - L| < 4$? Comparing 3 and 4, which way does the inequality go?*

This illustrates the important, but often overlooked, **transitive** property of inequalities. If we know that $a < b$ and we know that $b < c$, then we also know that $a < c$. We can think of this as collapsing the string of inequalities $a < b < c$.

Question. *Let's suppose I am trying to approximate a value L with a value a. Let's also assume that I need my approximation to have an error of less than 10^{-5}, that is $|a - L| < 10^{-5}$. In which (maybe more than one) of the following scenarios gives me enough information to know that my approximation is good enough?*

(1) $|a - L| < 10^{-6}$
(2) $|a - L| < 10^{-4}$
(3) $|a - L| = 0$
(4) $|a - L| < \frac{10^{-5}}{2}$
(5) $|a - L| < 2 \cdot 10^{-5}$
(6) $|a - L - 1| < 10^{-5}$
(7) $|a - L - 1| < 10^{-100}$
(8) $|a - L + 10^{-10}| < 10^{-10}$

This leads to our **Big Question!**

Question. *Say we are trying to approximate the limit of a series with the error*

$$\Big| \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} \Big|.$$

Suppose for any given N, we can find a positive value L_N so that

$$\Big|\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N}\Big| < L_N.$$

If we want to make sure we have the inequality

$$\Big|\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N}\Big| < 5,$$

is it enough to either find some N where $L_N > 5$ or else find an N such that $L_N < 5$? Suppose that a is some value such that $a > 5$ and b is some value such that $b < 5$. Then which of the following scenarios allow us to ensure that $\Big|\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N}\Big| < 5$?

- $L < a$
- $L > a$
- $L < b$
- $L > b$

You're done woohoo!

1.3. **Technicalities, or "Why do we care about error?"** Say we are considering an approximating process, say, a sequence $\{a_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} a_n = L$. In mathematics, too often, we get off too easy and can get away with just answering the question "Does $d(a_n, L)$ go to zero?" It usually doesn't work this way in the real world though! How fast we will approximate actually matters in most applications. It does me no good to write a computer program which outputs numbers converging to π if I don't know how fast we are converging. I can't just way " Maybe $n = 1000000$ is good enough" we could have $a_{1000000} = -30$ and I wouldn't want to use that value if I am trying to figure out how to build a bridge.
A more mathematical note on why we care about our error term, is that when we want to construct more delicate mathematical structures we often have to look under the hood. Note how the derivative

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x},$$

is very close to being a "growth/decay rate" comparison between $d(f(x), f(0))$ as a function of x and the linear function x.

Harder Question. *Suppose we have a function f which is differentiable at 0. Note that $lim_{x\to 0} x = 0$, and since f is differentiable at 0, it is also continuous at 0 and so $lim_{x\to 0} f(x) = f(0)$, and in particular $lim_{x\to 0} d(f(x), 0) = 0$. What, if anything, can we say about the decay rates of $d(f(x), 0)$ and x as x goes to zero?*

- *Can $d(f(x), 0)$ possibly decay slower than x?*
- *Can $d(f(x), 0)$ possibly decay faster than x?*

- *Can $d(f(x), 0)$ possibly decay at the same rate as x?*