TESTS FOR CONVERGENCE AND DIVERGENCE OF SERIES

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Here we will state the big theorems/tests we have learned to check for convergence and divergence of series. We will try to provide examples using a variety of valid justifications. We will also cover some important and common tricks you may see.
You should be comfortable with using all the tests in this document, but theorems preceded by a * can be considered optional. Also, footnotes are intended for clarification, this information is not very important for exams.

Trick (Telescoping Series)**.** Sometimes when the below tests will not work for us, we must resort to looking at the sequence of partial sums. Recall that $\sum_{n=1}^{\infty} a_n$ is a limit defined by

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty}\Big(\sum_{n=1}^{N} a_n\Big) = \lim_{N\to\infty}(a_1 + a_2 + a_3 + \cdots + a_N).$$

Telescoping series are a nice kind of series where the terms take the form $b_n - b_{n+1}$ for some sequence $\{b_n\}$ which converges to zero. We will present a concrete example to illustrate how to handle these series.
Q: Does the series $\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$ converge or diverge? A: We know that $\lim_{n\to\infty}\frac{1}{n+1} = 0$. Now,

$$\begin{aligned}\sum_{n=1}^{\infty}\Big(\frac{1}{n} - \frac{1}{n+1}\Big) &= \lim_{N\to\infty}\Big(\sum_{n=1}^{N}\frac{1}{n} - \frac{1}{n+1}\Big)\\ &= \lim_{N\to\infty}\bigg(\Big(\frac{1}{1} - \frac{1}{2}\Big) + \Big(\frac{1}{2} - \frac{1}{3}\Big) + \Big(\frac{1}{3} - \frac{1}{4}\Big) + \cdots + \Big(\frac{1}{n} - \frac{1}{n+1}\Big)\bigg)\\ &= \lim_{N\to\infty}\bigg(\frac{1}{1} + \Big(-\frac{1}{2} + \frac{1}{2}\Big) + \Big(-\frac{1}{3} + \frac{1}{3}\Big) + \cdots + \Big(-\frac{1}{n} + \frac{1}{n}\Big) - \frac{1}{n+1}\bigg)\\ &= \lim_{N\to\infty}\Big(\frac{1}{1} - \frac{1}{n+1}\Big)\\ &= 1\end{aligned}$$

So the series $\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$ converges, and in fact, $\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 1$.

Theorem. Suppose we have some sequence $\{a_n\}_{n=\alpha}^{\infty}$ where α is some integer. Then if we have integers β, γ such that $\beta, \gamma \geq \alpha$, then $\sum_{n=\beta}^{\infty} a_n$ converges if and only if $\sum_{n=\gamma}^{\infty} a_n$ converges.

The above assertion indicates that our convergence tests should not depend on the starting index. When applicable, we will use the notation $\{a_n\}$ and $\sum a_n$ to emphasize this fact.

Theorem (Test for Divergence)**.** If we have some sequence $\{a_n\}$ such that $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n$ exists and is not 0, then $\sum a_n$ diverges.

Example. Q: Does the series $\sum_{n=1}^{\infty}(1 - \frac{1}{n})$ converge or diverge?
A: The series $\sum_{n=1}^{\infty}(1 - \frac{1}{n})$ diverges.

Notice that $\lim_{n\to\infty}(1-\frac{1}{n} = 1$ and so by the test for divergence, the series $\sum_{n=1}^{\infty}(1-\frac{1}{n})$ diverges.

Q: Does the series $\sum_{n=1}^{\infty}(-1)^n$ converge or diverge?
A: The series $\sum_{n=1}^{\infty}(-1)^n$ diverges.
Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined by the rule $a_n = (-1)^n$. Then we have that $\lim_{n\to\infty} a_n$ does not exist, and hence $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(-1)^n$ diverges.

Theorem (Geometric Series)**.** Here we consider a special type of series. Here, since we consider the value we converge to, the initial index does matter. Suppose we have some series $\sum_{n=1}^{\infty} b_n$ where we can choose some real number a and some real number r such that $b_n = ar^{n-1}$. Then $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} ar^{n-1}$ is called a **geometric series**. If $|r| < 1$, then $\sum_{n=1}^{\infty} ar^{n-1}$ is convergent and

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

If $|r| \geq 1$ then $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.[1]

Theorem (Integral Test)**.** We want to determine the convergence or divergence of some series $\sum_{n=1}^{\infty} a_n$.
Suppose f is a continuous[2], positive, decreasing function on $[1,\infty)$ such that $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent. In other words:

- If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
- If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Theorem (p-series)**.** This is just a name for a certain type of sequence. A series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ with $p > 0$ is called a p-**series**. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $0 < p \leq 1$.

The above theorem follows directly from the integral test and you should be comfortable proving it.

Theorem (Direct Comparison Test)**.** The intuition: Here we are considering series $\sum a_n$ and $\sum b_n$ where the sequences $\{a_n\}$ and $\{b_n\}$ have only nonnegative terms and seeing how comparisons of the term size can allow us to compare the convergence/divergence properties of the sequences.
Suppose $\{a_n\}$ and $\{b_n\}$ are sequences such that for every n, $a_n \geq 0$ and $b_n \geq 0$. Then the following statements hold.

[1]The reason for the notation using r^{n-1} is to make so that we can use $n = 1$ for our initial term and have first term in the sequence be a. You may also see geometric series notated by $\sum_{n=0}^{\infty} ar^n$. This is exactly the same series.

[2]The continuity property is really just to ensure that we can integrate $f(x)$. We can extend this in a number of ways, the easiest being that can allow $f(x)$ to have finitely many discontinuities. It is usually easier to just check for continuity though.
Also, if we have some positive integer b such that the above hypotheses hold for f on $[b,\infty)$ then we can replace the 1 in both the integral and the sum with b and the theorem still holds. It is usually easier to just use 1.

- If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

- If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

Theorem (Limit Comparison Test)**.** The intuition: Here we are considering series $\sum a_n$ and $\sum b_n$ where the sequences $\{a_n\}$ and $\{b_n\}$ have only nonnegative terms and seeing how comparisons of the growth rate of $\{a_n\}$ and $\{b_n\}$ can allow us to compare the convergence/divergence properties of the sequences.
Suppose $\{a_n\}$ and $\{b_n\}$ are sequences such that for every n, $a_n \geq 0$ and $b_n \geq 0$. Then if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either $\sum a_n$ and $\sum b_n$ both converge or $\sum a_n$ and $\sum b_n$ both diverge.

Now we present an unsurprising but useful extension to the limit comparison test. Learning it is optional, but I think it could make your life easier.

***Theorem** (Generalized Limit comparison Test)**.** Suppose $\{a_n\}$ and $\{b_n\}$ are sequences such that for every n, $a_n \geq 0$ and $b_n \geq 0$. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

Where c is a finite number (possibly 0), then the following statements hold.

- If $\sum b_n$ converges then $\sum a_n$ also converges.
- If $\sum a_n$ diverges then $\sum b_n$ also diverges.

Alternatively, if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c,$$

where $c > 0$ and possibly $c = \infty$, then the following statements hold.

- If $\sum a_n$ converges then $\sum b_n$ also converges.
- If $\sum b_n$ diverges then $\sum a_n$ also diverges.

Series From Sequences With Some Negative Terms. So far we have looked at tests which apply to series generated by sequences with nonnegative terms. We we discuss how to deal with those generated sequences which have some negative terms and introduce a new test.

Definition. Given a series $\sum a_n$, we say that $\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges.

Theorem. If $\sum a_n$ is a series which converges absolutely, then $\sum a_n$ also converges in the usual sense.

This result is extremely useful. If we are dealing with a series $\sum a_n$ where $\{a_n\}$ is a sequence with some negative terms, then we automatically can not use many of the useful tests described above.[3] However, we may be able to apply the tests to $\sum |a_n|$. A common question we ask is whether a series converges absolutely, converges (but not absolutely), or

[3]The careful reader might notice that the problems only really come up for sequences which have both infinitely many positive terms and infinitely many negative terms. For sequences where this problematic condition does not hold, the problem of absolute convergence is equivalent to the problem of convergence.

diverges. Now we look at some ways to tackle the question of convergence possibly without absolute convergence.

Definition. We call $\sum a_n$ an **alternating series** if the terms of a_n alternate between nonnegative and nonpositive. That is, if there is some sequence $\{b_n\}$ such that $b_n \geq 0$ for all n, and either $a_n = (-1)^n b_n$ for all n or $a_n = (-1)^{n+1} b_n$ for all n.

Theorem (Alternating Series Test)**.** If we have an alternating series, $\sum(-1)^n b_n$, where $\{b_n\}$ is a nonnegative sequence such that b_n is monotone decreasing and $\lim_{n\to\infty} b_n = 0$, then $\sum(-1)^n b_n$ converges.
This theorem also applies to alternating series of the form $\sum(-1)^{n+1} b_n$.

Theorem (Ratio Test)**.** Suppose we have some series $\sum a_n$ where $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ converges.

- If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum a_n$ is absolutely convergent.
- If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, then the series $\sum a_n$ is divergent.

For the above theorem, if we have $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then we say that "the ratio test is inconclusive for $\sum a_n$." The ratio test does not give us information about the convergence or divergence of these series. This test can be thought of as measuring how much a series acts like a geometric series.