Integration techniques

F.T.C.

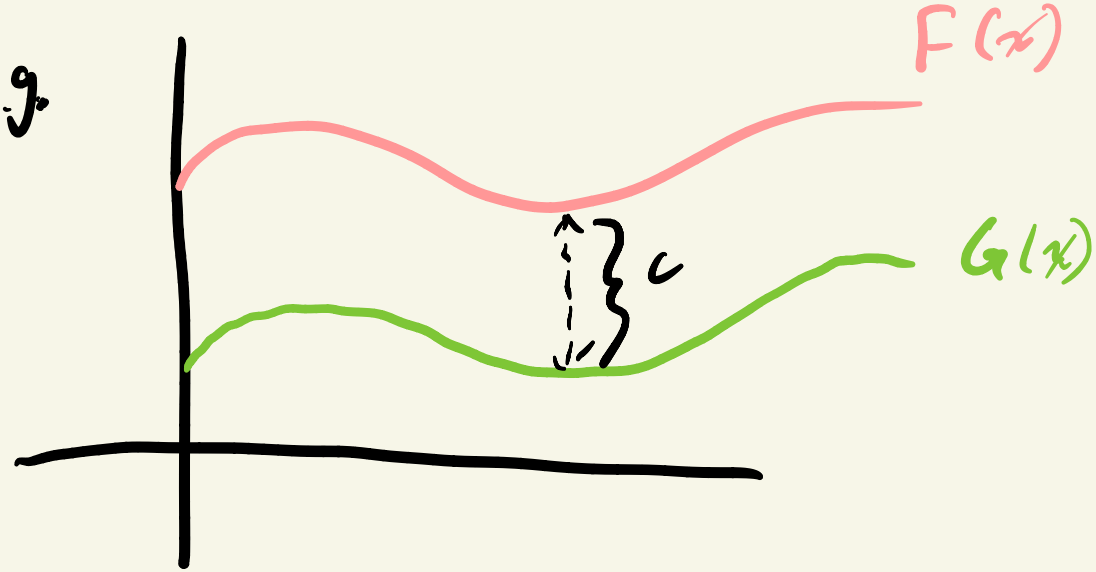
If we have functions, f, F, s.t. (such that) $\frac{d}{dx}[F(x)] = f(x)$, then we say that $F(x)$ is <u>an antiderivative</u> for $f(x)$.

Note: $F(x)+c$ would also be an antiderivative.

Fact: If $F(x)$, $G(x)$ are both antiderivatives for $f(x)$, then there is some real number c $(c \in \mathbb{R})$ s.t.

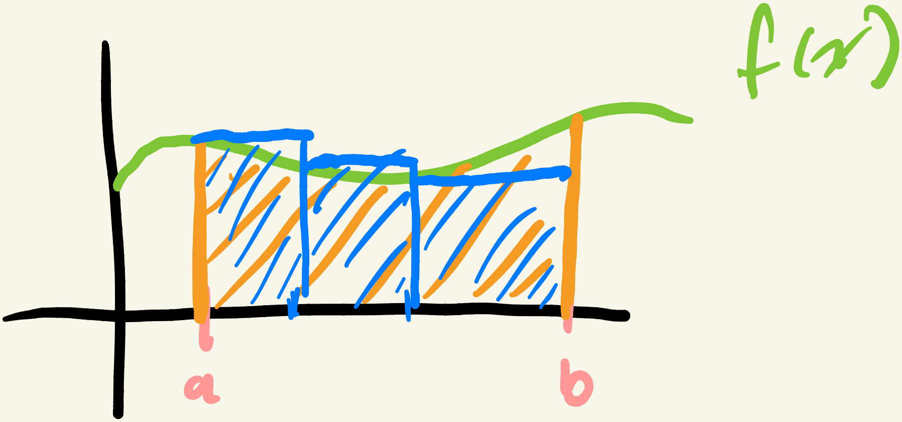
$$F(x) = G(x) + c$$

e.g.



Integration:

Given $f(x)$, $\int_a^b f(x)\,dx$ comes from the Riemann integral.



Def

$$\int f(x)\,dx = \text{"}F(x) + C\text{"}$$

where $F(x)$ is an antiderivative of $f(x)$

Fact (F.T.C.):

If F is any antiderivative for f,

then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Note If $F(a) = 0$, then

$$\int_a^x f(y)\,dy = F(x)$$

I can change my question from

- "what is the (Riemann) integral of f"
- "what differentiates to f"

"$\int_0^1 x\,dx$"?

"what differentiates to x"?

$\left(\frac{1}{2}x^2\right|$

"what differentiates to $f + g$"

- fg
- $f(g(x)) = f \circ g$
- f^g

"U-sub"

$f \circ g(x)$

chain rule

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$$

If F is antiderivative for f,

~~G " " g, then~~

looking for antiderivative of

$$\underline{f(g(x)) \cdot g'(x)},$$

try $\underline{F(g(x))}$

$$\frac{d}{dx}\left[F(g(x))\right] = F'(g(x)) \cdot g'(x)$$

$$= f(g(x)) \cdot g'(x)$$

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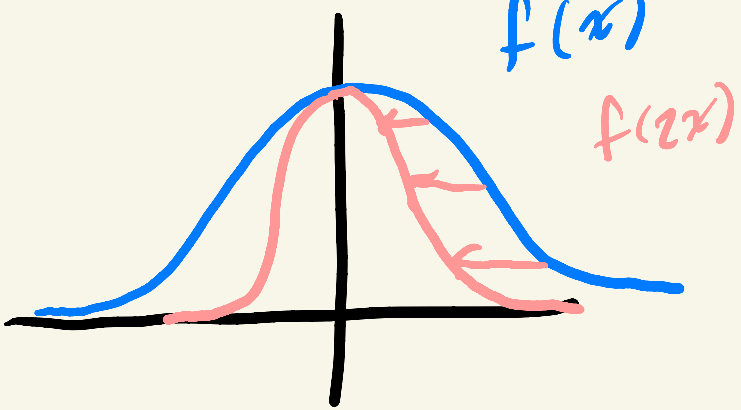
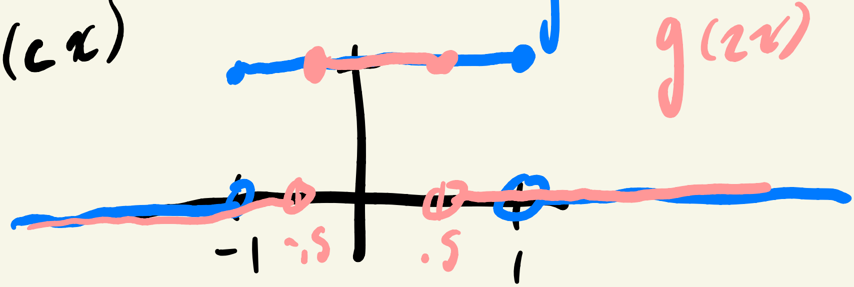
look at

$c = 2$

$f(cx)$



$g(cx)$

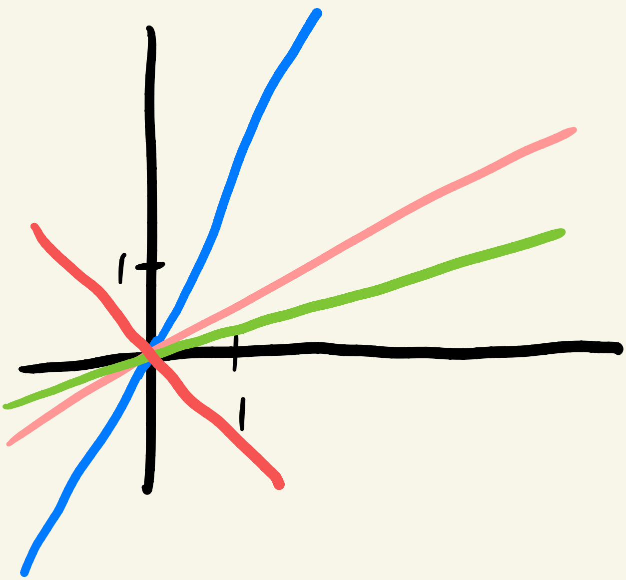


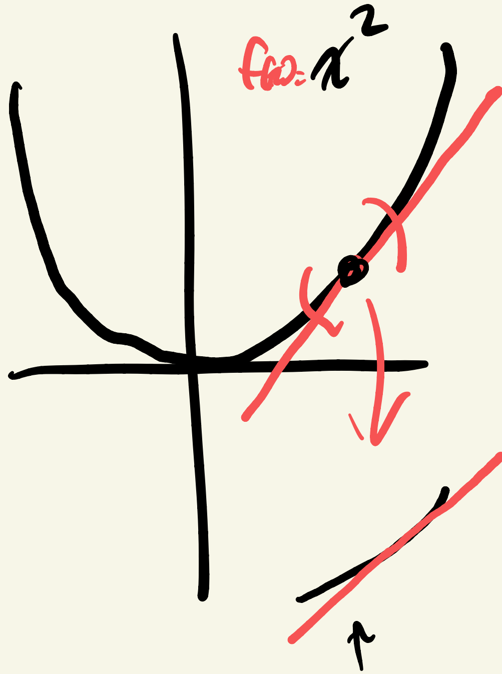
$$\int_{-\infty}^{\infty} g(x)\,dx = 2$$

$g(f(x))$

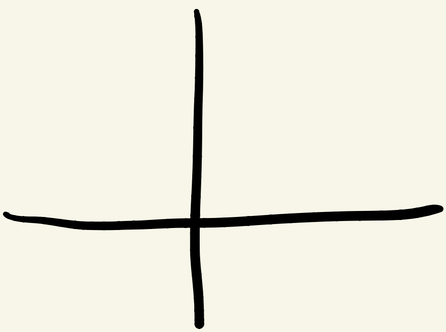
$f(x) = 2x$

$$\int_{-\infty}^{\infty} g(2x)\,dx = 1$$





Slope 2



$g(f(x)) \approx \underline{g(2x+c)}$

"to make up for the squished area, we have to multiply by the derivative of the inside"

$$f'(g(x)) \cdot g'(x)$$

"squishes the x axis"

"stretches the y axis"

$$\int f(x)\, g(x)\, dx$$

what can we say about this?

easiest cases:

$$\int x \cdot g(x)$$

$\sin(x)$

e^{x}

$x^2 + 3$ or other poly?

What can we say about integrals of products?

- product rule

$$\int_a^b (fg)' \, dx = fg(b) - fg(a)$$

$$(fg)' = f'g + fg'$$

$$\int (fg)' \, dx = \int f'g \, dx + \int fg' \, dx$$

$$\int f'g \, dx = \int (fg)' \, dx - \int fg' \, dx,$$

$$\int f'g \, dx = fg - \int fg' \, dx$$

$$\int (fg)' \, dx = fg + C$$

$$\int x \cdot \cos x \, dx = \int \frac{d}{dx}\left[\tfrac{1}{2}x^2\right] \cos x \, dx$$

for f',

$$= \int \frac{d}{dx}[\sin x] \cdot x \, dx$$

what is $\int x$ or $\int \cos x$

$= \frac{1}{2}x^2$ $\qquad$ $= \sin x$

$$\underline{\int f'g \, dx} = fg - \underline{\int f g' dx}$$

say $fg(b) = 0$ or c & $fg(a) = 0$ or c

$$\int_a^b f'g \, dx = -\int_a^b f \cdot g'$$

$$\int_a^b f'g\,dx = \left[fg\right]_a^b - \int_a^b f g'\,dx$$

$$\left(fg(b) - fg(a)\right)$$

$$\int x \cdot \cos x\,dx = \int \frac{d}{dx}\left[\tfrac{1}{2}x^2\right] \cos x\,dx \quad ①$$

$$= \int \frac{d}{dx}\left[\sin x\right] \cdot x\,dx \quad \boxed{2}$$

$$\int f'g\,dx = fg - \int f g'\,dx$$

$$\int \frac{d}{dx}\left[\sin x\right] \cdot x = x \sin x - \int \sin x \cdot \frac{d}{dx}[x]\,dx$$

$$= x \sin x - \int \sin x\,dx$$

$$= x \sin x + \cos x + C$$

$$\int \frac{d}{dx}\left[\frac{1}{2}x^2\right]\cos x\,dx = \underbrace{\frac{1}{2}x^2\cos x + \int \frac{1}{2}x^2 \sin x\,dx}$$

$$= \text{''} \quad \text{''} + \int \frac{d}{dx}\left[\frac{1}{6}x^3\right]\sin x\,dx$$

d/dx

$$\int x^2 e^x\,dx = \int x^2 \frac{d}{dx}\left[e^x\right]dx$$

$$= \left[x^2 e^x\right] - \int 2x\,e^x\,dx$$

$$= \left[x^2 e^x\right] - \int 2x \frac{d}{dx}\left[e^x\right]dx$$

$$= \left[x^2 e^x\right] - \left(\left[2xe^x\right] - \underbrace{\int 2e^x\,dx}\right)$$

$$= \left[x^2 e^x\right] - 2xe^x + 2e^x + C$$

question:

write your own test question!

Find an indefinite integral that you need both IBP (integration by parts) and u substitution to solve.

(Hint: start with a u sub problem)

Harder question:

$$\int e^x \sin x \, dx \quad ???$$

(Hint: What symmetries arise from applying IBP?)

Office hour tomorrow @ 9 AM