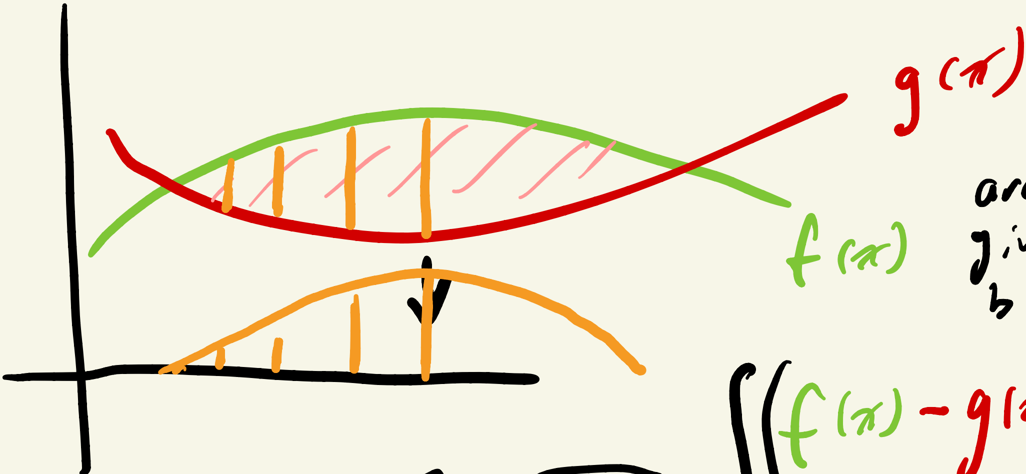
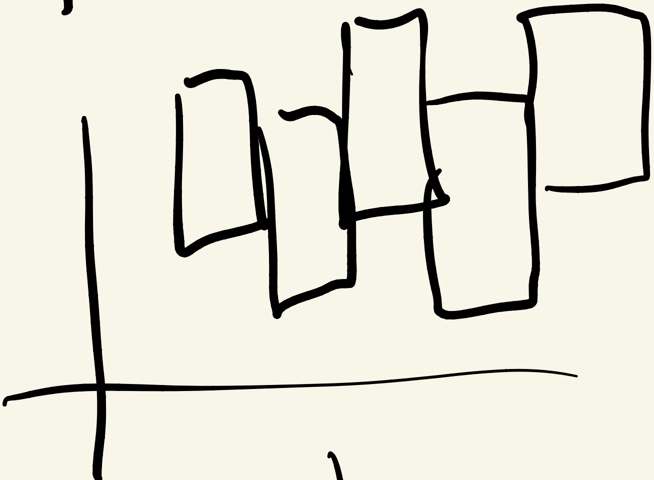
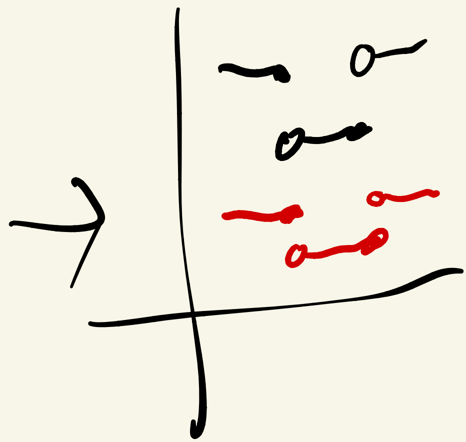
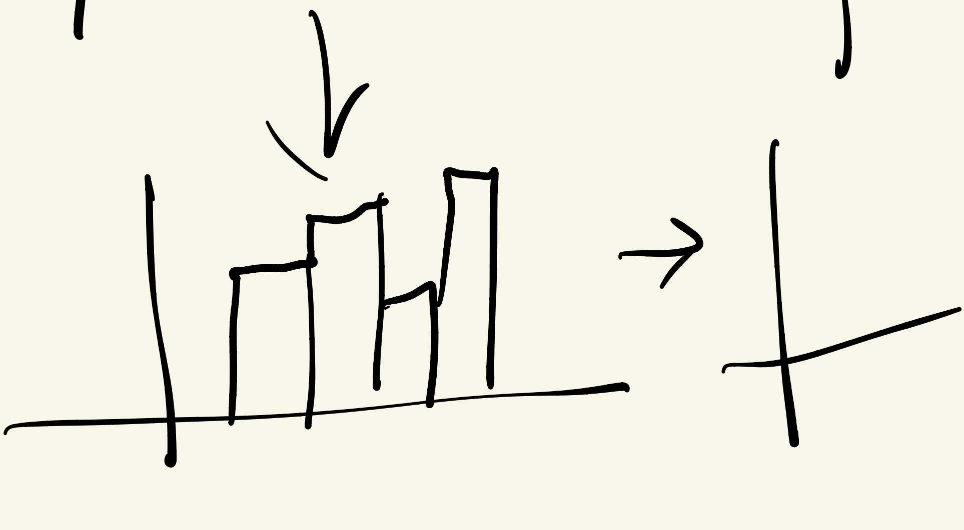


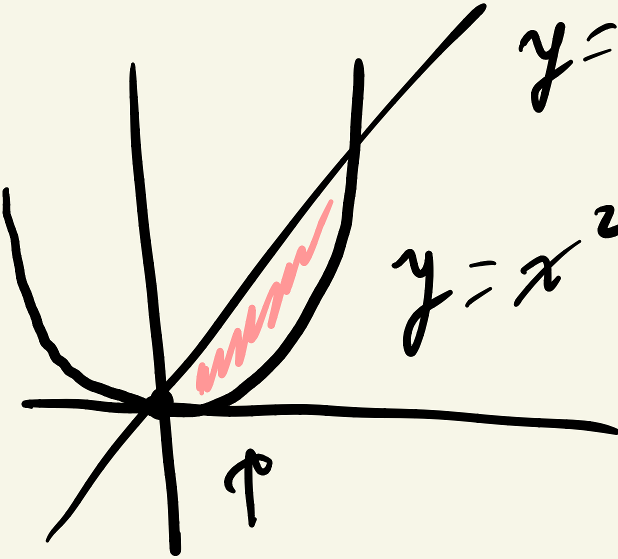
area given by

$$\int \left(f(x) - g(x)\right) dx$$









$y = x = g(x)$

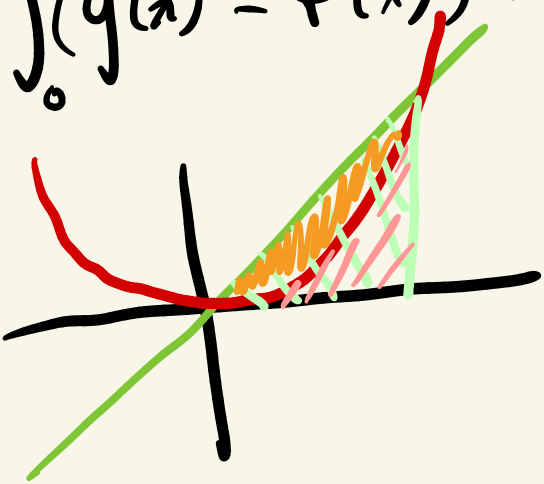
$y = x^2 = f(x)$

when does $g(x) = f(x)$?

$$\boxed{\int_0^1 (x - x^2)\,dx}$$

$x = 0, 1$

$$\int_0^1 (g(x) - f(x))\,dx = \int_0^1 g(x)\,dx - \int_0^1 f(x)\,dx$$



Growth rates

Given functions $f(x)$, $g(x)$, we will be considering

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \quad \text{or} \quad \lim_{x\to\infty} \frac{g(x)}{f(x)}$$

In particular, we usually consider functions f, g which are positive, & either increasing or decreasing

language:

decreasing

$0 < \lim_{x \to \infty} \frac{f(x)}{g(x)} < \infty$.

we say
f shrinks at the same rate as g

$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$

f shrinks faster
g shrinks slower

$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty$

g shinks faster
f shrinks slower

increasing

$0 < \lim_{x \to \infty} \frac{f(x)}{g(x)} < \infty$

here
"grows at the same rate"

$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$

g grows faster
f grows slower

$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty$

f grows faster
g grows slower

why not

$$\lim_{x \to \infty} (f(x) - g(x)) ?$$

$$\lim_{x \to \infty} \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \to \infty} \frac{x}{x^2}$$

$$= \lim_{x \to \infty} \frac{1}{x}$$

$$= 0$$

$$\int_1^\infty \frac{1}{x^2} dx$$

$$\int_1^\infty \frac{1}{x} dx$$

Consider x^n $\frac{1}{x^n}$

Suppose $n > m$

Compare x^n x^m

$$\lim_{x \to \infty} \frac{x^n}{x^m} = \lim_{x \to \infty} x^{n-m}$$

$$= \infty$$

x^n grows faster than x^m

Compare $\frac{1}{x^n}$ & $\frac{1}{x^m}$

$$\lim_{x\to\infty} \frac{\left(\frac{1}{x^n}\right)}{\left(\frac{1}{x^m}\right)} = \lim_{x\to\infty} \frac{x^m}{x^n}$$

$$= \lim_{x\to\infty} x^{m-n}$$

$$= 0$$

$\frac{1}{x^n}$ shrinks faster than $\frac{1}{x^m}$

$\boxed{\text{Ex! Show that if } f \text{ grows faster than } g\text{, then } \frac{1}{f} \text{ shrinks faster than } \frac{1}{g}}$

How does this relate to improper integrals

Say we have positive functions f, g, h, such that $\int_a^\infty f(x)\,dx$ converges, $\int_a^\infty g(x)\,dx$ diverges. The following statements relate to convergence/divergence of $\int_a^\infty h(x)\,dx$.

- If $\lim_{x\to\infty} \frac{h(x)}{f(x)} < \infty$ then $\int_1^\infty h(x)\,dx$ converges

- If $0 < \lim_{x\to\infty} \frac{h(x)}{g(x)}$ then $\int_1^\infty h(x)\,dx$ diverges.

$$\int_1^\infty \frac{\ln(x)}{x^3}\,dx$$ Compare with $\frac{1}{x^n}$ (hoping $n > 1$)

Compare with $\frac{\ln x}{x} \cdot \frac{1}{x^2}$

$$\lim_{x \to \infty} \frac{\frac{\ln(x)}{x^3}}{\frac{1}{x^n}} = \lim_{x \to \infty} \frac{\ln(x)}{x^{3-n}}$$

If $n < 3$, then L'Hopital

$$= \lim_{x \to \infty} \frac{\frac{1}{x}}{(3-n) \cdot x^{2-n}}$$

$$\int_1^\infty \frac{\ln(x)}{x^3}\,dx$$

Converges

because

$$\int_1^\infty \frac{1}{x^2}\,dx$$

does.

$$= \lim_{x\to\infty} \frac{1}{(n-3)\,x^{3-n}}$$

If $n<3$

then

$$\lim_{x\to\infty} \frac{1}{(n-3)\,x^{3-n}}$$

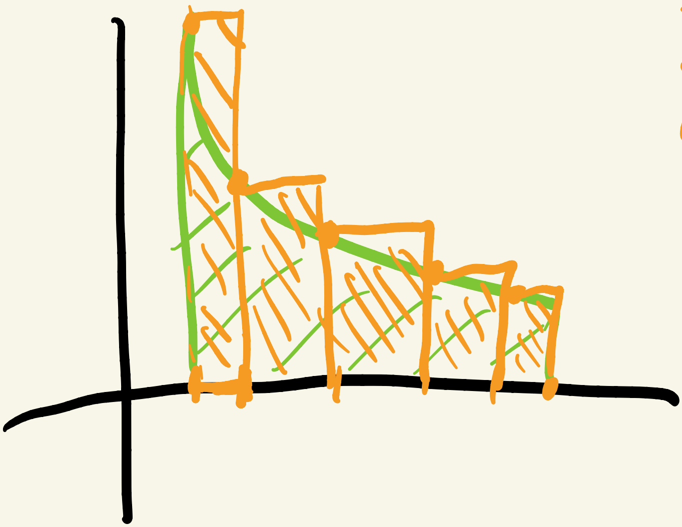
$$= 0$$

$\log(x)$

$\log(\log(x))$

$\log(\log(\log(x)))$.

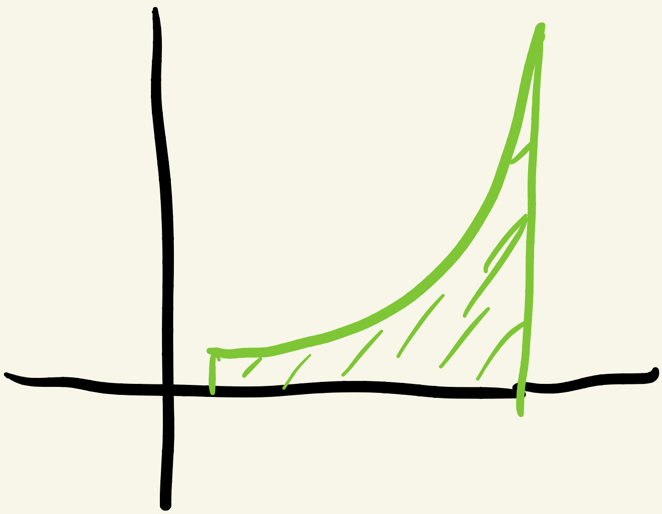
Left end point



If the orange (than green) area was greater

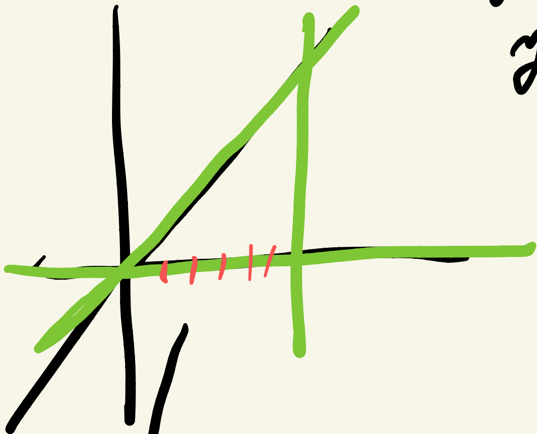
it would be an overapproximation

if less than green, we call it an underapproximation

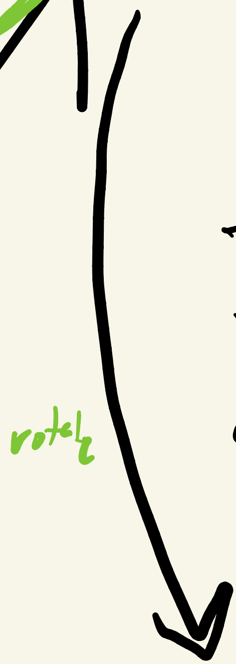


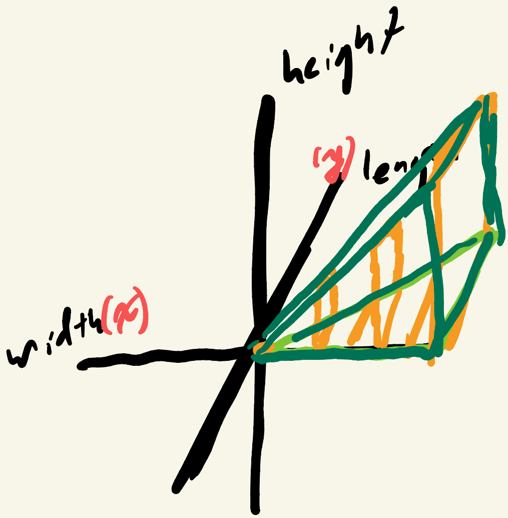
region bounded by

$y=x$, $x=1$ $y=0$



for every line in the region parallel to y axis, define a square cross section



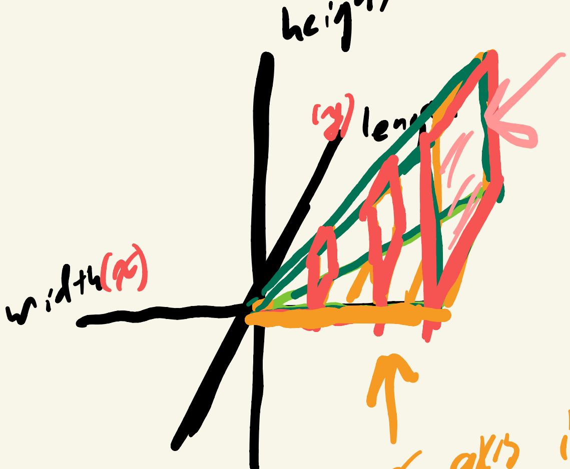


Trying to compute Vol

Subdivide x, appx?

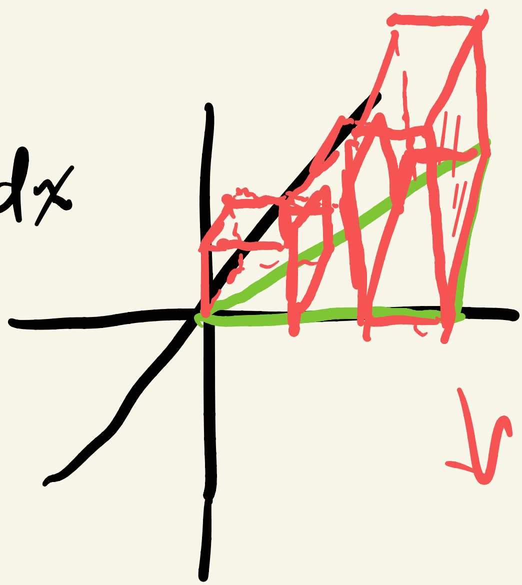
$S(x)$

function which gives area of cross section at x

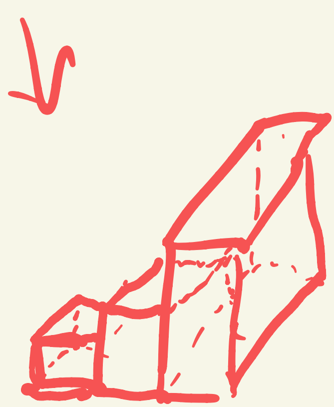


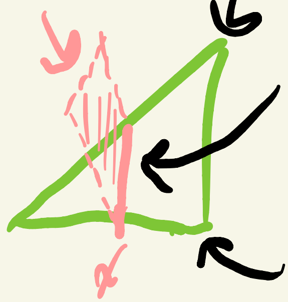
x axis interval $[a,b]$

$$\int_a^b S(x)\,dx$$



$$\int_0^1 x^2\,dx$$





length is $x - 0 = x$

area of cross section is x^2

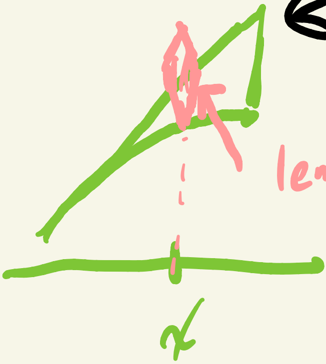
Consider the region bounded by

$y = x$, $y = \sin(x)$, $x = \frac{\pi}{2}$

with square cross sections having one leg parallel to y axis

Need an interval ($[0, \frac{\pi}{2}]$)

function which gives cross sectional area at points on that interval



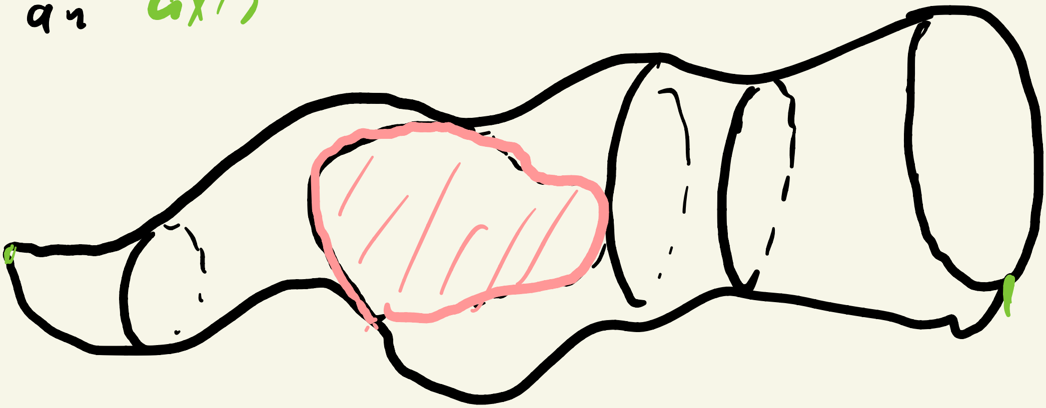
length of side is $|x - \sin(x)|$

area of cross section is

$$\boxed{(x-\sin x)^2}$$

$$\int_0^{\frac{\pi}{2}} (x-\sin x)^2 \, dx$$

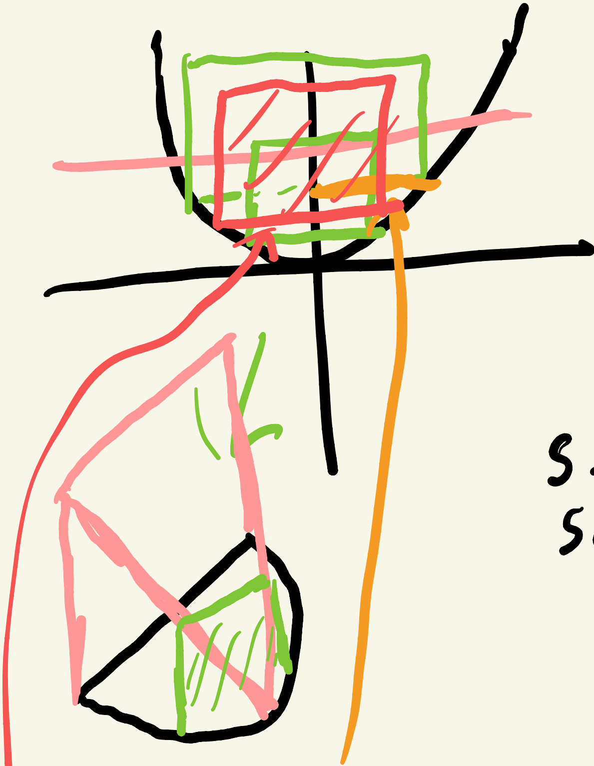
Say we have a region and an axis



Say at a given w, the area of a cross section being

$$(3w\sin(w) - e^w)$$

$$\int_3^{100} (3w\sin(w) - e^w)\,dw$$



region bounded by $y = x^2$

$y = 1$

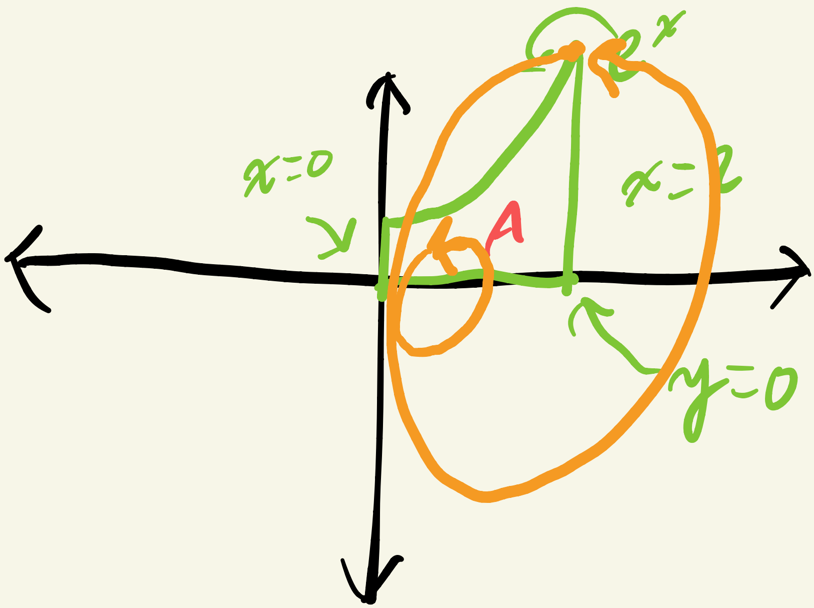
Solid defined by Square cross sections parallel to x-axis

dist is x at height y^2

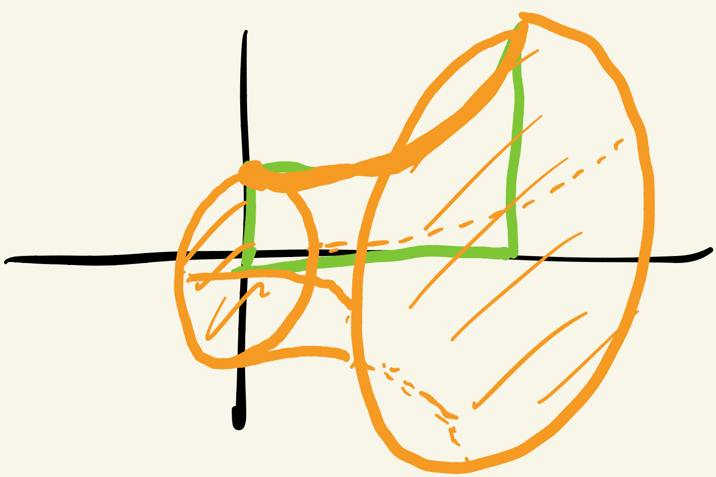
so dist at $\boxed{y \text{ is } \sqrt{x}}$

dist at y is $2\sqrt{y}$ area is $4y = (2\sqrt{y})^2$

Volume is $$\int_0^1 (2\sqrt{y})^2 \, dy$$



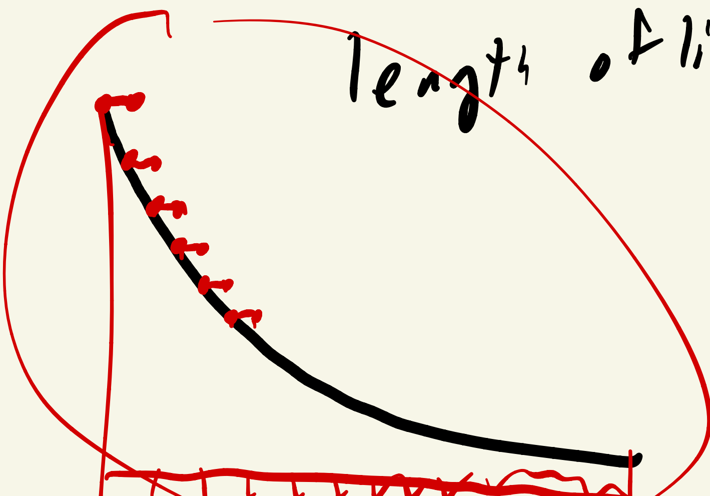
- define a solid by rotating A about the x-axis

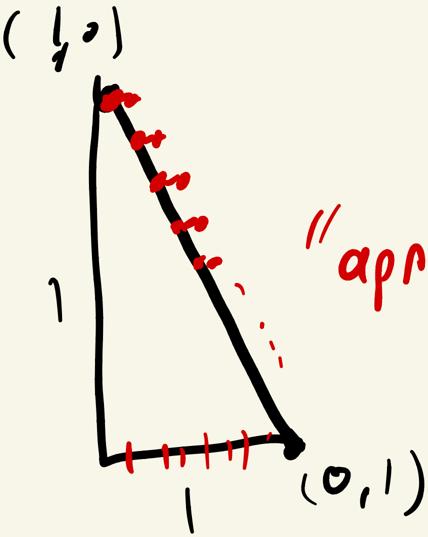


define a solid by rotating this region about the x axis

Warning about approximation

length of line



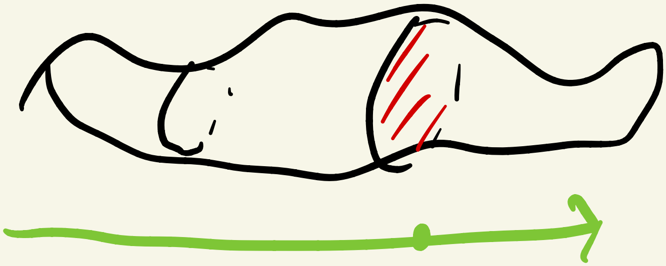


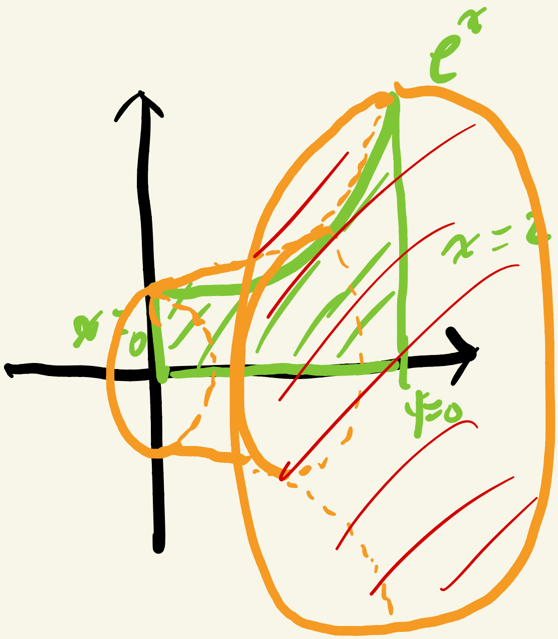
"approximation" is always 1

length is $\sqrt{2}$

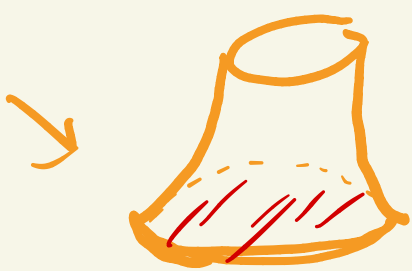
Volume

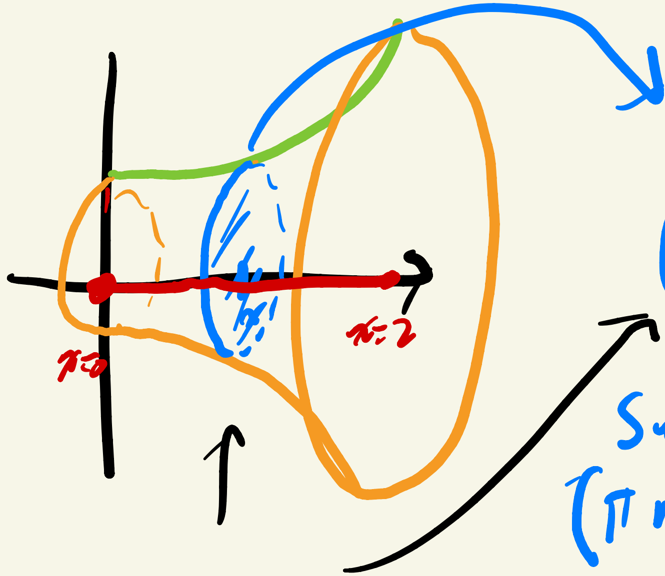
- pick an axis
- integrate surface area over the axis

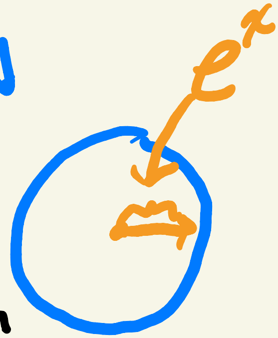




Form a solid by rotating this region about the x-axis







Surface area is

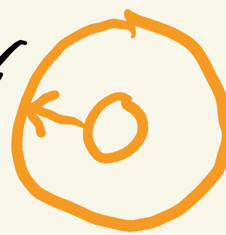
(πr^2)

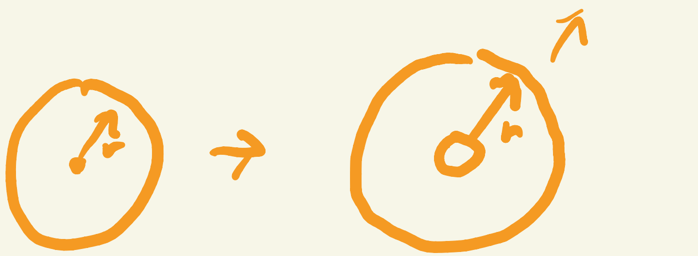
$$\pi (e^x)^2 = \pi e^{2x}$$

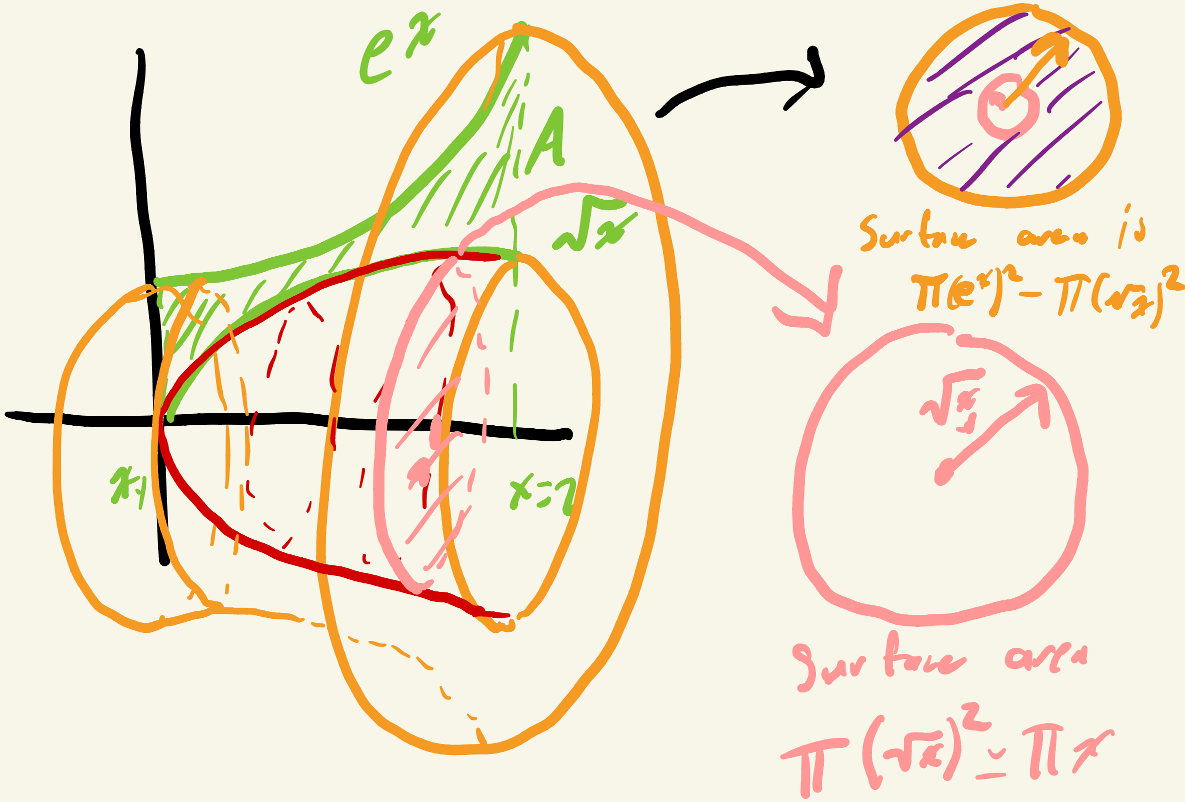
Cross section looks like a generalized "washer"

Volume will be

$$\int_0^2 \pi (e^x)^2 \, dx$$







For <u>surface area</u> of the region A

(Not volume of solid or surface area of cross sections)

$$\int_0^2 (e^x - \sqrt{x})\,dx$$

Volume should be

$$\int_0^2 \pi (e^x)^2 \, dx - \text{"what is scooped out"}$$

Volume of "what is scooped out" is

$$\int_0^2 \pi (\sqrt{x})^2 \, dx$$

---

overall volume is

$$\int_0^2 \pi (e^x)^2 \, dx - \int_0^2 \pi (\sqrt{x})^2 \, dx$$

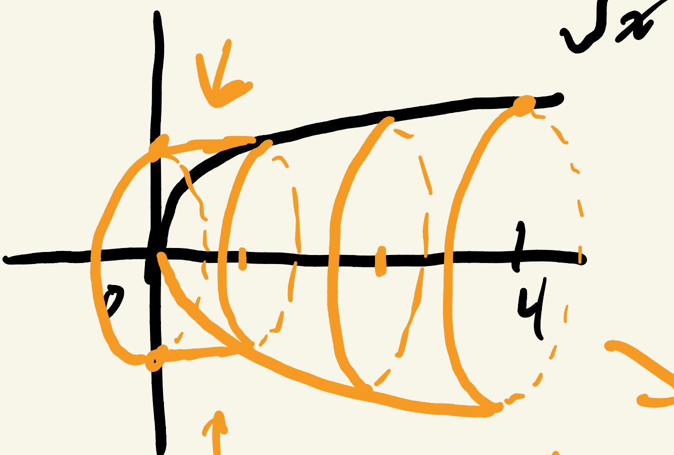
$$= \int_0^2 \left(\pi (e^x)^2 - \pi (\sqrt{x})^2 \right) dx$$

It is not!!!

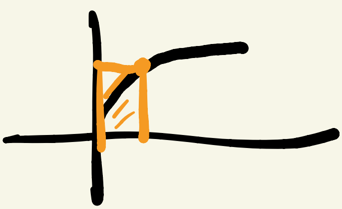
$$\cancel{\int_0^2 \pi \left(e^x - \sqrt{x}\right)^2 dx}$$

No!!

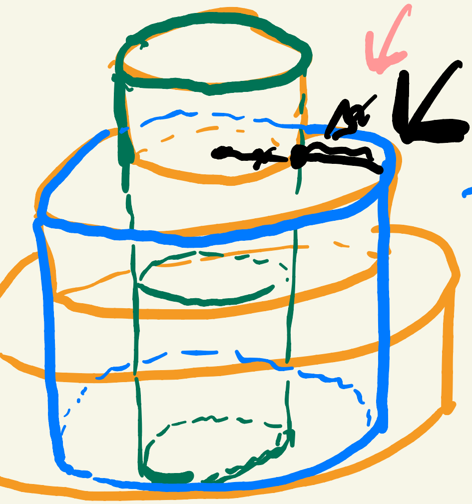
$$\int_0^4 \pi(\sqrt{x})^2\,dx$$

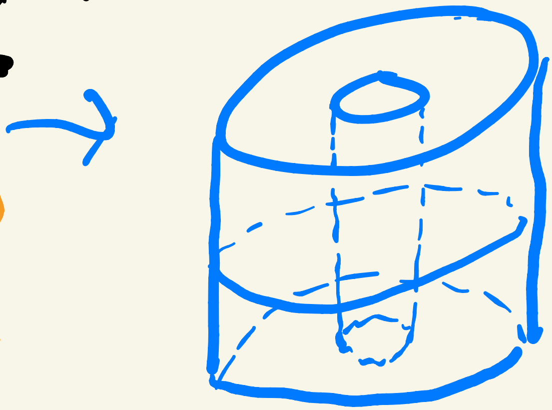


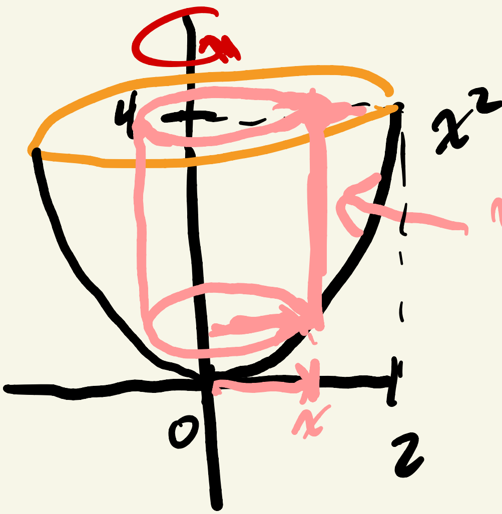
cylinder instead of



$$\left(\pi (x+\Delta x)^2 - \pi (x)^2\right) \cdot f(x)$$







what is the surface area of the cylinder?

$2\pi x \cdot \text{height}$

$\downarrow$

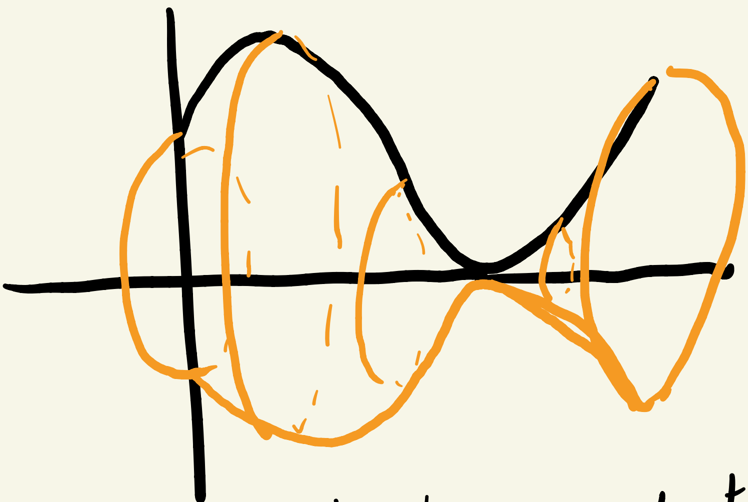
$4 - x^2$

$2\pi x(4-x^2)$

$$\int_0^2 2\pi x(4-x^2)\,dx$$

check in

Consider the solid formed by rotating $\sin(x)+1$ from $x=0$ to $x=2\pi$ about the x-axis



which integral represents the volume?

A $\int_0^2 \cos(x)+1 \, dx$

B $\int_0^{2\pi} \sin(x)+1 \, dx$

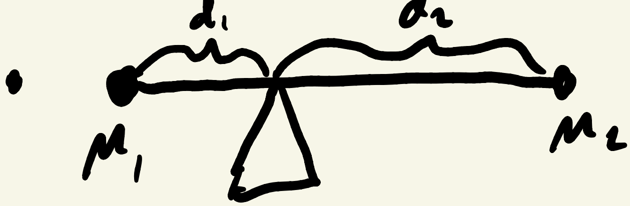
C $\int_0^{2\pi} \pi(\sin(x)+1)^2 \, dx$

D $\int_0^{2\pi} 2\pi(\sin(x)+1) \, dx$

MWF in person option

T Th all online

Center of mass / Balance point

• 

This will balance if $m_1 d_1 = m_2 d_2$

(or if we are considering signed distance so d_1 is negative, if

$$m_1 d_1 + m_2 d_2 = 0$$
)

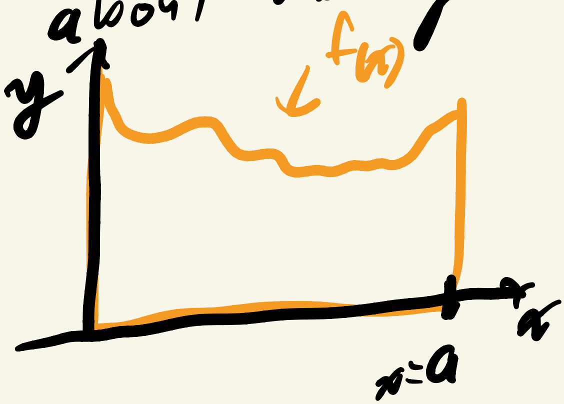
$F_1 = g \cdot m_1$ $F_2 = g \cdot m_2$ $F_1 d_1 + F_2 d_2 = 0$

• We can think of Mass·distance as being a measure of "rotational pressure" or almost as torque

- Find "rotational pressure" (book calls this moments) with rotational axis being x or y axis

The center of mass will be the point where, if all our mass was concentrated there, the "rotational pressure" would be the same.

- "Rotational pressure" or "$M \cdot d$" about the y axis is



$$\int_0^a x f(x)\,dx$$

The "balance point" for axis parallel to y-axis will be the line

$$\boxed{x = \frac{\cancel{\rho}\int_0^a x\, f(x)\,dx}{\cancel{\rho}\int_0^a f(x)\,dx}}$$

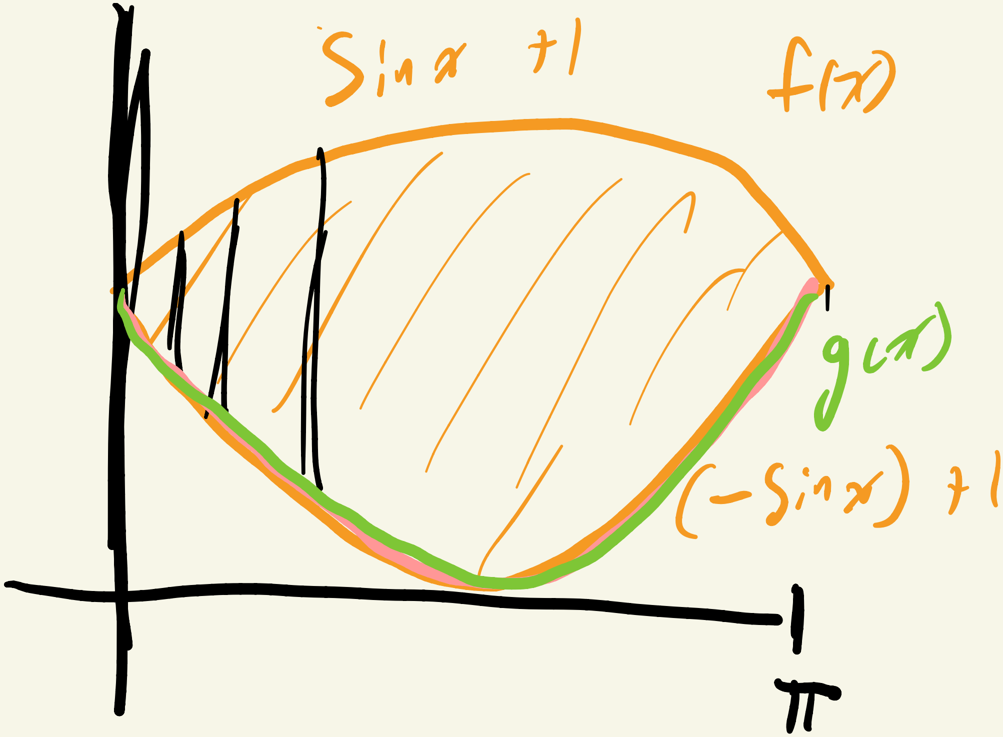
It ρ is density

"M d" about x-axis is

$$\int_0^a \frac{1}{2}\left(f(x)^2\right) dx$$

So balance at

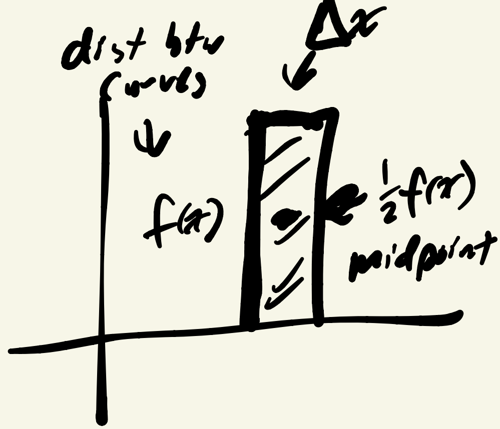
$$\boxed{y = \frac{\cancel{\rho}\int_0^a \frac{1}{2}\left(f(x)^2\right) dx}{\cancel{\rho}\int_0^a f(x)\,dx}}$$



x balance pt

$$\frac{\int_0^{\pi} x\left(\left(\sin x+1\right)-\left(\left(-\sin x\right)+1\right)\right)dx}{\int_0^{\pi}\left(\sin x+1-\left(\left(-\sin x\right)+1\right)\right)dx}$$

y balance point



$$\frac{\int_0^a \frac{1}{2}(f(x))^2\,dx}{\int_0^a f(x)\,dx}$$

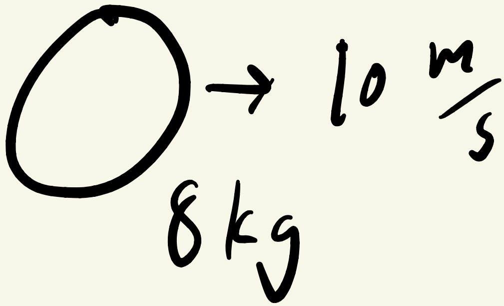
all surface area

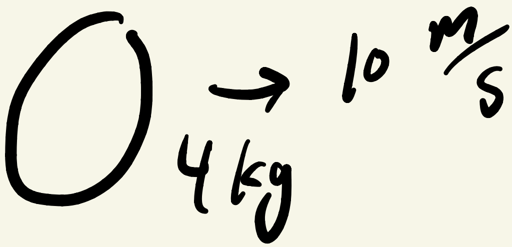
$f(x)-g(x)$

$\frac{f(x)+g(x)}{2}$

$$\int_0^{\pi} (\text{dist btw curves})\cdot(\text{midpoint})\,dx$$

$$=\int_0^{\pi}\left(\sin x+1-((-\sin x)+1)\right)(1)$$

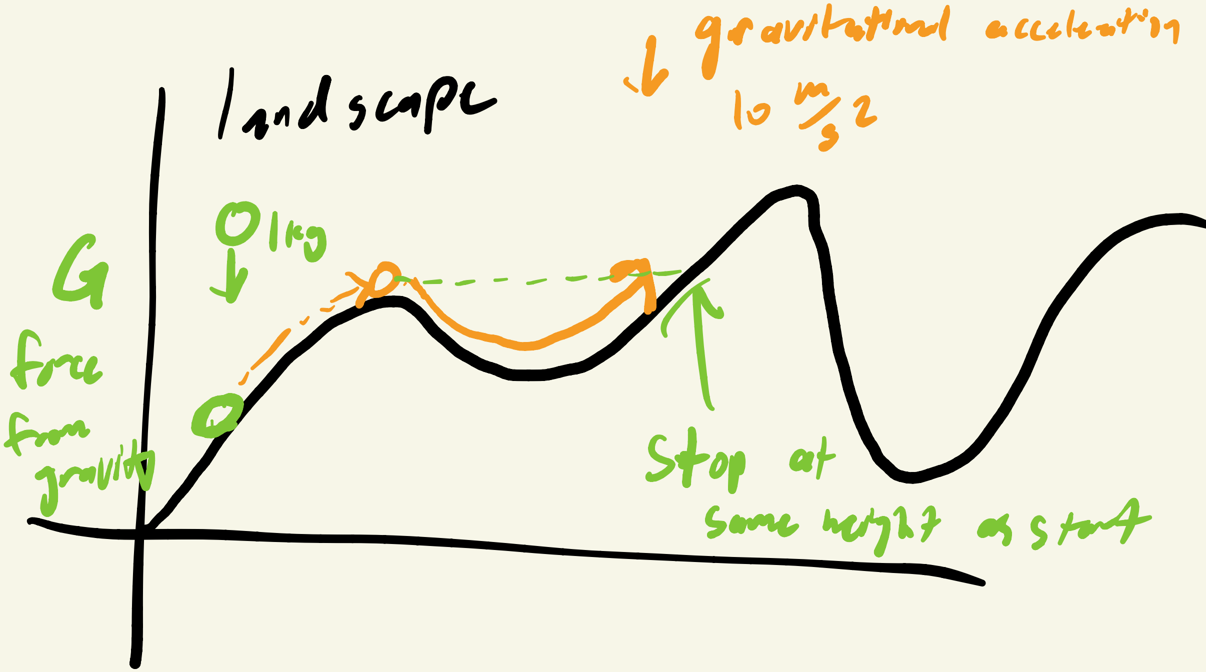




- Work will be a measure of transfer of energy

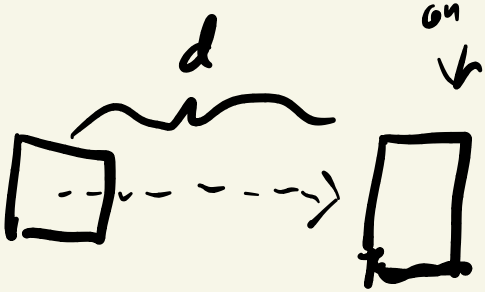
- we will want

$$\text{Force} \cdot \text{distance}$$

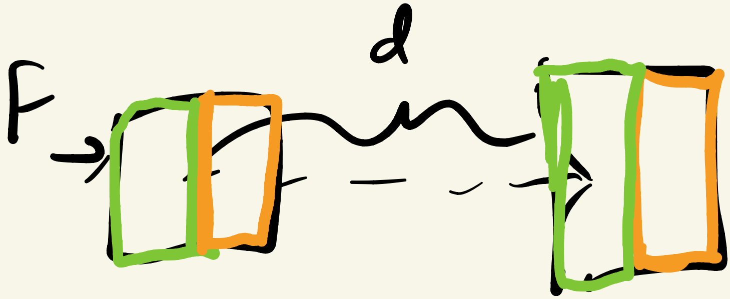


work will be computed by taking a force put on an object and multiplying by the distance across which that force is applied

constant force
→ F

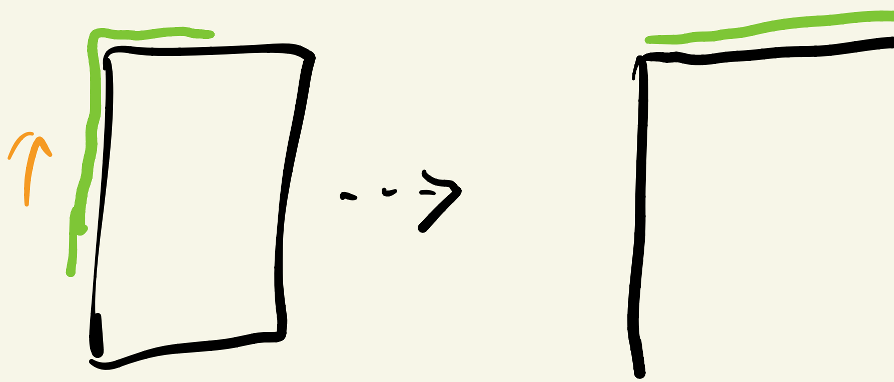


work done on the box to this point is Fd

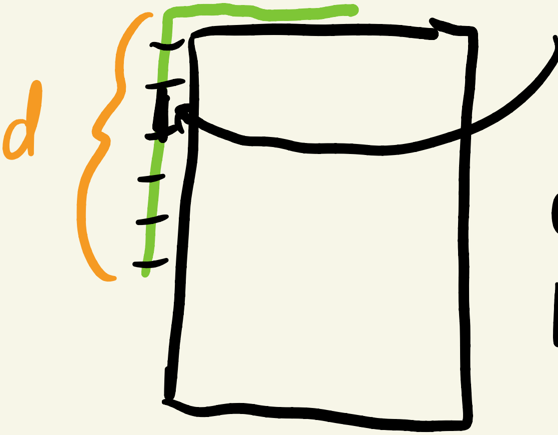


$$\frac{F}{2}\cdot d + \frac{F}{2}\cdot d = F\cdot d$$

Say I have a rope hanging off a building

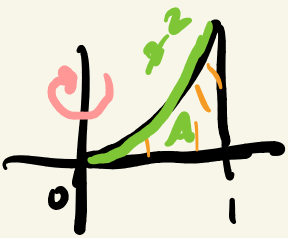


How much work does it take to pull up the rope



approx how much work is done on this portion?

Consider region A



rotate A about the y-axis to form a solid

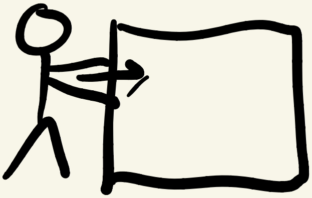
Which of the following gives the volume for our solid?

A $\displaystyle\int_0^1 \pi (x^2)^2 \, dx$

B $\displaystyle\int_1^1 \pi (\sqrt{y})^2 \, dy$

C $\displaystyle\int_0^1 2\pi \; x \cdot x^2 \, dx$

D $\displaystyle\int_0^1 2\pi \, x(1-x^2) \, dx$

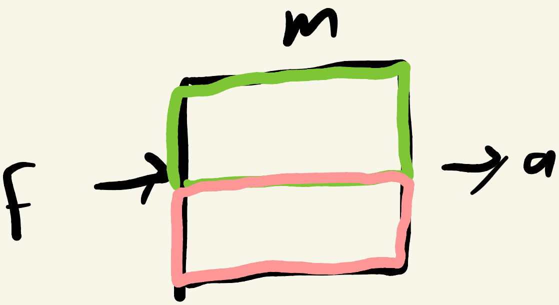


$$F = ma$$

force

mass

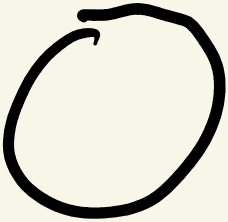
acceleration



a is the same for each box,

m is $\frac{1}{2}$

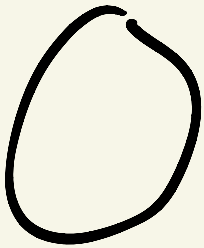
so F on each smaller box will be $\frac{1}{2}$ the original



2kg

Our idea for <u>work</u> was

4kg



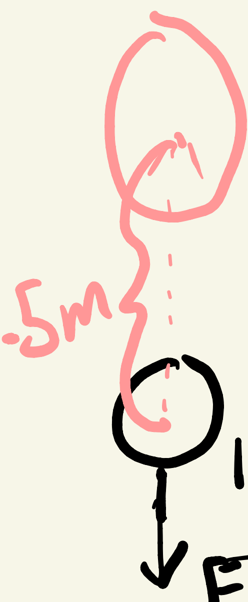
(Force exerted) · (dist across which the force is exerted)

(In the direction of the force)

$F \cdot d$

gravitational acceleration

$a = 10 \frac{m}{s^2}$



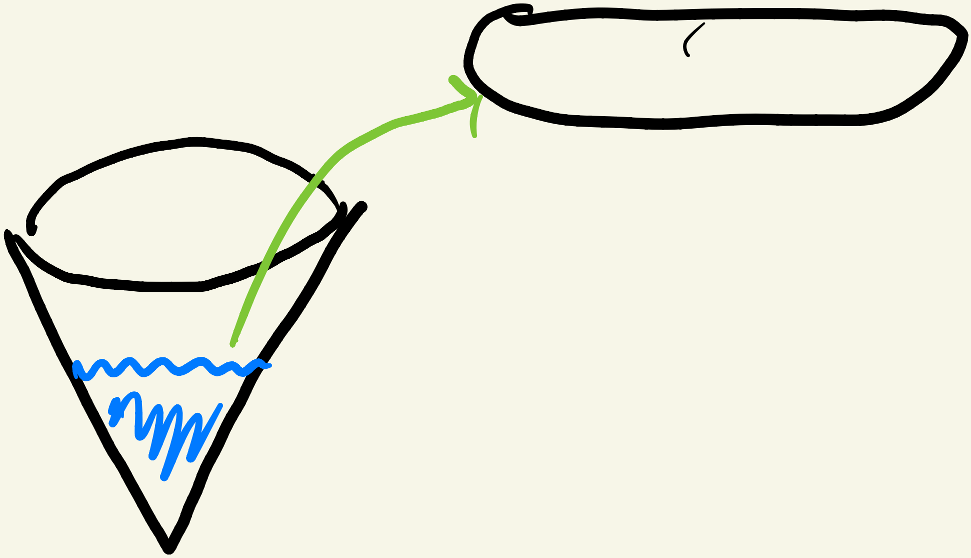
1kg

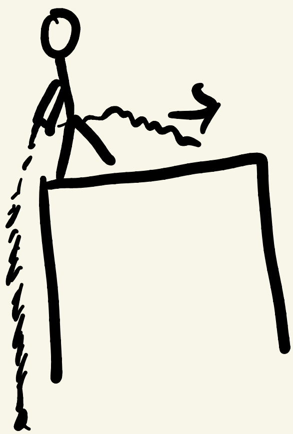
$F = 10 \frac{m}{s^2} \cdot kg = 10\,N$

10

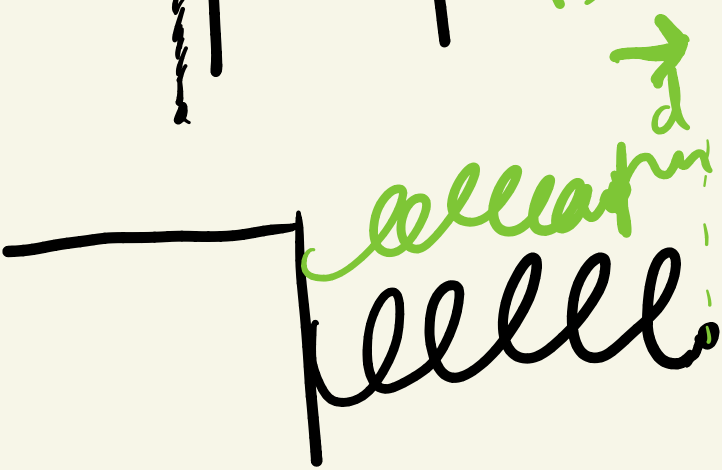
work done is

$F \cdot d = 10 \cdot (-5) = -50$



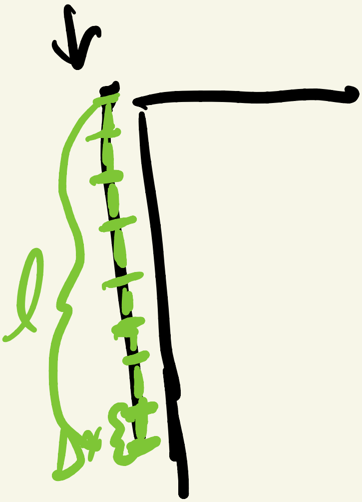


$F = dk$



density for rope is $\boxed{\text{kg}/\text{m}}$

$g = 10 \frac{m}{s^2}$



you might see

$g = 9{,}8 \frac{m}{s^2}$

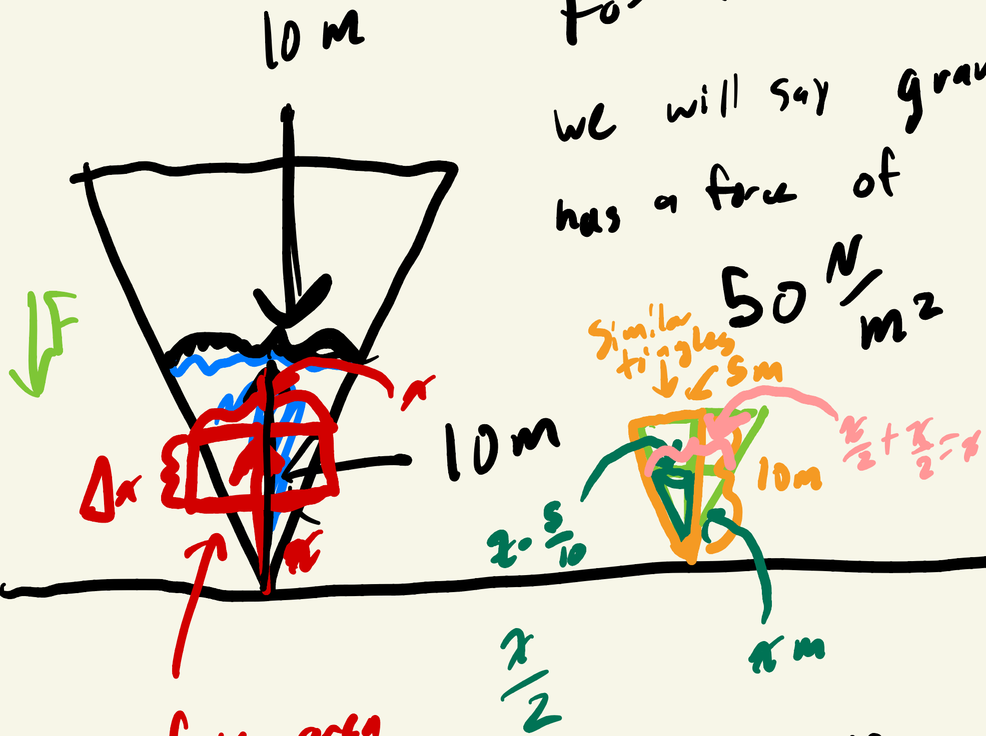
Δx Force on small segment is

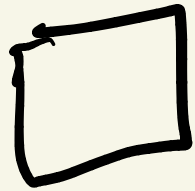
$$10 \cdot \Delta x^{(m)} \cdot 1^{(\text{kg}/m)} = 10 \cdot \Delta x$$

$$\int_0^l \underline{\quad ? \quad} = \int_0^l 10 (x)\, dx$$

$$= \int_0^l 10 \cdot (-(l - x))\, dx$$

For the fluid, we will say gravity has a force of $50 \, N/m^2$





Surface area

$\Delta x \cdot x$

$$\text{work} = \int_0^{10} x \cdot x \cdot 50 \, dx$$

$$= \int_0^{10} 50x^2 \, d\cancel{x}$$

- Force

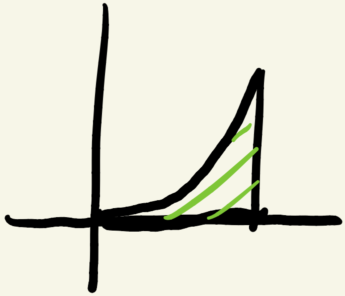
$$Sa \cdot 50 = \boxed{\Delta x \cdot x \cdot 50}$$

- dist traveled will be x
  - appx work will be $x \cdot \Delta x \cdot x \cdot 50$

Check in will be take home

Due before start of quiz

Q: Determine the center of mass for the surface bounded by $y = x^3$, $y = 0$, and $x = 1$



Bonus: Consider the following tank of fluid where gravity acts on the fluid with a force of $1\,N/m^2$.

- Determine the work done by gravity to drain the tank.

- Determine the height, H, of the center of mass of the fluid in the tank above ground level. Determine the total force, F, acting on the fluid in the tank. What is $F \cdot H$? Why is this nice?

$\int$ acceleration $\rightarrow$ Velocity

$\int$ velocity $dt \rightarrow$ position (dist traveled)

$\int$ work per unit length $\rightarrow$ work

velocity is $\boxed{\text{dist per unit time}}$

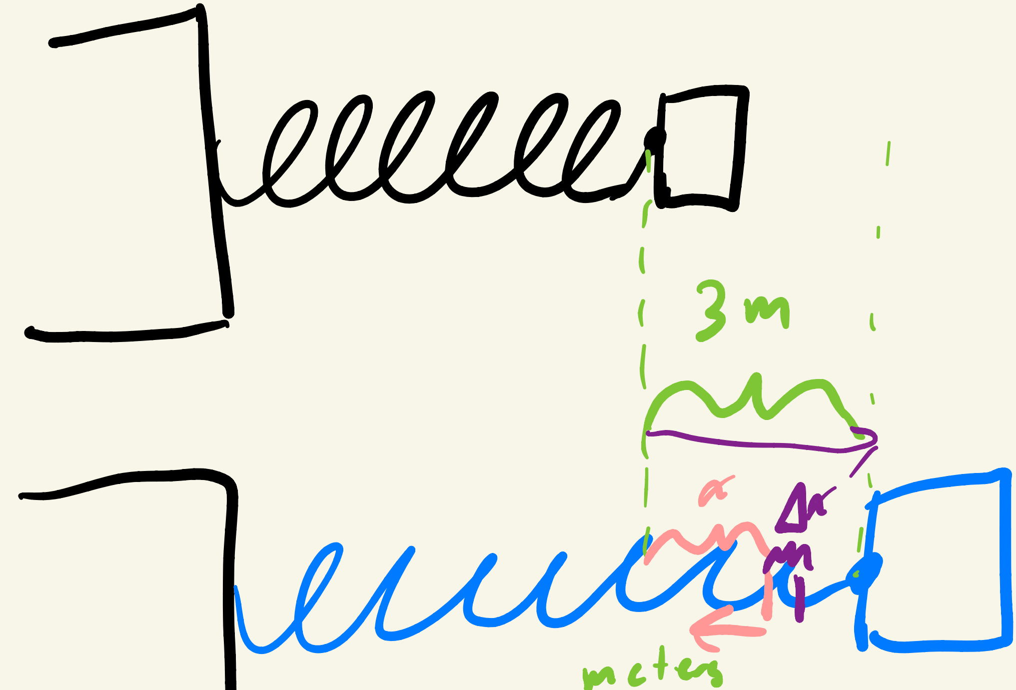
d/t integrate w.r.t. t
to get d

acceleration is velocity change per
unit time

d/t^2 $(d/t)/t \rightarrow d/t$

In general if we want to compute some quantity with units U with an integral with respect to x with units U' we want to integrate a quantity with units

$$U/U'$$

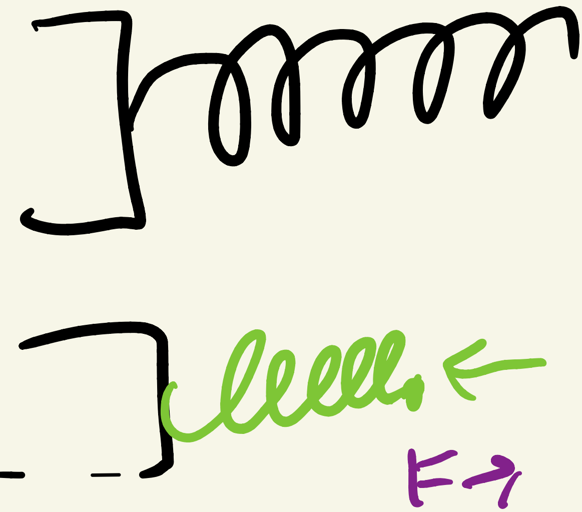


$$F_{spring} = 2 \cdot d$$

$$F = 2 \cdot x$$

$$W \approx 2 \cdot x \cdot \Delta x$$

$$\text{work} = \int_0^3 -2x \, dx$$



Differential equations

First: Algebraic "pointwise"
equations

Think about solving

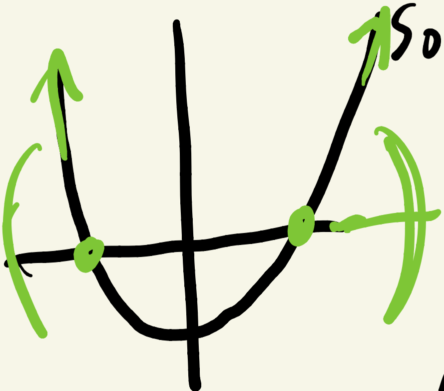
$x^2-1=0$ $\qquad$ $x^2+1=0$

$(x+1)(x-1)=0$

So this is true if

Existence

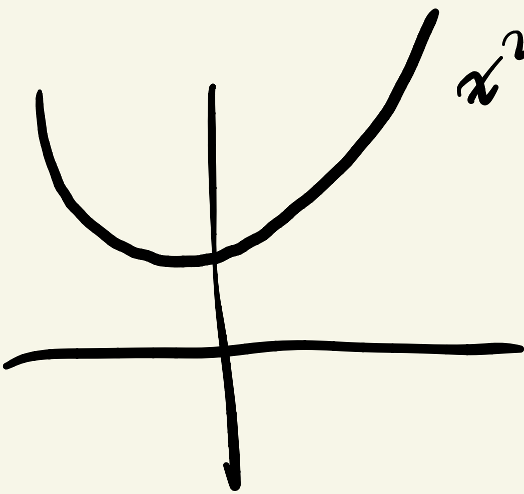
$x = 1$ or -1



$(1)^2 - 1 = 0$ ✓

$(-1)^2 - 1 = 1 - 1 = 0$ ✓

Uniqueness



x^2+1

In terms of real numbers:

· no solution!

If we allow imaginary numbers

$x^2+1=0$ is solved by "$-i$ or i"

$i^2+1=-1+1=0$

Now differential

$$f'(x)=\underline{f(x)}$$

$$f'(x)=x^2$$

I could check

$\boxed{f(x)=x^2}$ as a solution?

$f(x)=e^x$

$$f'(x) = \frac{d}{dx}\left[x^2\right] = 2x$$

$$f(x) = x^2$$

$$2x \stackrel{?}{=} x^2 \quad \underline{\text{No!}}$$

The function $f(x) = x^2$ does not satisfy the differential equation

$$f'(x) = f(x)$$

What is

$$f'(x) = x^2$$

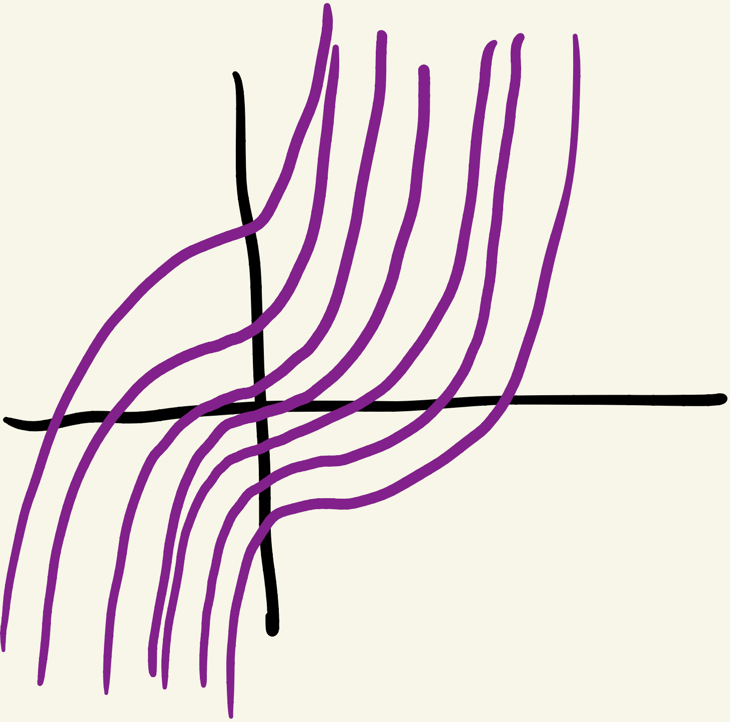
asking?

- what is an/the antiderivative of x^2
- How do we solve

$$\int x^2 dx$$

$$= \boxed{\frac{1}{3}x^3 + C}$$

what is this equal to at $x=0$? C!



I do not have uniqueness,

but maybe I can restrict information more to get it?

"Initial value problem (IVP)"

$$\begin{cases} f'(x) = x^2 \\ f(0) = 2 \end{cases}$$

Is there some / only one function of the form

$f(x) = \frac{1}{3}x^3 + C$ is 2 at $x = 0$?

$$f(0) = \frac{1}{3}(0)^3 + C = C$$

$f(0) = 2$ if and only if

$C = 2$, and so

$$f(x) = \frac{1}{3}x^3 + 2$$

In general, given a fixed function

$g(x)$ (ex. $g(x) = x^2$ $\quad g(x) = e^x \quad g(x) = \sin x$)

we approach

$$f'(x) = g(x)$$

by

$$\int g(x)\,dx$$

$f'(x) = x^2$

$f'(x) = e^x$

$f'(x) = \sin(x)$

Harder is when we depend on both f and f'

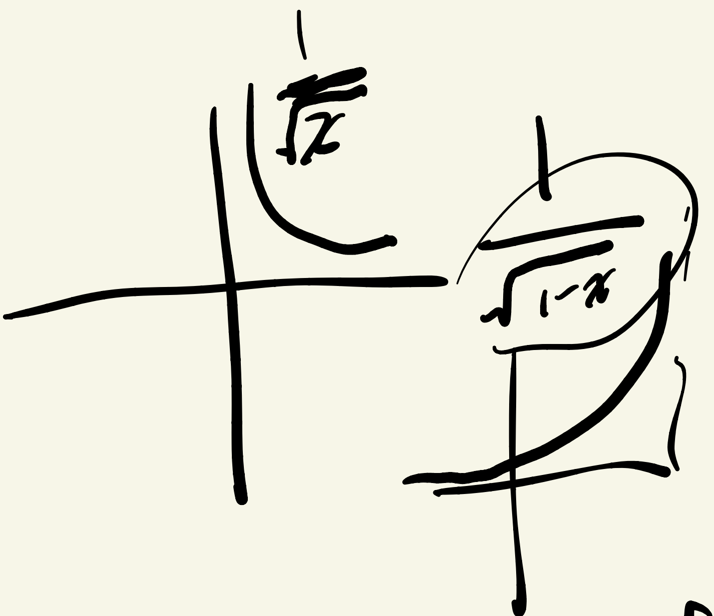
$$f'(x) = f(x)$$

Solved by $f = e^x$ / $f(x) = Ae^x$ (constant $\downarrow$ A)

Once we have a solution different things can go wrong

- Nicest case

  solution has arbitrarily many derivatives, exists for all time, and is unique



$$\frac{1}{2} \cdot \frac{1}{(1-x)^{\frac{3}{2}}}$$

$$f'(x) = \frac{1}{2} f^3$$

$$f'(x) = 2|f(x)|^{\frac{1}{2}}$$

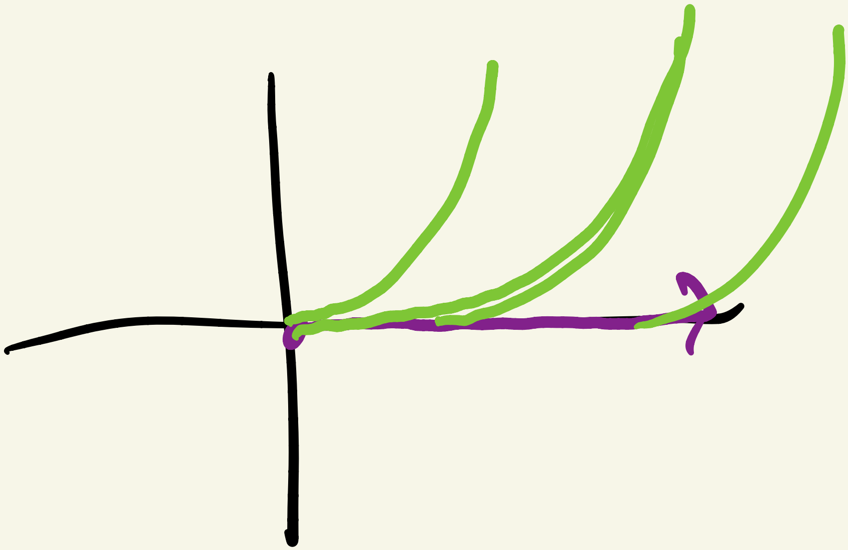
$$f(x) = 0$$

$$f(x) = x^2$$

$$f'(x) = 2x$$

$$2|x^2|^{\frac{1}{2}} = 2|x|$$

$$f'(x) = 3 + f^2(x)$$



$y' =$

$y' = f(y)\,g(x)$

Existence questions for particular DEs

we can solve either

(DE)

$$f'(x) = g(x)$$

or

(IVP)

$$\begin{cases} f'(x) = g(x) \\ f(0) = a \end{cases}$$

where f is our unknown.

Because this is just an antiderivative question. (ie what is the antiderivative of g?)

---

we could look at the de

$$y' = xy \longleftrightarrow f'(x) = x f(x)$$

Separable DE

$$y' = N(y) \cdot M(x)$$ or $$L(y)\, y' = M(x)$$

$$L(y) = \frac{1}{N(y)}$$

where N & M are fixed functions (not unknowns).

Examples

$$y' = \sin(x) \cdot \cos(y)$$

$$y' = y^2 \cdot \ln x$$

$$y' = x \cdot y^2 + x \cdot \sin(y)$$
$$= (x)(y^2 + \sin(y))$$

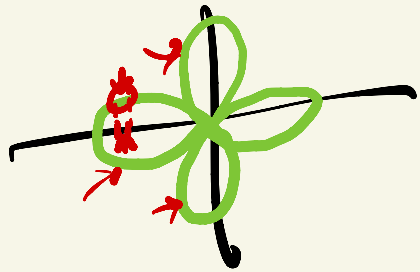
Non example

$$y' = x + y$$ X

Implicit differentiation:

Given some curve, and we assume the curve is locally represented by a function

$$y(x) = y = f(x)$$



EX

Say y is a function of x, and

$$g(y) = h(x)$$

$$g(y(x)) = h(x)$$

$$g(f(x)) = h(x)$$

differentiate both sides

$$g'(f(x)) \cdot f'(x) = h'(x)$$

$$g'(y) \cdot y' = h'(x)$$

$$L(y) \cdot y' = N(x)$$

If $L(y) = g'(y)$ & $N(x) = h'(x)$

then If $g(y) = h(x)$,

this relation "satisfies the DE"

If g has an inverse, then $y = g^{-1}(h(x))$ is a function, and should solve our DE.

Suppose I had a function $y(x)$ satisfying

$$L(y)\, y' = N(x)$$

then

$$L(y(x)) \cdot y'(x) = N(x)$$

→ Take an antiderivative in terms of x

$$\int L(y(x))\,y'(x)\,dx = \int N(x)\,dx$$

$$\Downarrow$$

$$\int L(y)\,dy = \int N(x)\,dx$$

Note We could also look at

$$\int L(y)\,dy = \left(\int N(x)\,dx\right) + C$$

for any constant C

Completing the Solution

once we integrate or antidifferentiate, we just have to do algebra/pre-calc to solve for y

---

$$y' = \frac{x}{y} \rightarrow y \cdot y' = x$$

$$y(x) \cdot y'(x) = x$$

$$\int y(x)\, y'(x)\, dx = \int x\, dx$$

$$\boxed{y = \sqrt{x^2 + C}}$$

$$y' = \frac{1}{2} \frac{1}{\sqrt{x^2+C}} \cdot 2x$$

$$\frac{x}{y} = \frac{x}{\sqrt{x^2+C}}$$

$$y = -\sqrt{x^2+C}$$

$$y' = -\frac{x}{\sqrt{x^2-C}}$$

$$\frac{x}{y} = \frac{x}{-\sqrt{x^2-C}}$$

$$\int y\,dy = \int x\,dx$$

$$\Downarrow$$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$

$$\Downarrow$$

$$\boxed{y^2 = x^2 + C}$$

$$\Downarrow$$

Should be a solution

$$y = \pm\sqrt{x^2+C}$$

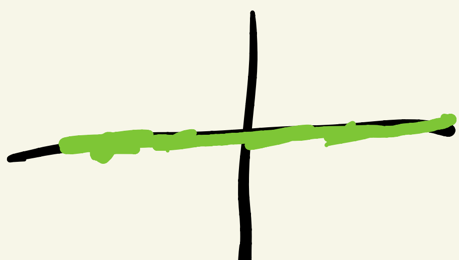
$$\frac{dy}{dx} = N(x) \cdot M(y)$$

$$\downarrow$$

$$\frac{1}{M(y)}\, dy = N(x)\, dx$$

$$\downarrow$$

$$\int \frac{1}{M(y)}\, dy = \int N(x)\, dx$$



$$y' = y \cdot f(x)$$

$\downarrow$ $\quad y=0$ case $\quad y' = (0) \cdot f(x)$

$$y' = 0$$

$$\frac{y'}{y} = f(x)$$

$y=0$ is a solution

(think about $y \neq 0$)

$\downarrow$

$$\int \frac{1}{y}\,dy = \int f(x)\,dx$$

Assume $F(x)$ is some antiderivative of $f(x)$

$$\ln|y| = F(x) + C$$

$$|y| = e^{F(x)+C}$$

$$y = \pm e^{F(x)+C}$$

All of our solutions are

$$y = 0$$

$$y = \pm e^{F(x)+C}$$

$$y = \pm e^{C} \cdot e^{F(x)}$$

we actually get the solutions

$$y = Ae^{F(x)}$$

for any real number A

Thm: If $F'(x) = f(x)$,

then

$$y = Ae^{F(x)}$$

solves

$$y' = y\,f(x)$$

for any A in the real numbers.

General solutions

$y' = x^2$

solutions

$y = \frac{1}{3}x^3 + c$

If I pick a point (x_1, y_1), I can plug in

$y_1 = \frac{1}{3}(x_1)^3 + c,$ and find

some c which makes this

true.

Full Idea of General Solution

is

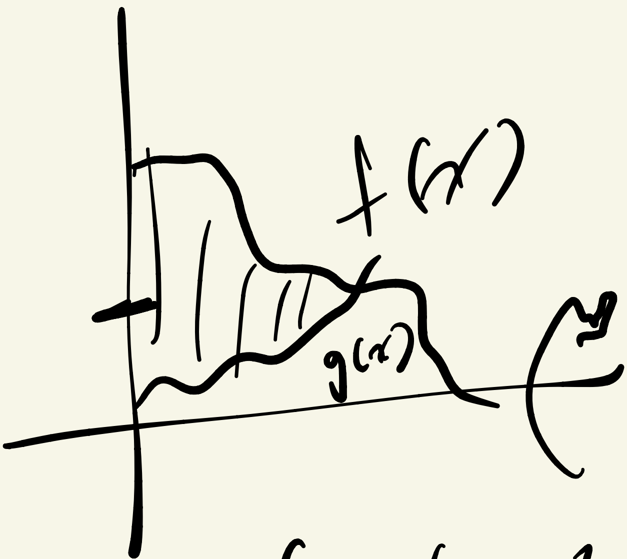
set of solutions / curves to a differential equation, so that for any given point (x_1, y_1), you can find a curve passing through the point.

what you should think (especially for separable)

set of solutions relying on some constant

↑

For separable, always the "+c" from integration step



$$\frac{\int \frac{1}{2}\left(f(x)^2 - g(x)^2\right) dx}{\int f(x) - g(x)\, dx}$$

$$\begin{cases} y' = y + 3xy \\ \boxed{y(0) = 3} \end{cases}$$

$$y' = y(1 + 3x)$$

$$\frac{1}{y} y' = 1 + 3x$$

$$\int \frac{1}{y}\, dy = \int (1 + 3x)\, dx$$

$$\ln|y| = x + \frac{3}{2}x^2 + C$$

$$\boxed{y = Ae^{x+\frac{3}{2}x^2}}$$

want

$y = 3$ when $x = 0$

$$3 = Ae^{0+\frac{3}{2}0^2}$$

$$3 = A$$

Solution to the IVP

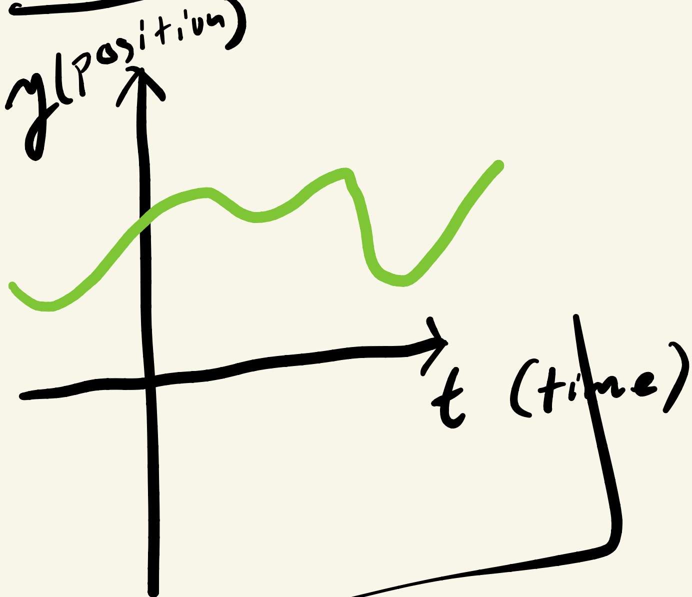
$$\begin{cases} y' = y + 3xy \\ y(0) = 3 \end{cases}$$

is given by

$$y = 3e^{x + \frac{3}{2}x^2}$$

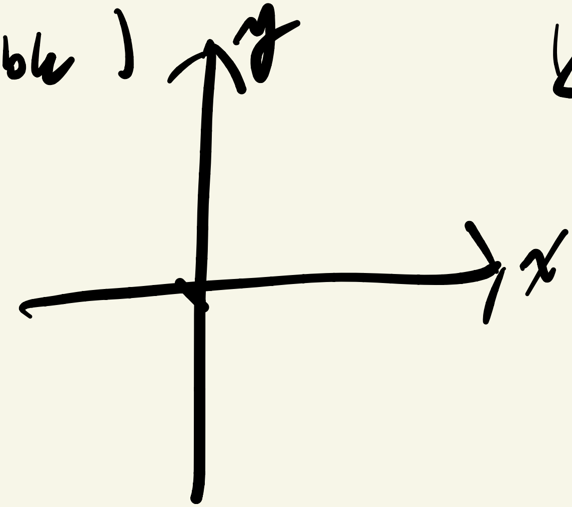
$$y' = \frac{3x}{y^2}$$

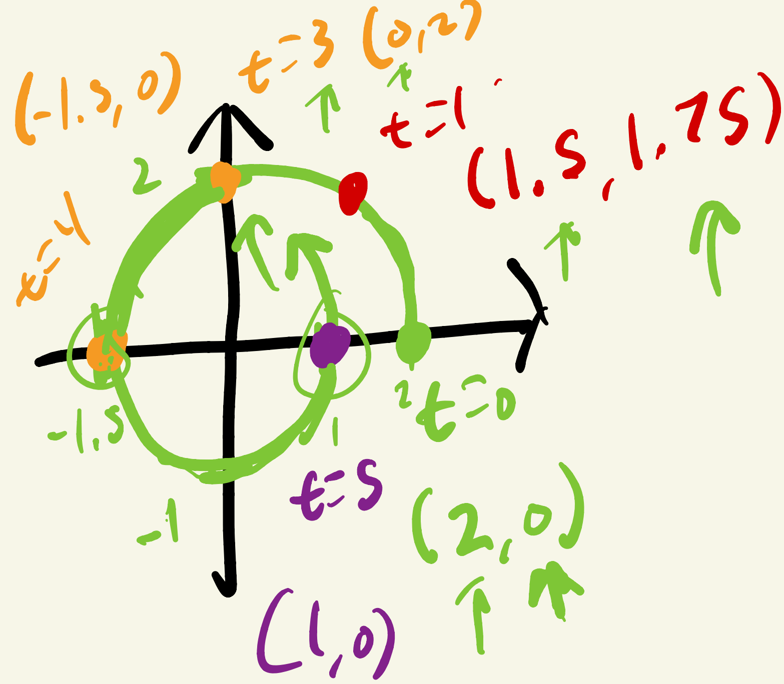
Parametric equations

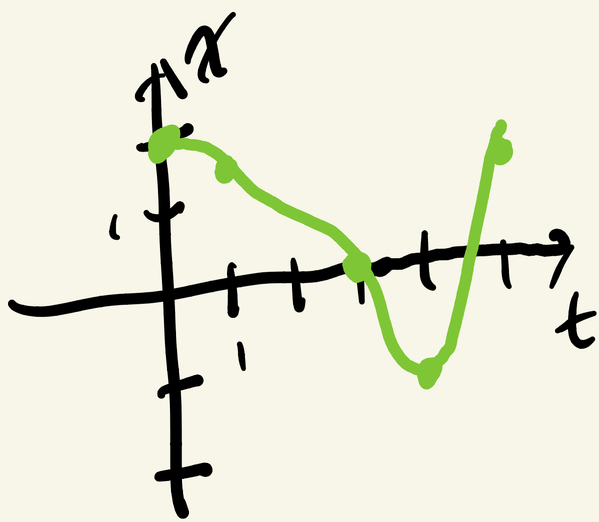


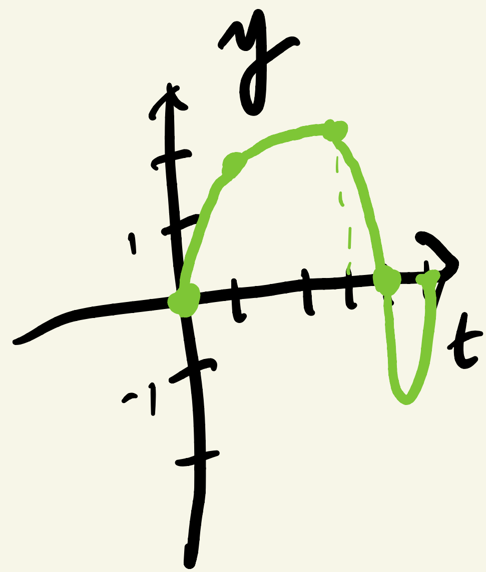
distinct

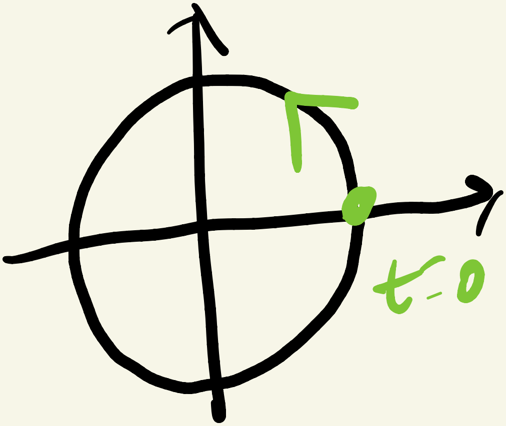
(table)

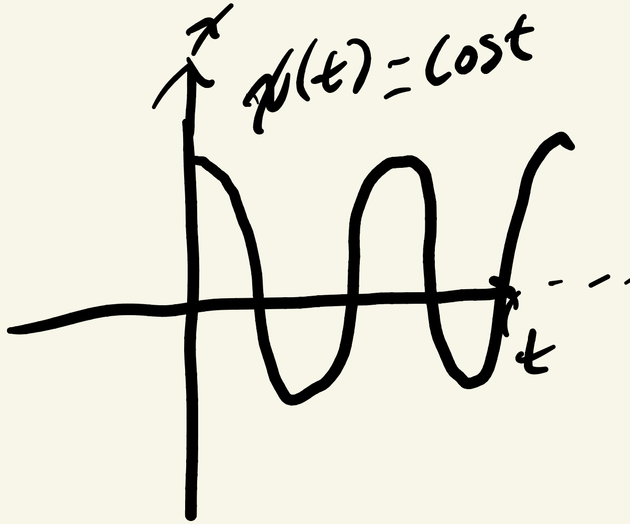



(-1.5, 0)
t=3
(0,2)
t=1
(1.5, 1.75)
t=4
2
-1.5
1
t=0
t=5
-1
(2,0)
(1,0)

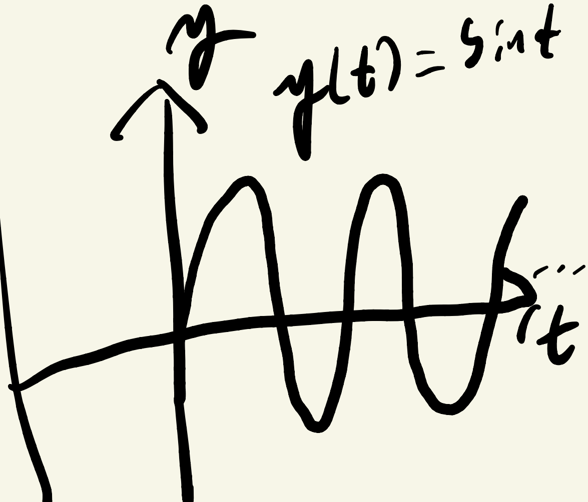

x
1
t
1


y
1
-1
t





$$\begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases}$$



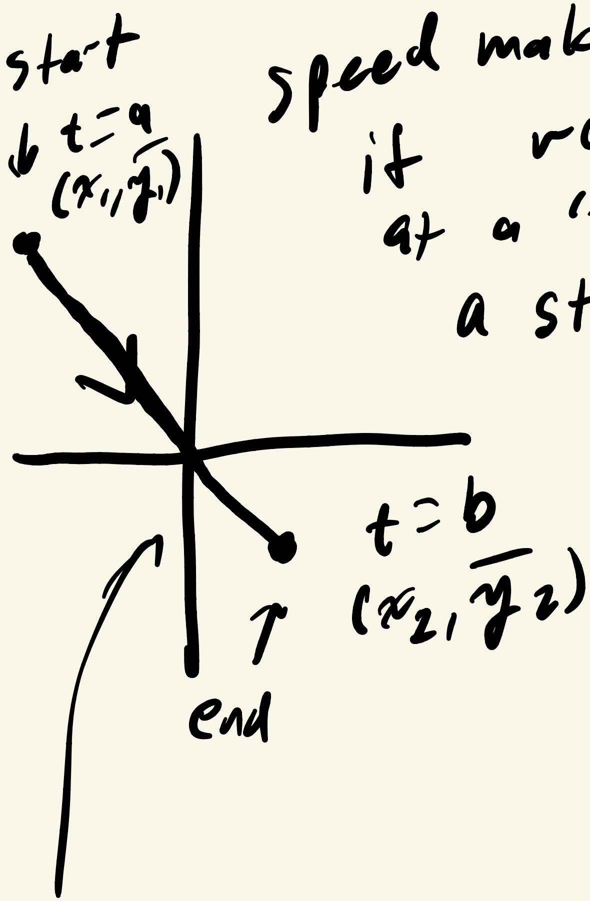
$$\begin{cases} x(t) = \cdots \\ y(t) = \cdots \\ z(t) = \cdots \end{cases}$$

Calculus questions for motion

☆ speed? ← differentiation questions
acceleration? ←

☆ distance traveled? ← integration question

speed makes sense already if we are traveling at a "constant rate" in a straight line.

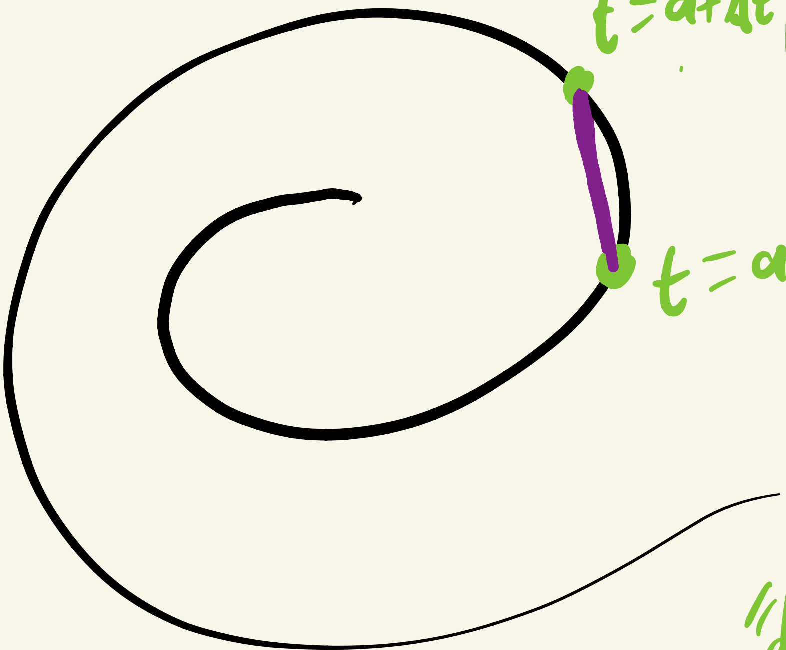


Speed is $\dfrac{\text{dist traveled}}{\text{per unit of time}}$

length of line $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

time traveled for $b-a$

$$\text{Speed} := \left(\frac{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}{b-a}\right)$$



what is the appx speed?

Idea:

close up "differentiable" curves are "almost linear"

look at a small change in time "Δt"

approximate speed by

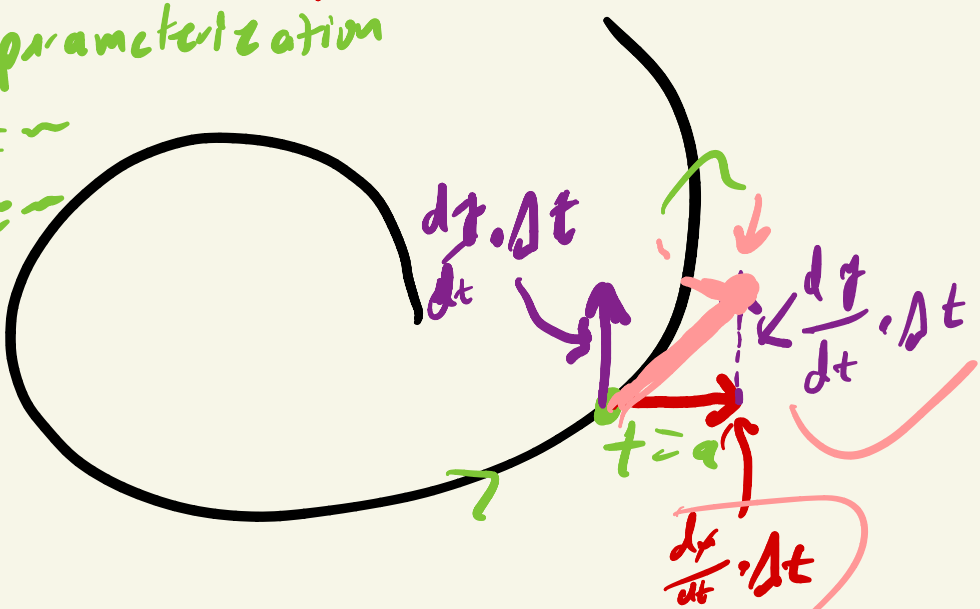
$$\frac{\text{dist}(\text{point at } t=a,\ \text{point at } t=a+\Delta t)}{\Delta t}$$

$$= \frac{\sqrt{\left(x(a+\Delta t) - x(a)\right)^2 + \left(y(a+\Delta t) - y(a)\right)^2}}{\Delta t}$$

Idea: looking at the limit of this as $\Delta t \to 0$, should give a good definition of speed

Have parameterization

$$\begin{cases} x(t) = \sim \\ y(t) = \sim \end{cases}$$



Easier question

How far have I traveled in the x direction after Δt?

$\frac{dx}{dt}$ approximates dist traveled in x direction after given time.

$\frac{dy}{dt}$ approximates dist traveled in y direction

Speed from x, y appx

$$\frac{\sqrt{\left(\frac{dx}{dt}\cdot\Delta t\right)^2 + \left(\frac{dy}{dt}\cdot\Delta t\right)^2}}{\Delta t}$$

$$= \frac{\sqrt{(\Delta t)^2\left(\frac{dx}{dt}^2 + \frac{dy}{dt}^2\right)}}{\Delta t}$$

$$= \frac{\Delta t}{\Delta t}\cdot\sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}$$

A bit surprising that it actually works but we can de fine speed of a parametrization by

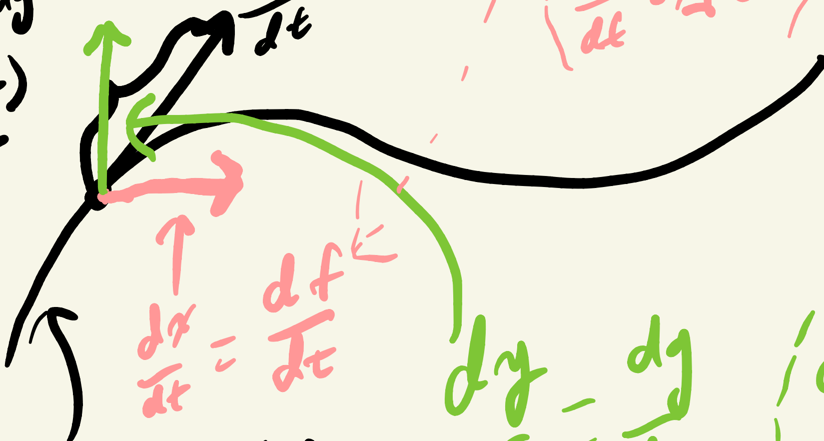
$$\text{Speed} := \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

---

length is $\frac{d?}{dt}$

$\left(\frac{dx}{dt} \cdot \Delta t\right)$

$\frac{d(\text{dist})}{dt}$

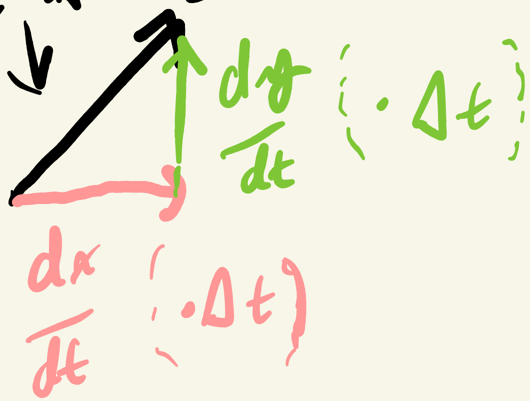


$\frac{dx}{dt} = \frac{df}{dt}$

$\frac{dy}{dt} = \frac{dg}{dt}$ $\left(\frac{dy}{dt} \cdot \Delta t\right)$

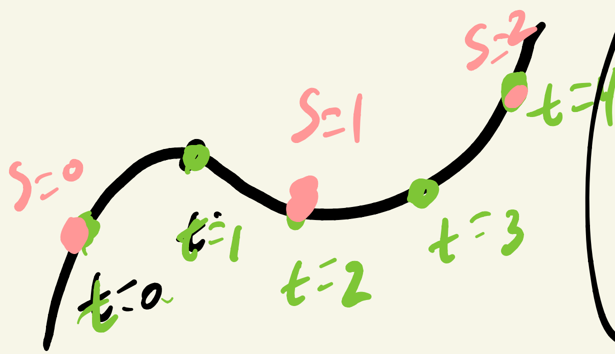
$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

appx change in plane after Δt



Two other ideas

- Slope of tangent line / construction of a tangent line
- Arc length



$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

$$\begin{cases} x = f(2t) \\ y = g(2t) \end{cases}$$

$$\begin{cases} x = f(2s) \\ y = g(2s) \end{cases}$$

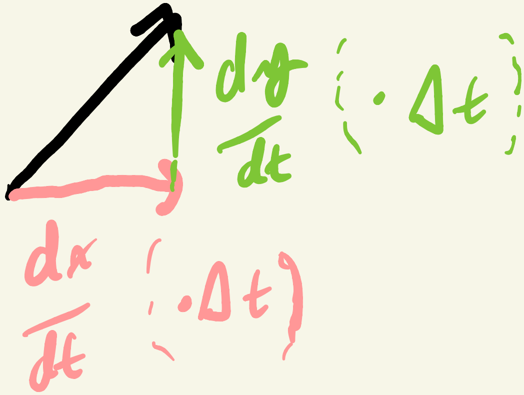
Speed: $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

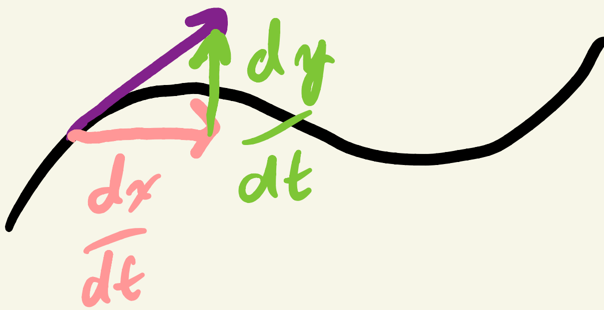
$$\sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2}$$

$$\sqrt{\left(\frac{d(f(2t))}{dt}\right)^2 + \left(\frac{d(g(2t))}{dt}\right)^2}$$

Check: speed of change of parameter will be twice speed under original parameterization.

Tangent line / tangent slope





Hope: slope maybe is $\left(\frac{dy}{dt} \Big/ \frac{dx}{dt} \right)$,

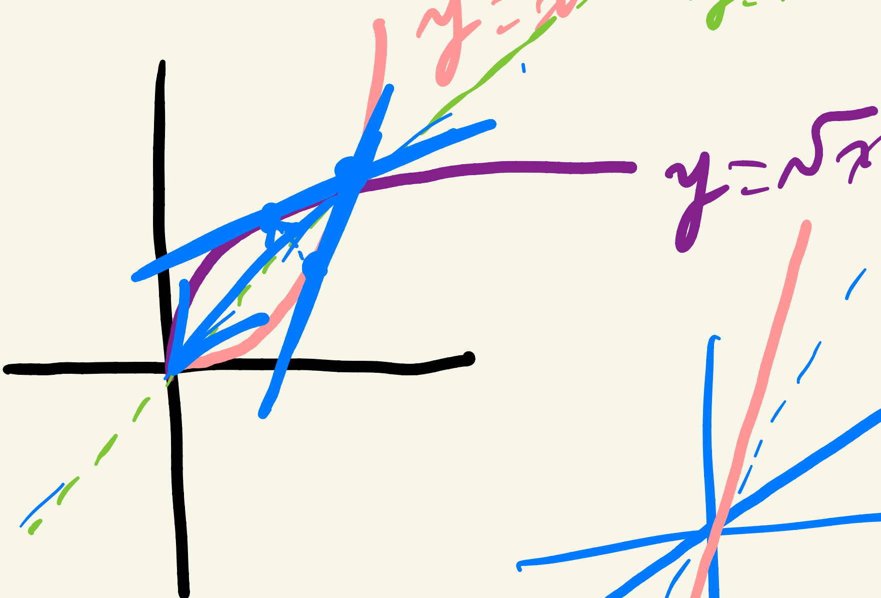
So we are hoping to have

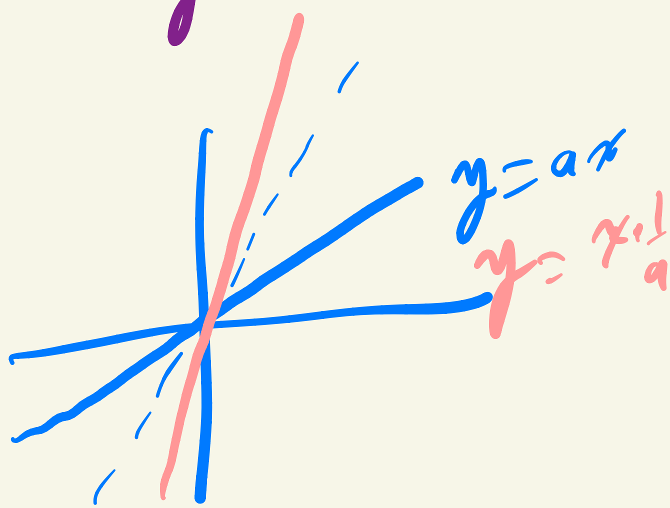
$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{1}{\left(\frac{dx}{dt}\right)}$$

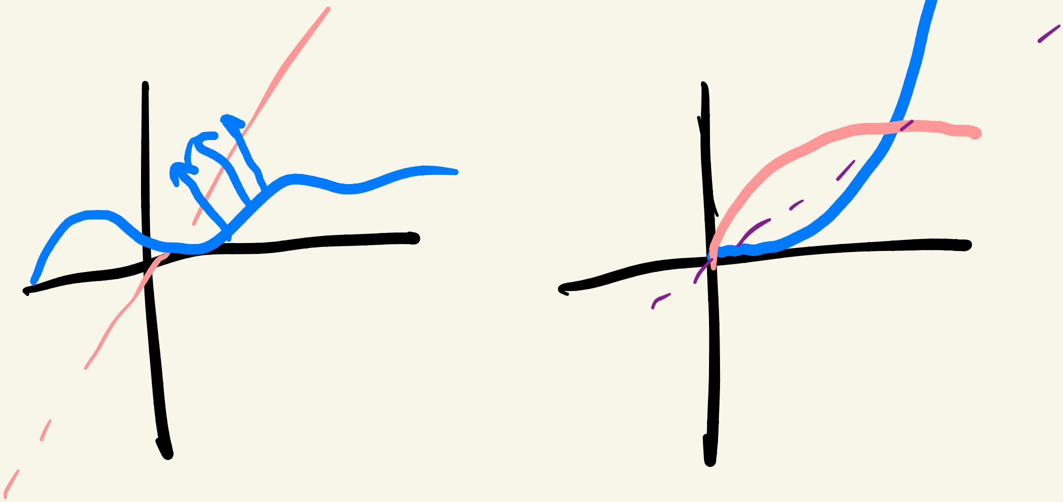
Note that by chain rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

Interesting question is $\frac{dt}{dx} = \frac{1}{\left(\frac{dx}{dt}\right)}$







looking at

$$\frac{dt}{dx} \text{ vs } \frac{dx}{dt}$$

$$\frac{dy}{dx} \text{ vs } \frac{dx}{dy} \quad \text{when}$$

y has an inverse

$$y = f(x) \qquad \boxed{\begin{matrix}\text{inverse} \\ \text{not} \\ \frac{1}{f(x)}!\end{matrix}} \qquad x = f^{-1}(y)$$

$$\frac{dy}{dx} = f'(x) \qquad \frac{dx}{dy} = \left(\frac{d}{dy} f^{-1}\right)(y)$$

want to use $\quad f(f^{-1}(x)) = x$

$$\frac{d}{dx}\left[f(f^{-1}(x))\right] = \frac{d}{dx}[x]$$

$$\left(\left(\frac{d}{dx} \overset{f'}{f}\right)(f^{-1}(x))\right) \cdot \left(\left(\frac{d}{dx} f^{-1}\right)(x)\right) = 1$$

$$\left(\frac{d}{dx}f\right)(f^{-1}(x)) = \frac{1}{\left(\frac{d}{dx}f^{-1}\right)(x)}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{1}{\left(\frac{dx}{dt}\right)}$$

Say $\left.\frac{dy}{dx}\right|_x$ is $\frac{dy}{dx}$ evaluated at x

or $\left(\frac{dy}{dx}\right)(x) = y'(x)$

want

$$\left.\frac{dy}{dx}\right|_{\underset{x(t)}{\overset{\uparrow}{x}}} = \left.\frac{dy}{dt}\right|_t \cdot \frac{1}{\left.\frac{dx}{dt}\right|_t}$$

$$\frac{1}{\left.\frac{dx}{dt}\right|_t} = \left.\frac{dt}{dx}\right|_{x(t)}$$

$$\left.\frac{dy}{dx}\right|_{x(t)} = \left.\frac{dy}{dt}\right|_{t} \cdot \left.\frac{dt}{dx}\right|_{x(t)}$$

true by chain rule.

by "$\frac{dt}{dx} = 1/\left(\frac{dx}{dt}\right)$",

we get

$$\left.\frac{dy}{dx}\right|_{x(t)} = \left.\frac{dy}{dt}\right|_{t} \cdot 1/\left(\left.\frac{dx}{dt}\right|_{t}\right)$$

$$= \frac{\left(\frac{dy}{dt} \Big|_t \right)}{\left(\frac{dx}{dt} \Big|_t \right)}$$

Consider the differential equation

$$f'' = -f.$$

Submit by 9:00

- does $f(x) = \sin(x)$ solve this equation?
- what about $f(x) = \cos(x)$?
- what about $f(x) = \sin(x) + \cos(x)$?

<u>Bonus</u>: Give two solutions to the IVP

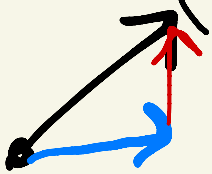
$$\begin{cases} f'' = -f \\ f(0) = 1 \end{cases}$$

Hint: Think about the first 3 parts.

Speed: $\sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}$

Slope $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$

$\left(\frac{dx}{dt}, \frac{dy}{dt} \right)$



What happens for linear

$$\begin{cases} x(t) = m_1 t + b_1 \\ y(t) = m_2 t + b_2 \end{cases}$$

$$\frac{dx}{dt} = m_1 \quad \frac{dy}{dt} = m_2$$

$$\left(\frac{dy}{dt}\right) \Big/ \left(\frac{dx}{dt}\right) = \frac{m_2}{m_1}$$

to determine a line

constant

- 2 pts

- 1 pt
  & slope

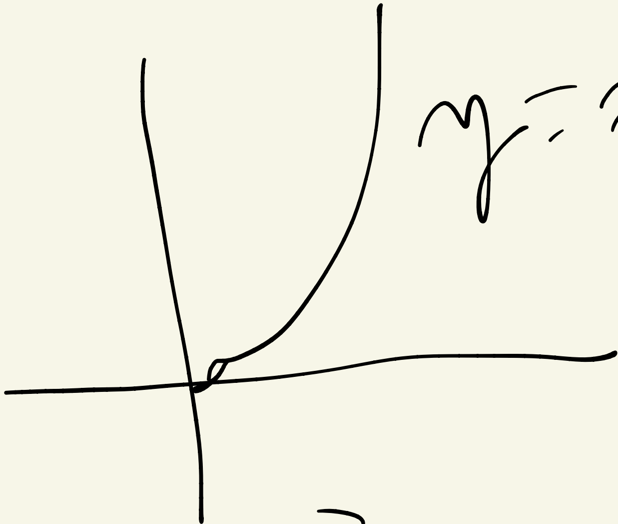
Want to pass through

(a_1, b_1) & (a_2, b_2)

$\uparrow$ $\uparrow$

$t=0$ $t=1$

$$\begin{cases} x(t) = (a_2 - a_1)t + \underline{a_1} \\ y(t) = (b_2 - b_1)t + \underline{b_1} \end{cases}$$



$y = x^2$

y & x as a function of t

set $t = x$

Speed?

$$\sqrt{1+(2t)^2}$$

$$\begin{cases} x(t) = t \\ y(t) = t^2 \end{cases}$$

Arc length

recall speed is $\frac{\text{dist}}{\text{time}}$

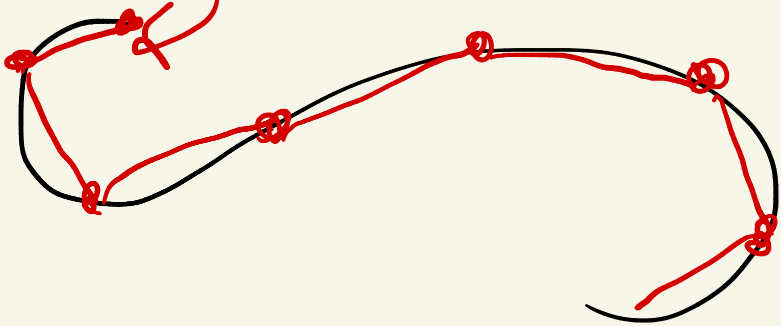
1 dim case

$$p'(t) = v(t)$$

$$\int v(t)\,dt = \text{position}$$

In general

$$\text{dist traveled} = \int_a^b \text{speed}\,dt$$



for parametric or arc length

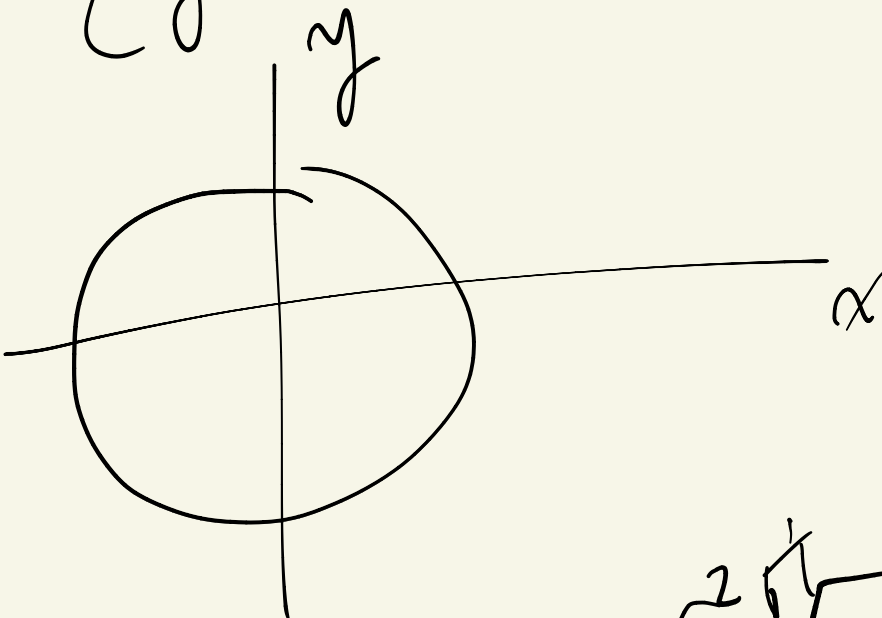
dist traveled $t = a$ to $t = b$ is given by

$$\int_a^b \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}\, dt$$

$$\begin{cases} x(t) = \cos(t) \\ y(t) = \sin(t) \end{cases}$$

1 revolution

$t = 0$ to 2π



$$\int_0^{2\pi} \text{Speed}\, dt = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = -\sin(t) \quad \frac{dy}{dt} = \cos(t)$$

$$\int_0^{2\pi} \sqrt{\sin^2(t) + \cos^2(t)}\, dt$$

$$\int_0^{2\pi} 1\, dt = 2\pi$$

Say we have $y = f(x)$ &

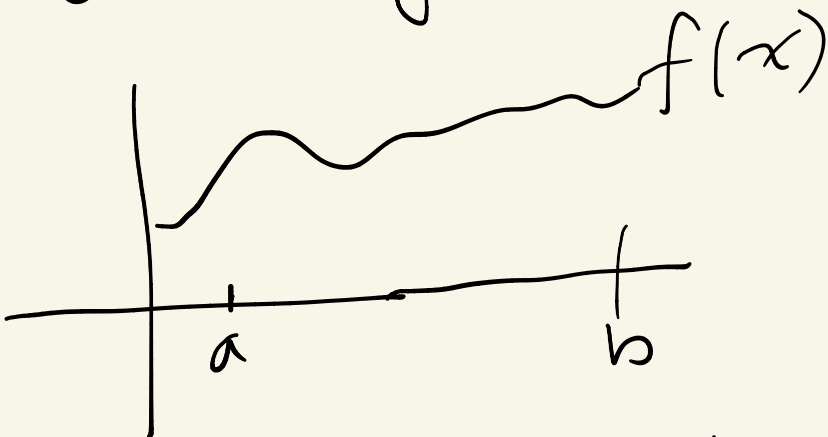
we parameterize

$$\begin{cases} x = t \\ y = f(t) \end{cases}$$

speed:

$$\sqrt{1 + \left(\frac{dy}{dt}\right)^2}$$

arc length from $x = a$ to b



Same as arc length for

$$\begin{cases} x = t \\ y = f(t) \end{cases} \quad t \text{ from } a \text{ to } b$$

arc length

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Compute tangent line to

$$\begin{cases} x = t^2 \\ y = t^3 \end{cases}$$

$t = \pm\sqrt{x}$

$t = \sqrt[3]{y}$

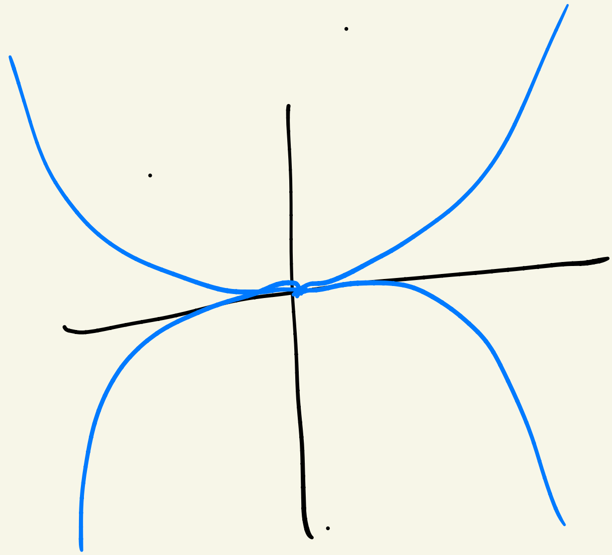
at $t = 2$

$\sqrt[3]{y} = \pm\sqrt{x}$

$y = (\pm\sqrt{x})^3$

"tricky part"

$\frac{dy}{dx}$



$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$$

$$\frac{dy}{dt} = 3t^2 \qquad \frac{dy}{dx} = \frac{3t^2}{2t}$$

$$\frac{dx}{dt} = 2t$$

$$\frac{dy}{dx} = \frac{3(2^2)}{\underline{2(2)}} = 3$$

line passing through

$(x(2), y(2)) = (4, 8)$

with slope 3

---

what ab parametric?

x line passing through

4 @ $t = c$ $= (0)$

$dx/dt = 4$

y line passing through

8 @ t = c = (0)

$$\frac{dy}{dt} = 12$$

$$\begin{cases} x(t) = 4t + 4 \\ y(t) = 12t + 8 \end{cases}$$

Summary:

(x, y)
$\downarrow$
(r, θ)

For parametric

$$\begin{cases} x(t) = f(t) \\ y(t) = g(t) \end{cases}$$

$$\text{Speed} := \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{f'(t)^2 + g'(t)^2}$$

Tangent slope w.r.t. x, y

$$\frac{dy}{dx} = \left(\frac{dy}{dt}\right) \Big/ \left(\frac{dx}{dt}\right)$$

Arc length (t from a to b)

$$:= \int_a^b \text{"speed"}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

for function y of x

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

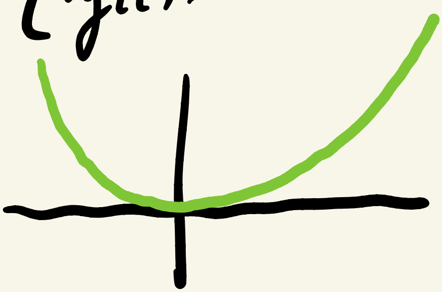
• Note if we have $\begin{cases} x(t) = f(t) = m_1 t + b_1 \\ y(t) = g(t) = m_2 t + b_2 \end{cases}$

If f, g are both lines then this parametric curve is also a line.

For a general parametric curve, we need to be a bit careful w/ arc length.

$y = x^2$

$$\begin{cases} x(t) = t \\ y(t) = t^2 \end{cases}$$

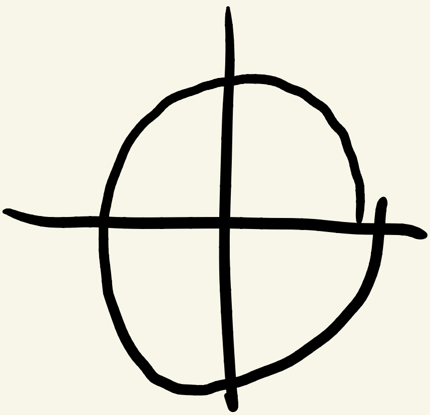


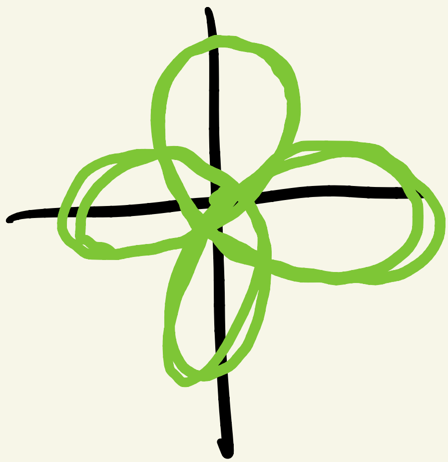
"Curve $y = x^2$"

$$\begin{cases} x(t) = (t^2) \\ y(t) = (t^2)^2 \end{cases}$$

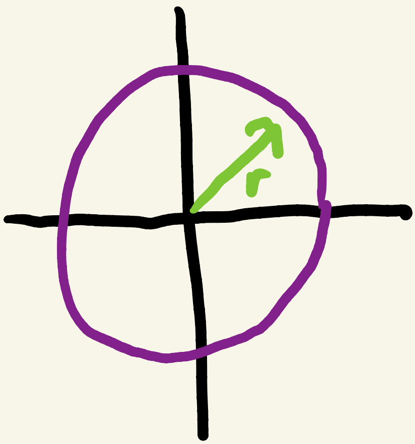
If I ask for arc length between $t=-1$, and $t=1$ what do I mean

- distance traveled
- length of the traced line





Polar co-ordinates



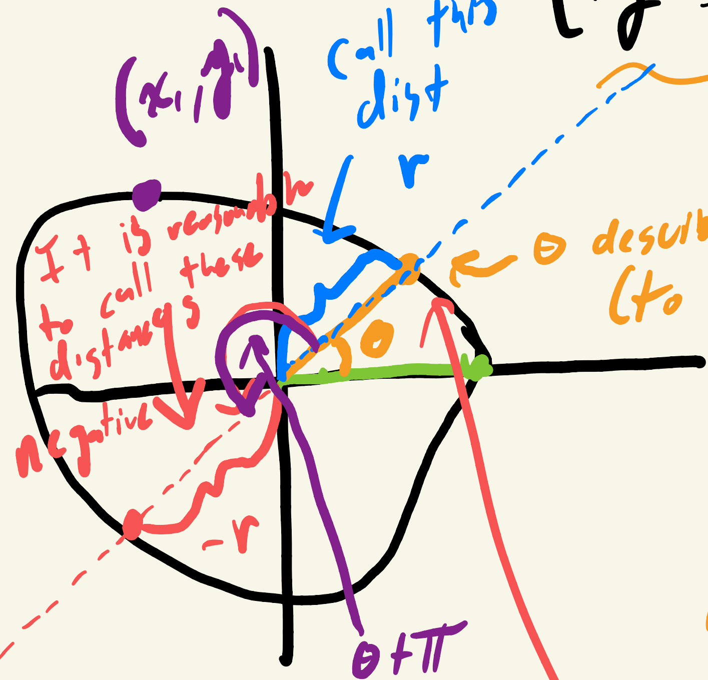
If we want to talk about this curve with functions, we would (at this point) usually use parametric functions

$$\begin{cases} x = r\cos(t) \\ y = r\sin(t) \end{cases}$$

here t is describing an angle

Parameterize in terms of θ. (θ is a nice name for an angle)

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$



θ describes this point (to an extent)

what other information do I need to <u>completely</u> determine the point?

Q: what points have the same angle

this point could be described with

θ	r
θ	r
$\theta + \pi$	$-r$
$\theta + 2\pi$	r

One way we can deal with this weirdness (for the most part) is restrict

- $0 \leq \theta < 2\pi$,

and

- $0 \leq r$

or

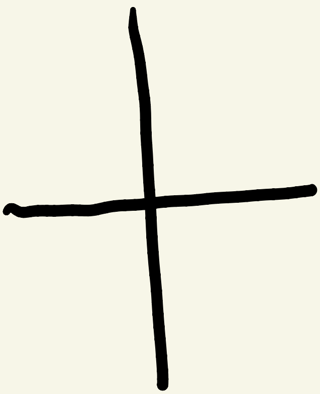
- $0 \leq \theta < \pi$
- no restriction on r

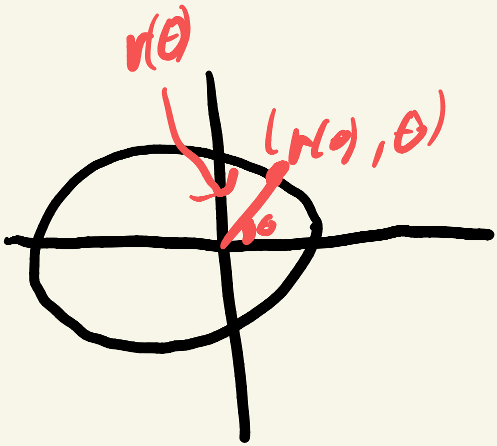
We can't help the weirdness at 0, but it turns out that is not so bad.

We can form new co-ordinates

$\cancel{(\theta, r)}$ (or (r, θ)) we'll use this

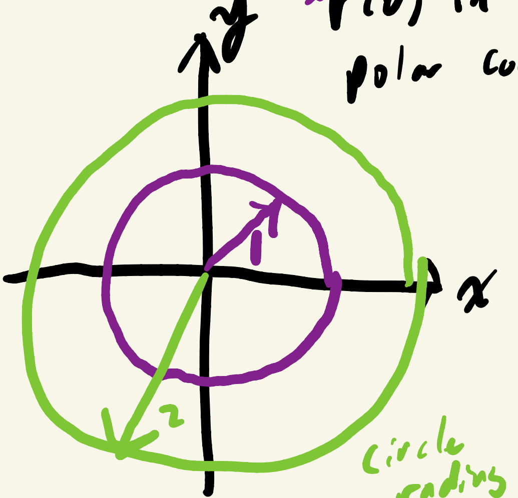
(x, y) (or (y, x))



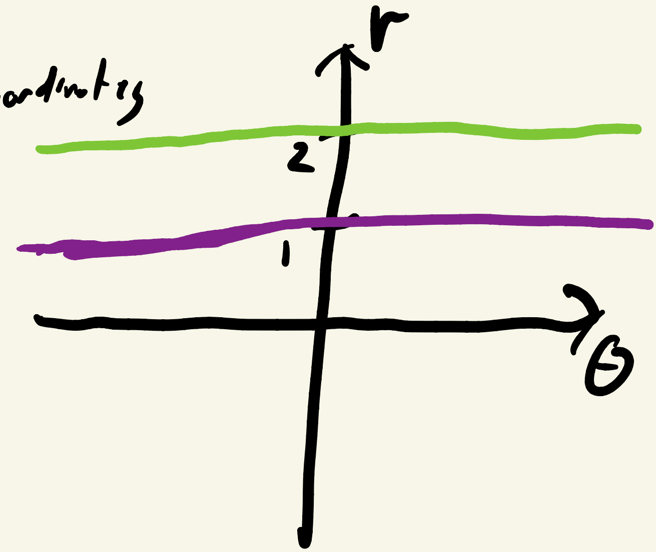


Often we want to write radius as a function of angle.

$r(\theta)$ in polar coordinates

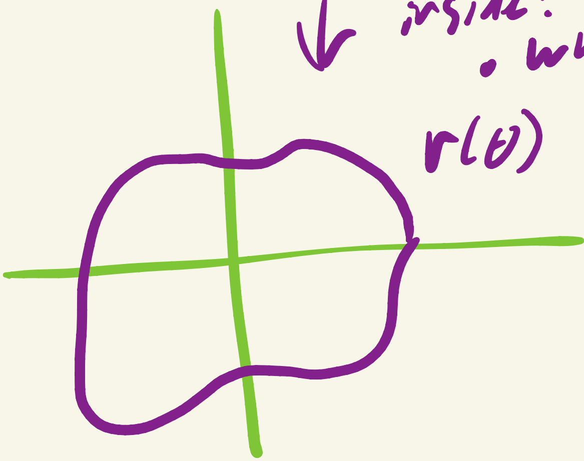


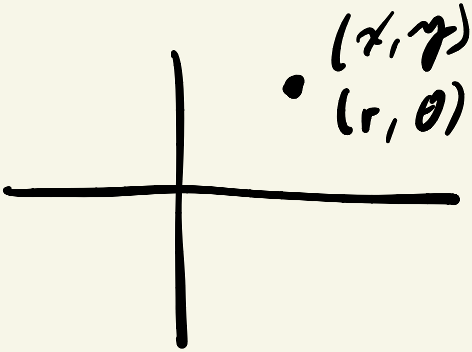
circle radius 2



- $r(\theta) = 1$
- $r(\theta) = 2$

- what is the area of the region inside?
- what is the arc length?





$$r^2 = x^2 + y^2$$

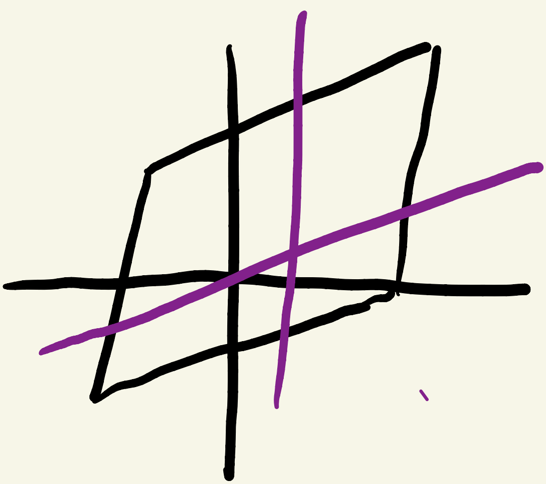
$$\tan(\theta) = \frac{y}{x}$$

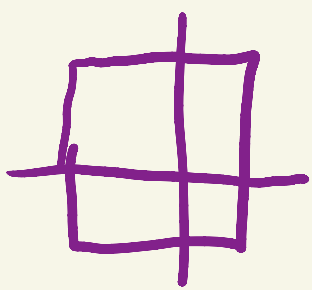
These relations connect the systems

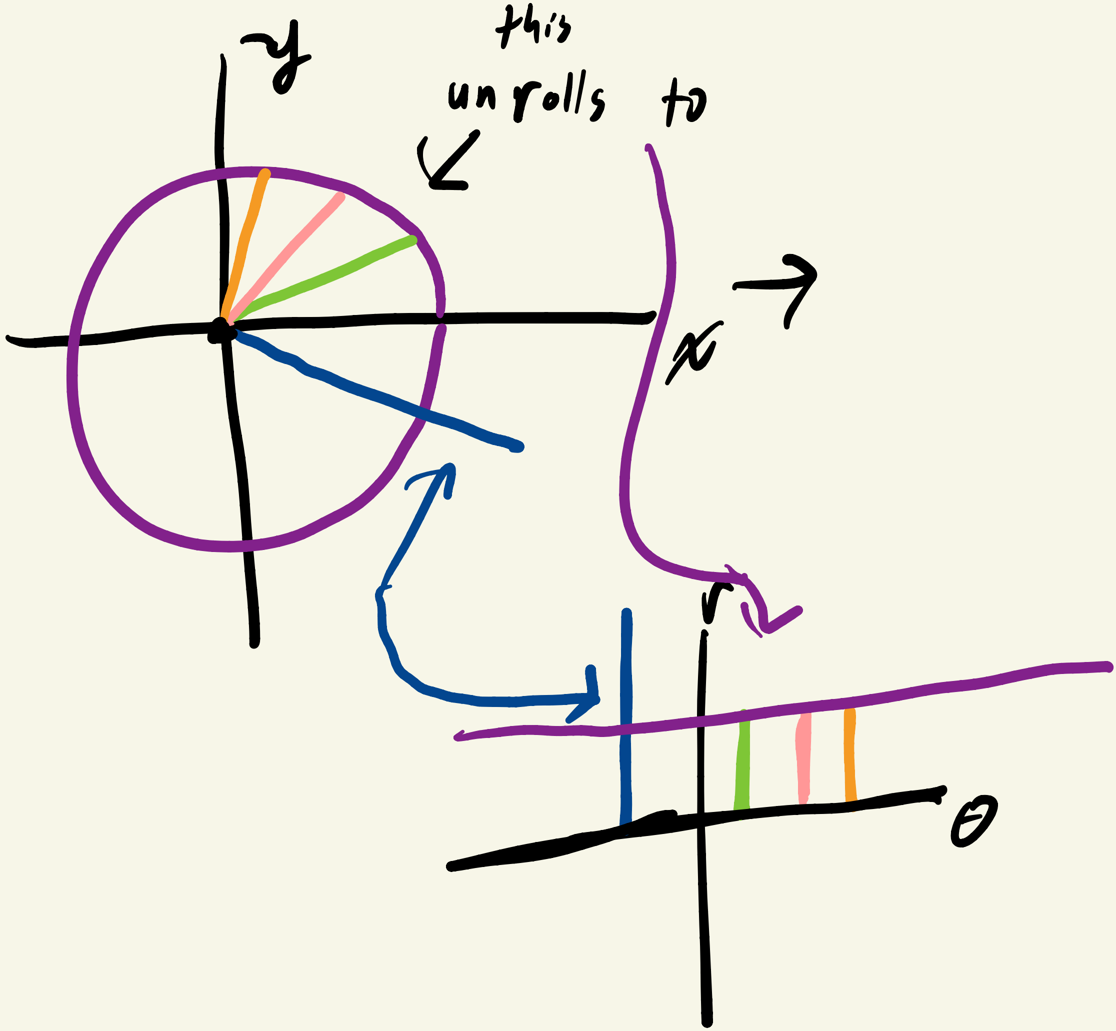
Idea of changing coordinates

- u-sub was a 1-dimensional "change of coordinates"

Could we choose 2 different axis to "regularize" this?

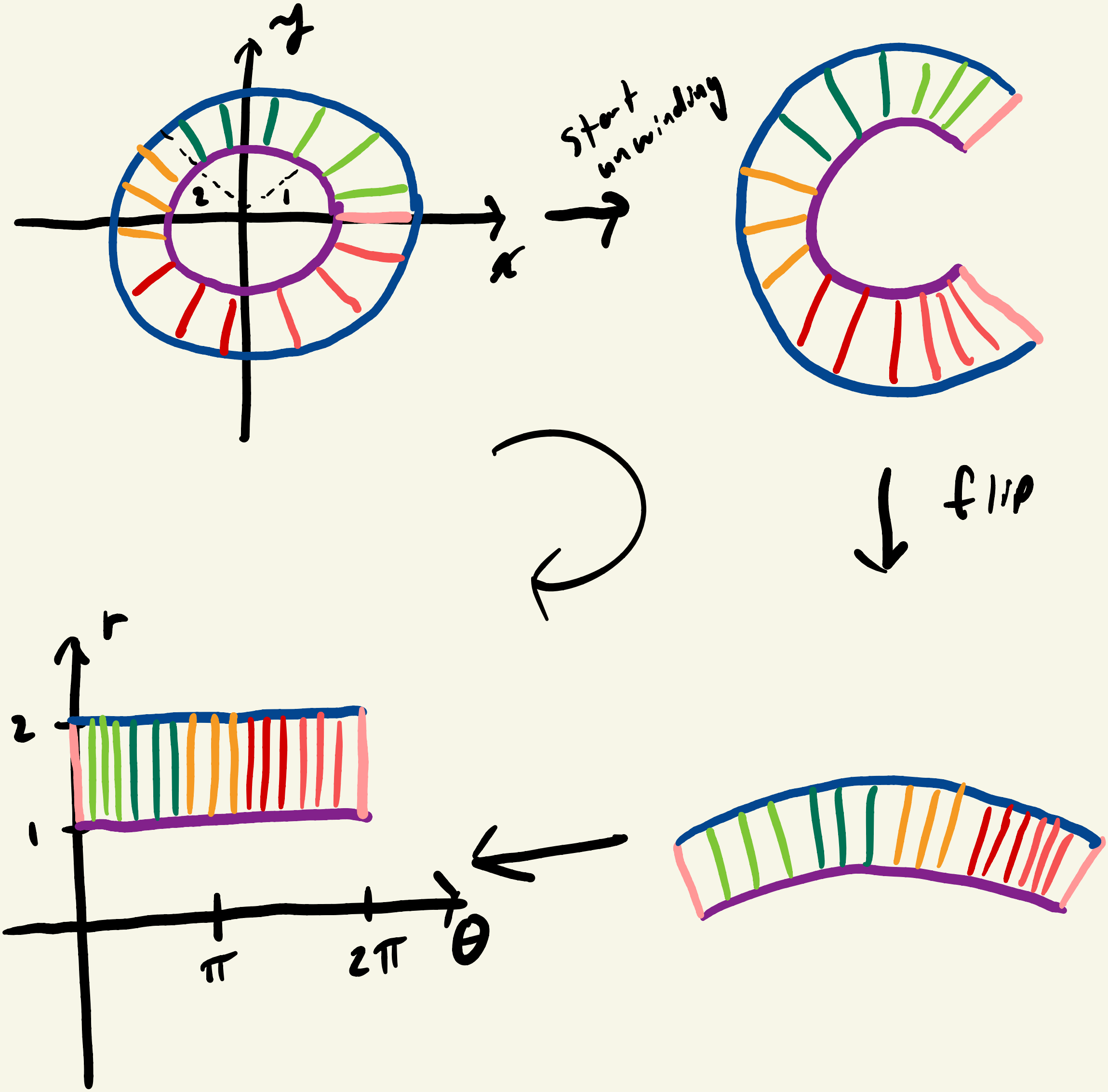




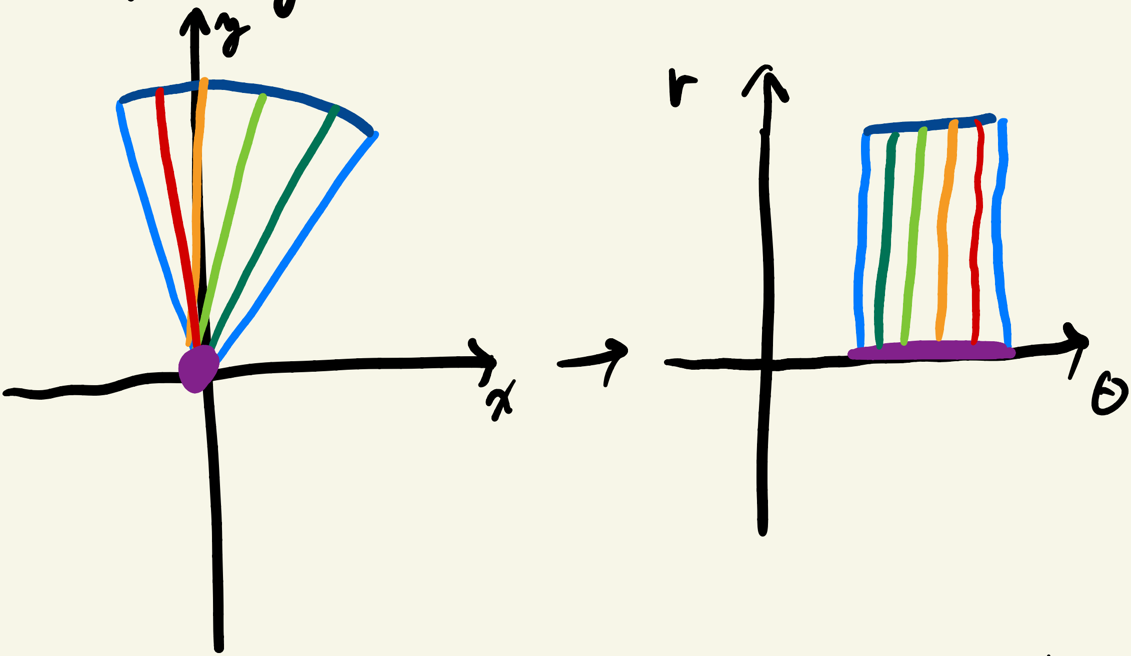


y
this
unrolls
to
x
r
θ

I think avoiding the origin gives the most clean geometric picture of the "unwrapping" we are doing from polar to cartesian



So overall we get the mappings from "wedges" to rectangles



but the origin in the x-y plane has to "blow up" to become the θ-axis