Series Convergence

For series w/ positive & negative terms

$$\sum_{n=1}^{\infty} r^n$$

(converges if $-1 < r \le 0$)

(diverges if $r \le -1$).

If $\{a_n\}$, $\{b_n\}$ have positive terms,
and $a_n \le b_n$ for every n, then
if $\sum_{n=1}^{\infty} a_n$ diverges, $\sum_{n=1}^{\infty} b_n$ does as well

" " $\sum_{n=1}^{\infty} b_n$ converges, $\sum_{n=1}^{\infty} a_n$ " " .

$$|a_n| \le |b_n|$$

If $\{a_n\}, \{b_n\}$ have positive terms,

if $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where $c > 0$

and c is finite, then either

$\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ both converge or both diverge

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If $\{a_n\}$ has positive terms, and

$f(x)$ is nonnegative and monotone

decreasing (if $x < y$, then $f(x) \geq f(y)$),

and for every natural number n,

$a_n = f(n)$, then

$\sum_{n=1}^{\infty} a_n$, $\int_1^{\infty} f(x)\,dx$ either both converge or both diverge.

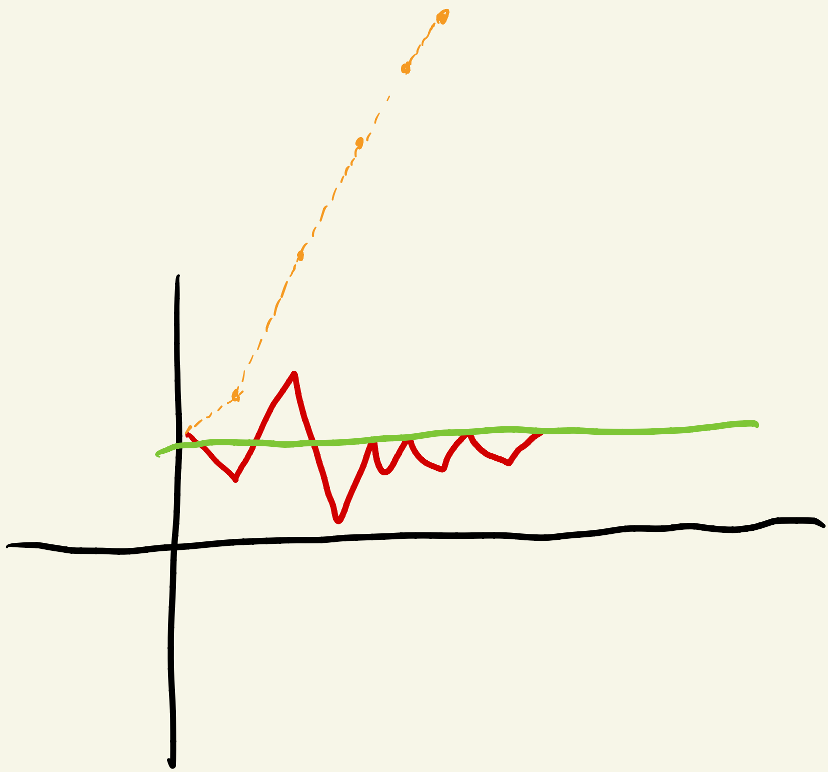
How can we deal w/ negative terms?

$\{b_n\}$ has maybe positive & negative terms.

$$\sum_{n=1}^{\infty} b_n$$

How can we "make" the terms positive?

$$\sum_{n=1}^{\infty} |b_n|$$



Idea: if $\sum_{n=1}^{\infty} |b_n|$ converges, then

$\sum_{n=1}^{\infty} b_n$ converge.

Definition: we say that if $\sum_{n=1}^{\infty} |b_n|$ converges, then $\sum_{n=1}^{\infty} b_n$ converges absolutely.

Thm: If a series converges absolutely, then the series also converges.

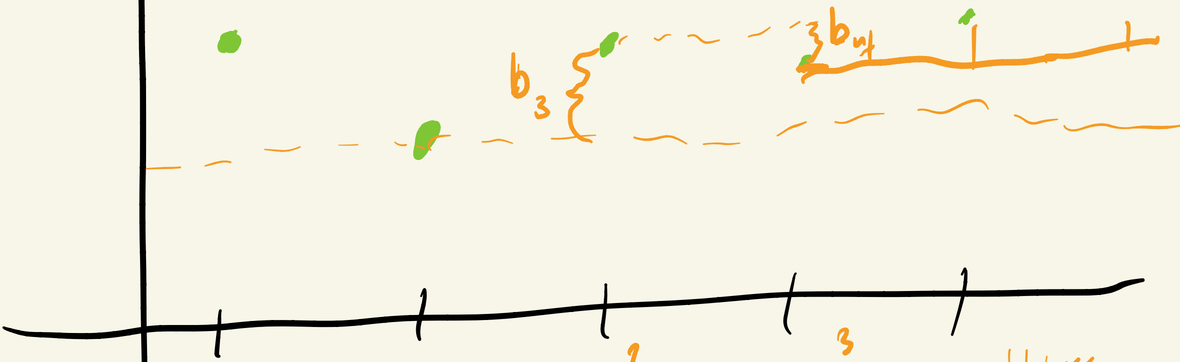
Alternating series, Series of the form

$\sum_{n=1}^{\infty} (-1)^n a_n$ or $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ where $\{a_n\}$ is a sequence with positive terms.

$$\sum_{n=1}^{N} (-1)^n a_n$$

$(-1)^n a_n = b_n$

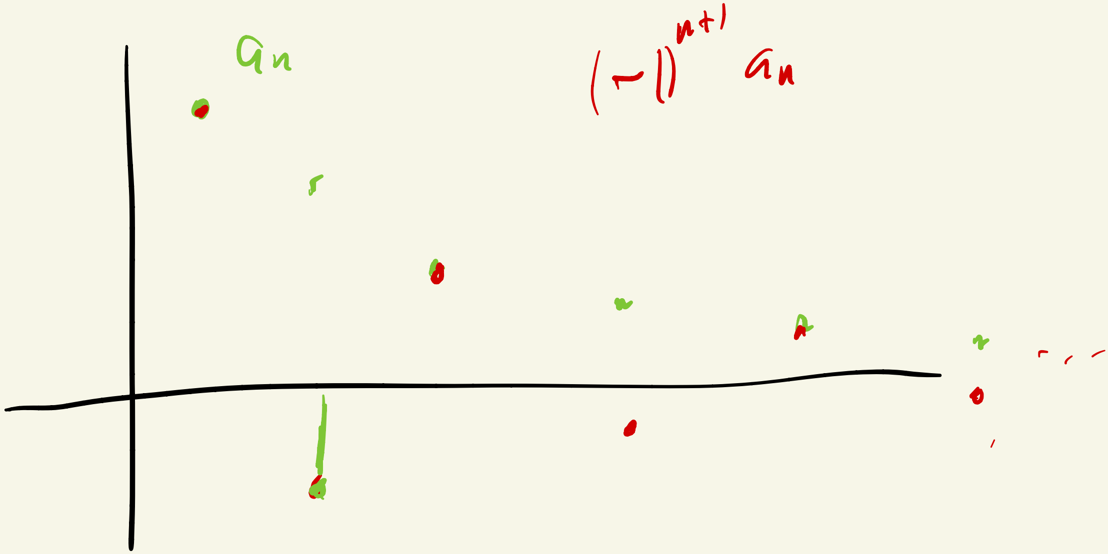
$b_4 \le b_3$



Jump from $\sum_{n=1}^{2} b_n$ to $\sum_{n=1}^{3} b_n$ positive

" " to $\sum_{n=1}^{4} b_n$ is positive.

usually we like decreasing sequence,



From this considering

$$\sum_{n=1}^{\infty} (-1)^n a_n, \quad \text{for any even}$$

m, and $N \geq n$,

$$\sum_{n=1}^{m} (-1)^n a_n \geq \sum_{n=1}^{N} (-1)^n a_n \wedge$$

when this ← exists

&

$$\sum_{n=1}^{m} (-1)^n a_n \geq \sum_{n=1}^{\infty} (-1)^n a_n \quad \text{and } N > n$$

for odd m,

$$\sum_{n=1}^{m} (-1)^n a_n \leq \sum_{n=1}^{N} (-1)^n a_n$$

$$\sum_{n=1}^{m} (-1)^n a_n \le \sum_{n=1}^{\infty} (-1)^n a_n$$

when this exists

for an odd m

$$\sum_{n=1}^{m} (-1)^n a_n \le \sum_{n=1}^{\infty} (-1)^n a_n \le \sum_{n=1}^{m+1} (-1)^n a_n$$

dist between is

a_{m+1}

It seems like things should close in

Thm:
Consider a sequence of positive terms $\{a_n\}$ such that $\{a_n\}$ is monotone decreasing and $\lim_{n\to\infty} a_n = 0$. Then

both
$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converge.

Recall absolute convergence:
Note that $a_n + |a_n|$ has positive terms, and is bounded from above by $2|a_n|$

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then

$\sum_{n=1}^{\infty} 2|a_n|$ converges, and by direct comparison test, $\sum_{n=1}^{\infty} |a_n| + a_n$ converges

By algebraic limit thm,

$$\sum_{n=1}^{\infty} a_n + |a_n| - |a_n| = \left(\sum_{n=1}^{\infty} a_n + |a_n|\right) - \left(\sum_{n=1}^{\infty} |a_n|\right)$$

Thm:

Consider a sequence of positive terms $\{a_n\}$ such that $\{a_n\}$ is monotone decreasing and $\lim_{n\to\infty} a_n = 0$. Then both

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converge.

$$\frac{|\cos(n)|}{n^2} \leq \frac{1}{n^2} \qquad \frac{|\cos(n)|}{n^2}$$

Do the following series converge?

- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges
- $\displaystyle\sum_{n=1}^{\infty} (-1)^n \log(n)$ diverge
- $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$
- $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\log(n)}$ converges $\left\{\frac{1}{n}\right\}$

$\{\log(n)\}$

$\lim_{n\to\infty} (-1)^n \log(n)$ diverges

<u>Thm</u>: If $\sum_{n=1}^{\infty} b_n$ is an alternating series, then if $\{|b_n|\}$ is monotone decreasing and $\lim_{n\to\infty} b_n = 0$, $\sum_{n=1}^{\infty} b_n$ converges. $\left(\lim_{n\to\infty} |b_n| = 0\right)$

$$\frac{1}{\log(n)}$$

Idea: Absolute convergence

For a general series $\sum_{n=1}^{\infty} b_n$ $\left(\sum_{n=5}^{\infty} c_n\right)$, consider instead $\sum_{n=1}^{\infty} |b_n|$ $\left(\sum_{n=5}^{\infty} |c_n|\right)$.

We say $\sum_{n=1}^{\infty} b_n$ converges absolutely if

$\sum_{n=1}^{\infty} |b_n|$ converges.

Fact/Thm: If $\sum_{n=1}^{\infty} b_n$ converges absolutely,

then $\sum_{n=1}^{\infty} b_n$ converges.

Q: If a series $\sum_{n=1}^{\infty} b_n$ converges,

does it have to converge absolutely? $\boxed{\text{NO!}}$

Thm (Alternating series test):

If $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n b_n$ is an alternating series,

(that is, there is a sequence $\{b_n\}$

with only non-negative terms $b_n \geq 0$,
such that $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$)

such that $|a_n| \geq |a_{n+1}|$ ($\{a_n\}$ is monotone decreasing), $b_n \geq b_{n+1}$, and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} |a_n| = \lim_{n\to\infty} b_n = 0,$$

then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n b_n$ converges.

Recall $\sum_{n=1}^{\infty} \frac{1}{n}$ & Converse to Divergence test is not true!!!

$\lim_{n\to\infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$\frac{1}{n}$ is monotone decreasing

↓ turn into alternating series

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ by AST (alternating series test),

we know $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges,

but since $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges,

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ does not converge absolutely

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Main tool for showing divergence for series w/ negative terms is

- Divergence test

Given $\sum_{n=1}^{\infty} b_n$, show $\lim_{n\to\infty} b_n \neq 0$,

or $\lim_{n\to\infty} b_n$ does not exist.

Ratio test

(If we consider $\sum b_n$, and

$|b_n|$ either decays "like or faster

than $\frac{1}{r^n}$ $(r<1)$" or if b_n

decays "much slower than $\frac{1}{r^n}$ $r<1$"

ex. $\frac{1}{r^n}$ $r>1$ then this

test is the way to go to determine

absolute convergence).

Ratio Test:

Suppose we have a series $\sum b_n$,

- If $\lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| = L$ $L < 1$

then $\sum b_n$ converges absolutely,

- If $\lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| = L$ $L > 1$, then

$\sum b_n$ diverges,

- If $\lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| = 1$ the theorem

says nothing. We say the "ratio test is inconclusive."

$$\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = L \qquad L<1$$

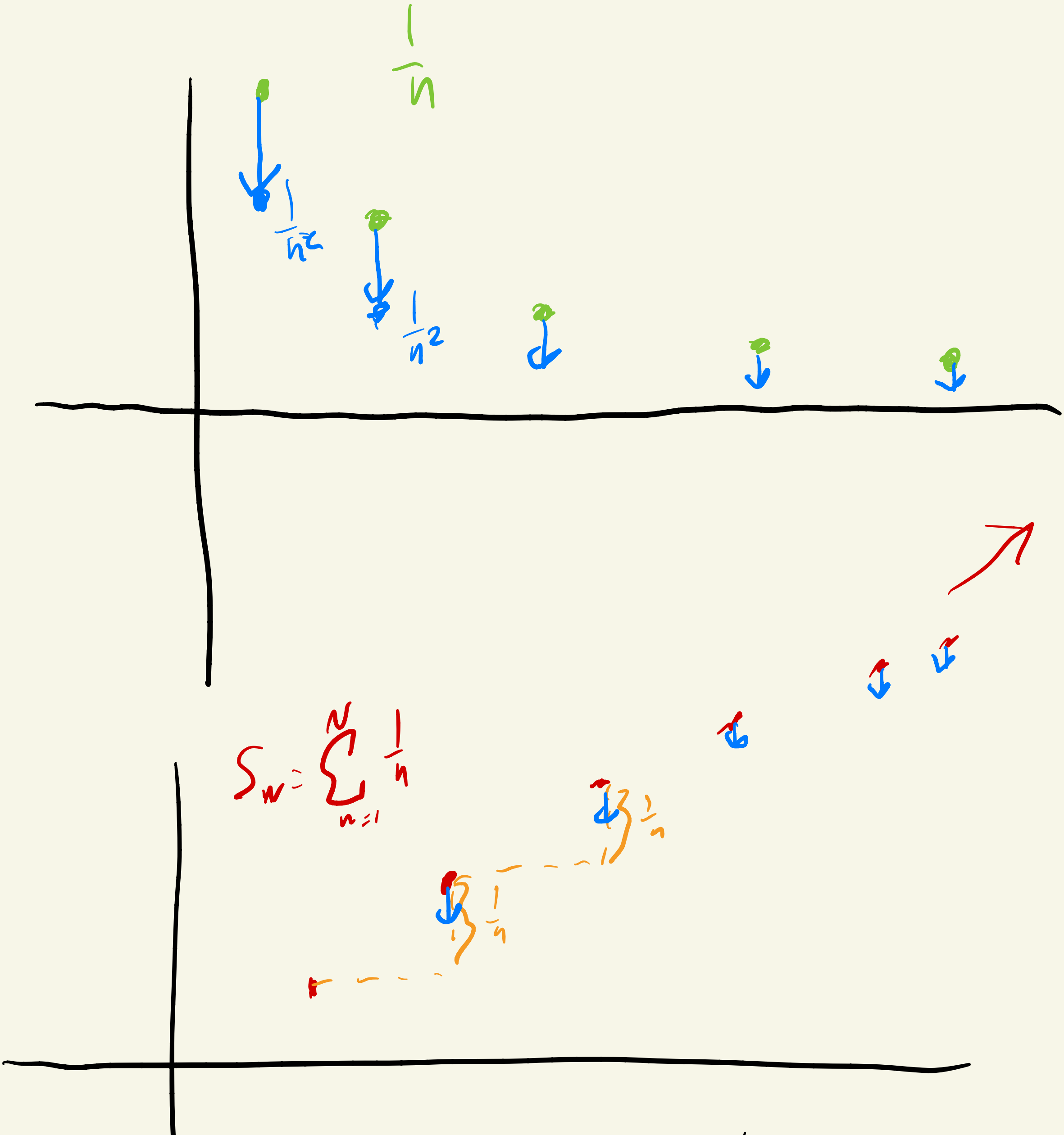
We can choose $L<\lambda<1$, so that

eventually $b_{n+1} < b_n\lambda$

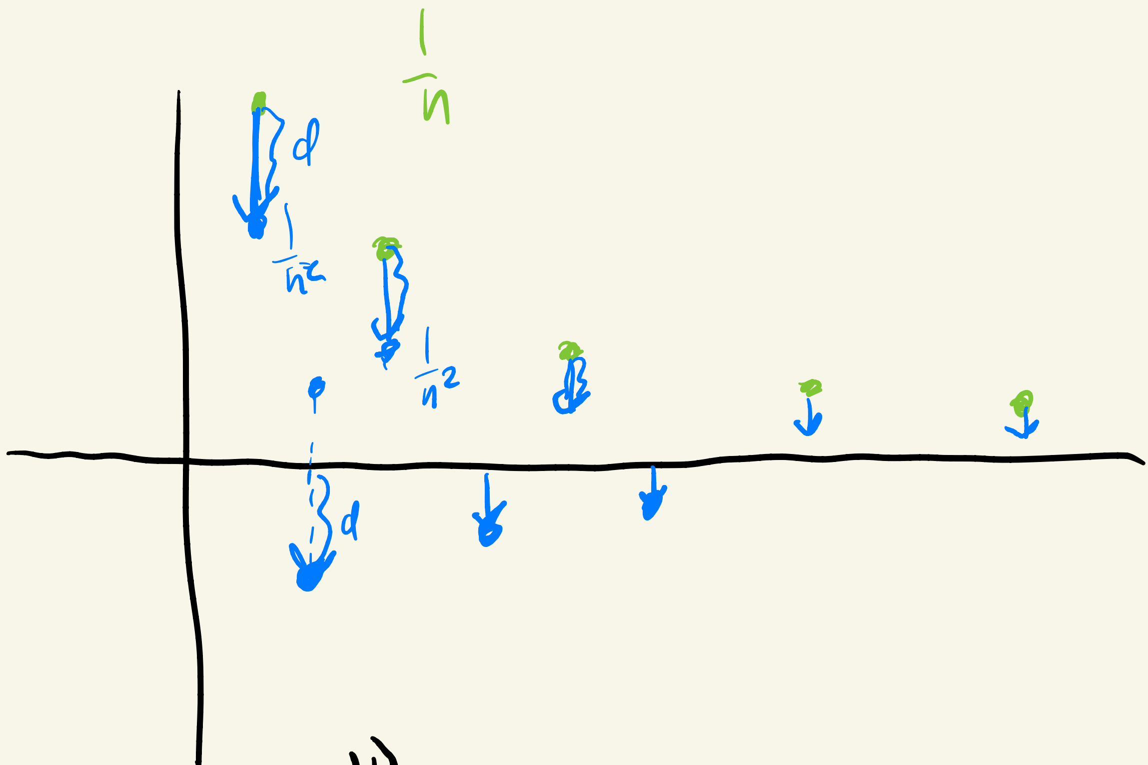
$$b_{n+2} < b_{n+1}\lambda < b_n\lambda^2$$

$$\vdots$$

$$b_{n+m} < b_n\lambda^m$$



If we shrank these enough we could converge



Think: (roughly) "If an alternating series converges, but does not converge absolutely, then a_{n+1} shrinks a_n enough so that

$$(a_1+a_2)+(a_3+a_4)+(a_5+a_6)+(a_7+a_8)+\dots \text{ converges"}$$

think of these "$a_n + a_{n+1}$" terms as the series terms

Ratio Test:

Suppose we have a series $\sum b_n$,

- If $\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = L$ $L<1$

  then $\sum b_n$ converges absolutely,

- If $\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = L$ $L>1$, then

  $\sum b_n$ diverges,

- If $\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = 1$ the theorem

  says nothing. We say the "ratio test is inconclusive."

$$\sum_{n=1}^{\infty} \frac{1}{n!} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \sum_{n=1}^{\infty} \frac{1}{n} \leftarrow \text{inconclusive}$$

Converges by ratio test

$$\lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{n!}{(n+1)\,n!}$$

$$= \lim_{n\to\infty} \frac{1}{n+1}$$

$$= 0 < 1$$

$$\lim_{n\to\infty} \left| \frac{\frac{1}{n+1}}{n} \right|$$

$$= \lim_{n\to\infty} \frac{n}{n+1} = 1$$

$$\lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)^2}}{n^2} \right| = \lim_{n\to\infty} \frac{n^2}{(n+1)^2}$$