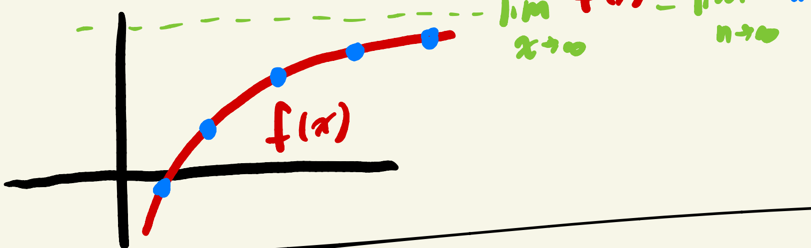
Some theorems regarding sequence convergence

• First recall that if we have a sequence $\{a_n\}$ and function $f(x)$ so that at the natural numbers, $f(n)=a_n$, and if $\lim_{x\to\infty} f(x)$ exists and is a real number, then $\lim_{n\to\infty} a_n$ exists and

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x).$$



(It turns out this also holds if $\lim_{x\to\infty} f(x)$ is ∞ or $-\infty$ but we will avoid using this if possible.)

Also recall that in class, we constructed functions corresponding to sequences w.r.t. convergence. That is, we asserted that

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) \quad \text{when} \quad f(x) = a_{(x)}.$$

This holds true if either limit exists, and, if one limit diverges, the other does as well.

The second construction immediately allows us to prove versions of some of our function convergence theorems for sequences.

Algebraic limit theorem for sequences:

Suppose we have sequences $\{a_n\}$, and $\{b_n\}$, such that $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ both exist, and in addition, we have some real number c. Then

- $\lim_{n\to\infty}(a_n + b_n) = \left[\lim_{n\to\infty} a_n\right] + \left[\lim_{n\to\infty} b_n\right]$
- $\lim_{n\to\infty}(c \cdot a_n) = c \cdot \left[\lim_{n\to\infty} a_n\right]$
- $\lim_{n\to\infty}(a_n \cdot b_n) = \left[\lim_{n\to\infty} a_n\right] \cdot \left[\lim_{n\to\infty} b_n\right]$

- $\lim_{n\to\infty} (a_n/b_n) = \left[\lim_{n\to\infty} a_n\right] / \left[\lim_{n\to\infty} b_n\right]$ if $\lim_{n\to\infty} b_n \neq 0$

Order limit theorem:

Suppose we have sequences $\{a_n\}$, and $\{b_n\}$ so that $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ both exist, and for all natural numbers n, $a_n \leq b_n$. Then

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n.$$

Squeeze theorem:

Suppose we have a sequence $\{b_n\}$, and sequences $\{a_n\}$, $\{c_n\}$ so that for all natural numbers n, $a_n \leq b_n \leq c_n$, and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$$

for some real number L. Then $\lim_{n\to\infty} b_n$ exists and

$$\lim_{n\to\infty} b_n = L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

Extra Check in questions:

- Evaluate $\lim_{n\to\infty} \frac{1}{n!}$ by considering the sequences $\{a_n\}$, and $\{c_n\}$ defined by $a_n = 0$ and $c_n = \frac{1}{n}$, and using the squeeze theorem.

- Suppose $\{a_n\}$ is a sequence whose terms are all positive real numbers. I it possible for $\lim_{n\to\infty} a_n$ to be negative? Why? (use a theorem)

- Are there any real numbers c so that if $\{a_n\}$ is defined by

$$a_n = \sin(cn),$$

then $\lim_{n\to\infty} a_n$ converges?

If so give an example. If not, explain why.

Harder questions

↓

- Given an arbitrary sequence $\{a_n\}$ such that $\lim_{n \to \infty} a_n = 0$. Determine a way to construct some sequence $\{b_n\}$ so that b_n decays faster than $\{a_n\}$

- Can we have some sequences $\{a_n\}$ and $\{b_n\}$ which have positive values, are monotone decreasing and both converge to 0, such that $\{a_n\}$ does not decay faster, slower, or at the same rate as $\{b_n\}$?

i.e. $\lim_{n \to \infty} \frac{a_n}{b_n}$ diverges in some other manner than "to ∞"

Hint: try to define $\left\{\frac{a_n}{b_n}\right\}$ sequence first, & then define the sequences $\{a_n\}, \{b_n\}$