INTEGRATION ALGORITHMS FOR SPECIAL FUNCTIONS

IAN MILLER

1. Introduction

We will present algorithms for computing antiderivatives of special functions. We will present an algorithm for functions of the form $\sin^n(x)\cos^n(x)$ with $n, m \geq 0$. We will also provide an algorithm for functions of the form $\sec^n(x)\tan^m(x)$ with $n \geq 1$ and $m \geq 0$. We will also provide a set of practice problems.
The general method for constructing these algorithms is to directly compute a class of integrals, and then to find ways to reduce all of the integrals in question to those basic integrals.

2. Integrating Functions of the Form $\sin^n(x)\cos^n(x)$

The integral is trivial when $n, m = 0$ so we can just consider cases where $n \geq 1$ or $m \geq 1$.

2.1. Computations when $n = 1$ or $m = 1$.

We will directly compute these cases. First we will consider $\sin(x)\cos^n(x)$. In this integral we will use the substitution $u = \cos(x)$.

$$\begin{aligned}\int \sin(x)\cos^n(x)dx &= -\int (-\sin(x))\cos^n(x)dx \\ &= -\int u^n du \\ &= \frac{-u^{n+1}}{n+1} \\ &= \frac{-\cos^{n+1}(x)}{n+1}\end{aligned}$$

Now we will consider $\sin^m(x)\cos(x)$. In this integral we will use the substitution $u = \sin(x)$.

$$\begin{aligned}\int \sin^m(x)\cos(x)dx &= \int u^m du \\ &= \frac{u^{m+1}}{m+1} \\ &= \frac{\sin^{m+1}(x)}{m+1}\end{aligned}$$

This is just a technical note, but notice that these computations are valid when the variable exponent is equal to zero.

2.2. **Computations when n is odd or m is odd.** Recall the identities $\sin^2(x) = 1 - \cos^2(x)$ and $\cos^2(x) = 1 - \sin^2(x)$. We will use these identities freely in the following computations. First suppose that m is odd. This implies that we can choose some nonnegative integer l, such that $m = 2l + 1$ and we have the following equality:

$$\sin^m(x)\cos^n(x) = (\sin^2(x))^l \sin(x)\cos(x) = \sin(x)\cos(x)(1 - \cos^2(x))^l.$$

Notice then that the integral

$$\int \sin^m(x)\cos^n(x)dx = \int \sin(x)\cos(x)(1 - \cos^2(x))^l dx.$$

will then decompose into sums and differences of integrals of a form that we have already solved. For example:

$$\begin{aligned}
\int \sin^5(x)\cos^4(x)dx &= \int \sin(x)\cos^4(x)(1 - \cos^2(x))^2 dx \\
&= \int \sin(x)\cos^4(x)(1 - 2\cos^2(x) + \cos^4(x))dx \\
&= \int \Big(\sin(x)\cos^8(x)dx\Big) - 2\Big(\sin(x)\cos^6(x)dx\Big) + \Big(\int \sin(x)\cos^4(x)dx\Big).
\end{aligned}$$

The case with n being odd uses a symmetric method applying the identity $\cos^2(x) = 1 - \sin^2(x)$ to reduce the maximum exponent on the sin function to 1.

2.3. **Computations when both n and m are even.** The following two identities will be useful.

$$\sin^2(x) = \frac{1}{2}(1 - \cos 2x) \qquad \text{and} \qquad \cos^2(x) = \frac{1}{2}(1 + \cos 2x)$$

The fundamental idea here is to apply both identities to obtain an expression comprised only of constants and exponents of $\cos(2x)$. Then we decompose this integral into the summands, and if there are any integrals containing only even powers of $\cos(2x)$, we apply the identity again to obtain an expression of $\cos(4x)$. We can repeat this until we only have to integrate constants and odd power cosine functions. Seeing an example is helpful for this.

$$\begin{aligned}
\int \sin^2(x)\cos^2(x)dx &= \int (\frac{1}{2}(1 - \cos 2x))(\frac{1}{2}(1 + \cos 2x))dx \\
&= \frac{1}{4}\int (1 - \cos^2 2x)dx \\
&= \frac{1}{4}\int 1dx - \frac{1}{4}\int \cos^2 2x dx \\
&= \frac{1}{4}\int 1dx - \frac{1}{4}\int (\frac{1}{2}(1 + \cos 4x))dx \\
&= \frac{1}{4}\int 1dx - \frac{1}{8}\int 1dx - \frac{1}{8}\int \cos 4x dx.
\end{aligned}$$

Notice that integrating $\cos^m(2^j x)$ has the same fundamental difficulties as the integral $\cos^m(x)$ after a linear substitution of $u = 2^j x$.
From these techniques we can now integrate all of the desired functions.

3. Integrating Functions of the Form $\sec^n(x)\tan^n(x)$

We will again break the problem down into cases. We will show how to integrate all functions of this form with $n \geq 1$ and $m \geq 0$

3.1. **The integral of secant.** The integral of $\sec(x)$ is $\ln(\sec(x) + \tan(x))$. This integral was, for a time in the seventeenth century, an open problem and is a difficult computation, but it is readily verified by differentiation.

3.2. **Integrating $\sec^2(x)\tan^m(x)$.** We will use the substitution $u = \tan(x)$.

$$\begin{aligned}\int \sec^2(x)\tan^m(x)dx &= \int u^m du \\ &= \frac{u^{m+1}}{m+1} \\ &= \frac{\tan^{m+1}(x)}{m+1}.\end{aligned}$$

3.3. **Integrating $\sec^n(x)\tan(x)$.** We are only concerned with the case when $n \geq 1$. We will use the substitution $u = \sec(x)$.

$$\begin{aligned}\int \sec^n(x)\tan(x)dx &= \int u^n du \\ &= \frac{u^{n+1}}{n+1} \\ &= \frac{\sec^{n+1}(x)}{n+1}.\end{aligned}$$

3.4. **Computations when n is even.** Recall the identity

$$\sec^n = 1 + \tan^2(x).$$

Using a similar technique to what we did for sine cosine functions with an odd exponent, we can use the given identity to decompose the integral into integrals of the form $\sec^2(x)\tan^m(x)$, which we know how to solve.

3.5. **Computations when m is odd.** Recall the identity

$$\tan^2(x) = \sec^2(x) - 1.$$

Using a similar technique to above, we can use the identity to decompose our integral into integrals of the form $\sec^n(x)\tan(x)$.

3.6. **Computations when n is odd and m is even.** This final case is the most difficult and will require new techniques.

4. Integrating Rational Functions

Coming this week