BASICS OF SEQUENCE CONVERGENCE AND DIVERGENCE

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0.1. **Definition.** Sequences allow us to take limits of discrete processes rather than those occuring over continuous time. One reason sequences are so useful is that humans often times have a discrete way of thinking. In experiments we take measurements at discrete times, we chunk our actions into discrete actions. Sequences allow us to use our calculus ideas with a "discrete way of thinking".
First let us define what it means to be a limit of a sequence. Your book takes the following statement to be the definition.

Definition. A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \, \text{as} \, n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty}$ exists we say the sequence **converges**(or is **convergent**). Otherwise we say the sequence **diverges**(or is **divergent**).

A couple notes about this definition. First is that when we say "we can make the terms a_n as close to L as we like by taking n sufficiently large", another way of thinking about this is to say "a_n is eventually as close to L as we would like". We will omit a formal definition since it tends not to be the best way to think about limits in the context of Calc 2. We will however make a note. Consider the sequence

$$\{a_n\} = (1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, \ldots),$$

where we we continually add one more 0 between the 1s as we go. This sequence diverges since, even though we eventually stay at 0 for as long of a time as we want, there will always be *some* time further along where we will visit 1 again.

0.2. **Connection to Function Limits.** You may have noticed that this concept is similar to our idea of taking limits of functions off to infinity. This analogy is useful for conceptualizing sequential limits, and we will present two results which make the analogy more concrete.

Theorem. If f is some function such that $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ for all positive integers n, then $\lim_{n\to\infty} a_n = L$.

Notice that f is a function whose domain is the real numbers and that a_n is a sequence. There is actually a stronger version of this theorem, it is sufficient that $f(n) = a_n$ "eventually" instead of for all n. This should not be too supprising since the limit only cares about the "tail" of the sequence and not anything that only occures at the "beginning" of the sequence.
The next theorem is useful for conception but is not typically useful for any application.

Theorem. We will use $\lfloor\cdot\rfloor$ to denote the floor function. Then if we have a sequence $\{a_n\}$ and define the function $f(x) = a_{\lfloor x\rfloor}$ with domain $[1,\infty)$, then

$$\lim_{n\to\infty} a_n = L \qquad \text{if and only if} \qquad \lim_{x\to\infty} f(x) = L.$$

Furthermore, $\lim_{n\to\infty} a_n$ does not exist if and only if $\lim_{x\to\infty} f(x)$ does not exist and

$$\lim_{n\to\infty} a_n = \pm\infty \qquad \text{if and only if} \qquad \lim_{x\to\infty} f(x) = \pm\infty.$$

After this, most of the results we will discuss should not be too supprising, and some of them follow directly as a result of this most recent theorem.

0.3. **Some Properties of Sequences and a Generalization of the Squeeze Theorem.** We will now state properties of algebraic combinations of sequence limits which are very similar to some properties of function limits which we proved in Calc 1.

Theorem. Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent sequences and that c is some real number. Then the following hold

$$\begin{aligned}
\lim_{n\to\infty}(a_n + b_n) &= \Big(\lim_{n\to\infty} a_n\Big) + \Big(\lim_{n\to\infty}\Big) \\
\lim_{n\to\infty}(c\cdot a_n) &= c\cdot \lim_{n\to\infty} a_n \\
\lim_{n\to\infty}(a_n\cdot b_n) &= \Big(\lim_{n\to\infty} a_n\Big)\cdot\Big(\lim_{n\to\infty} b_n\Big) \\
\lim_{n\to\infty}\frac{a_n}{b_n} &= \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} && \text{if} \lim_{n\to\infty} b_n \neq 0 \\
\lim_{n\to\infty} a_n^p &= \Big(\lim_{n\to\infty} a_n\Big)^p && \text{if}\, p > 0 \,\text{and}\, a_n \geq 0 \,\text{for}\, n \geq 0.
\end{aligned}$$

The squeeze theorem is another theorem for limits of functions for which there is an analogous theorem for sequences. We state it below.

Theorem. If we have sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ such that

$$a_n \leq b_n \leq c_n \qquad \text{for all}\, n \geq 1,$$

and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

then

$$\lim_{n\to\infty} b_n = L.$$

The following statement is an immediate corrolary from the "squeeze theorem for sequences". It would be good to check how you would prove the result yourself.

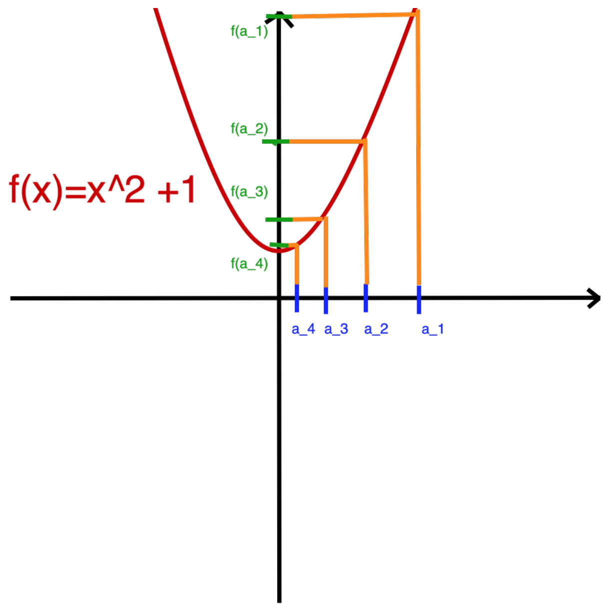
Theorem. If we have some sequence $\{a_n\}$ such that $\lim_{n\to\infty}|a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$.

0.4. **A Result regarding sequences and continuous functions.** Here we will state two results, one useful and one for amusment. Here is the useful result.

Theorem. If we have some function f which is continuous at L, and some sequence $\{a_n\}$ such that $a_n \to L$ as $n\to\infty$, then

$$\lim_{n\to\infty} f(a_n) = f(L).$$

The following picture may be useful for visualizing why this theorem is true.



Now this theorem intuitively states that if we approximate a domain value, L, and plug those values into the function, the outputs will approximate $f(L)$. It turns out that this is really what we want out of our continuous functions, and is in fact sometimes the way that continuity is defined. The following theorem illustrates this fact, but will not be used after this in the course.

Theorem. A function f is continuous at L if and only if given *any* sequence $\{a_n\}$ such that $\lim_{n\to\infty} a_n = L$, we also have that $\lim_{n\to\infty} f(a_n) = f(L)$.

This statement is a bit tricky because it can be hard to think of *every* possible sequence which converges to some given value.

0.5. **Convergence of monotone, bounded, functions.** This will be updated after we talk about it in class.