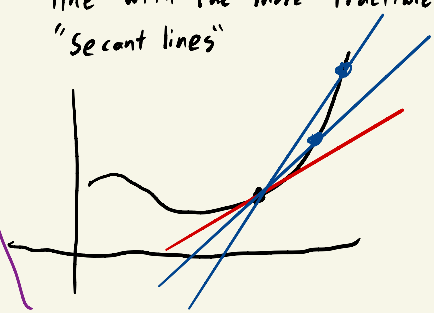
Taylor Polynomials and Series

- (1) Taylor Polynomials

In many calculus and analysis situations we obtain results and answer questions by approximating difficult to deal with structures by easier to deal with ones.

Just think: This is how we defined the derivative, by approximating the somewhat nebulous "tangent line" with the more tractible "secant lines"



In Cac 1 we discussed the idea of "linear approximations to functions at given points". A linear approximation is a linear function which shares both the function's value and derivative at a point.

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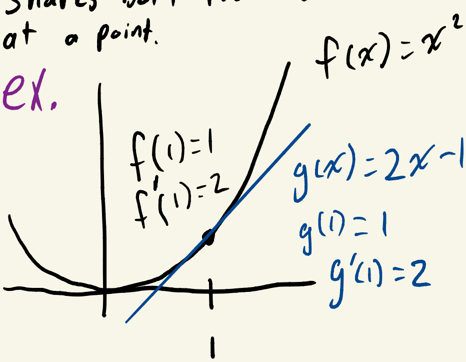


The reason we care about linear approximations is that sometimes the functions we are dealing with are hard to deal with, but lines are easy to deal with. If you were lucky in Calc 1, you got to work on questions where using linear approximations made the solution much easier.

Linear approximations are very nice but we only retain two pieces of information about the origional structure, the function value and derivative at a point.

We run into the question, are there any other "simple" structures we can compare a function to so that we can retain more information?

- It turns out, if we want to extend our idea of linear appx. in order to keep more information about the origional function, one natural structure to look at is the polynomial.

Recall that an n^{th} degree polynomial is a polynomial where the largest power of x with a nonzero coefficient is x^n. So, polynomials of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$.

e.g. x^2+1 is a degree 2 polynomial

$x^3+x^5+3x^4+1$ is a degree 5 polynomial

$0x^4+2x^3+4$ is a degree 3 polynomial

An n^{th} degree polynomial has at most n nonzero derivatives. For linear approximation, we approximated functions with lines sharing a first derivative with the function. For polynomial approximation, we will be approximating functions with polynomials which share first derivative and also higher degree derivatives with the function.

The following fact is almost too obvious to mention, but it demonstrates one reason that polynomials are very nice

Fact: Given a function, f, which is at least n times differentiable at some point a, there exists a unique polynomial of degree at most n such that for all i satisfying $0 \leq i \leq n$, $f^{(i)}(a) = p^{(i)}(a)$.

This means that we can always find a polynomial satisfying these conditions and once we do, we've found the only one.

It turns out that these polynomials are easy to compute if we can compute the derivatives of our function are easy to compute. The following definition tells us how to compute the polynomials.

Def: Given a function, f, which is at least n times differentiable at a point, a, the <u>n^{th} degree Taylor polynomial for f, centered at a</u> is the polynomial given by

$$p(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i.$$

Check for yourself that these polynomials do indeed have the properties described in the previous fact. Also, notice that the 1^{st} degree Taylor polynomial at a point is just the linear approximation at that point.

☆ <u>Check your understanding:</u> (Taylor Poly)

- Compute the 5^{th} degree Taylor polynomial for e^x, centered at 0.
- Compute the 4^{th} degree Taylor polynomial for $\sin(x)$, centered at 0. Is there a pattern for the n^{th} degree polynomial?
- Compute the 4^{th} degree Taylor polynomial for $\frac{1}{x}$, centered at 1

Suppose we have some function f and some number n, so that for $i \geq n$, $f^{(i)}(x) \geq 0$ at every x value. Are we garunteed that any of our Taylor polynomials are under approximations? Which ones? What if we only have that $f^{(n)}(x) \geq 0$ for all x, without information about higher derivatives?

Being able to construct garunteed over and under approximations is a very valuable tool, so the last question presents a rather simplistic example of a benefit of using polynomial approximation instead of linear approximation.

Now approximations are most useful when we can talk about error bounds or some sort of limiting behavior. We will circle back to this in the context of Taylor polynomials, but before we do, it will be useful to introduce a new idea.

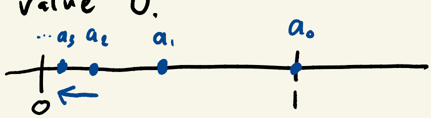
(2) On Limits of Functions

Typically we use limits to discuss numerical approximations.

E.g. when we consider the sequence
$\{a_n\} = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$,
and assert

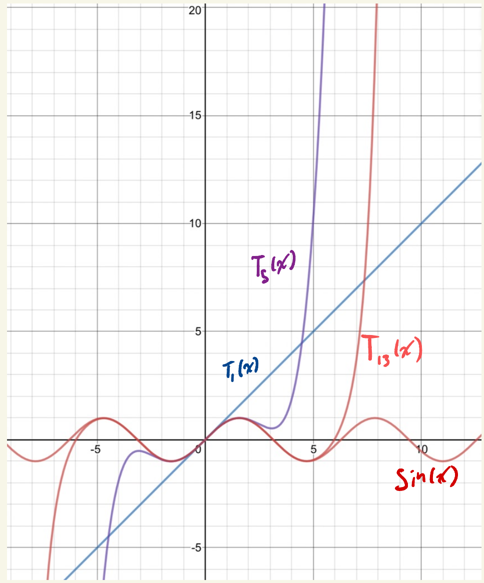
$$\lim_{n\to\infty} a_n = 0,$$

we are saying that the terms of a_n are "eventually" good approximations of the value 0.



We would sometimes write $a_n \to 0$.

Now consider the following graph of some Taylor polynomial approximations of $\sin(x)$.



Here, it appears that, in some way, the Taylor polynomials are approximating $\sin(x)$ better and better as the degree increases, at least for x values close to 0.

This leads to the question, "Can we extend our idea of limits which talk about numerical approximation in order to discuss limits of entire functions?" In other words, is there some rigorous sense in which

$$\lim_{n\to\infty} T_n = \sin(x)?$$

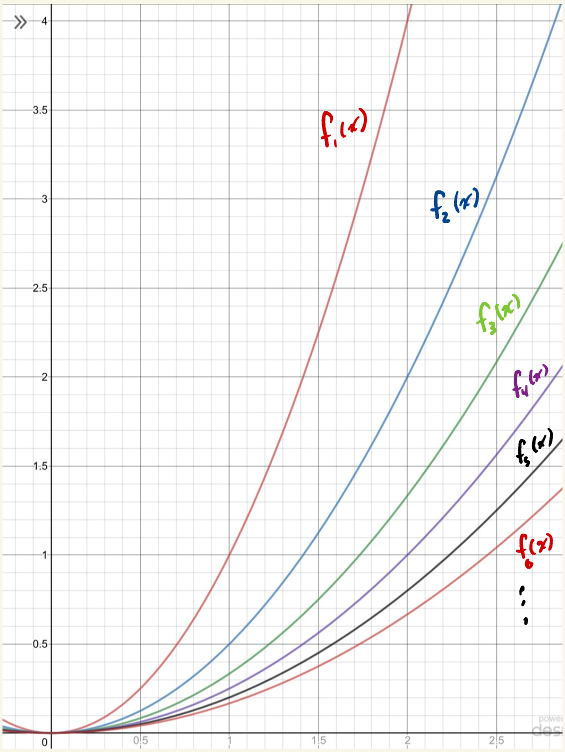
(Notice that the above statement is meaningless with our current definition of lim.)

It turns out that there are incredibly many useful ideas of limits of functions,

(This is partially due to the fact that one could come up with many reasonable ways to discuss "distance" between two functions but for this course, we only need one idea. Suppose we have a sequence of functions, $\{f_n\}_{n=1}^{\infty}$. Notice that this is just a list of functions, we will use the example $\{f_n\}_{n=1}^{\infty}$ given by $f_n(x) = \frac{x^2}{n}$, so

$$\{f_n\} = \left\{x^2, \frac{x^2}{2}, \frac{x^2}{3}, \dots\right\}.$$

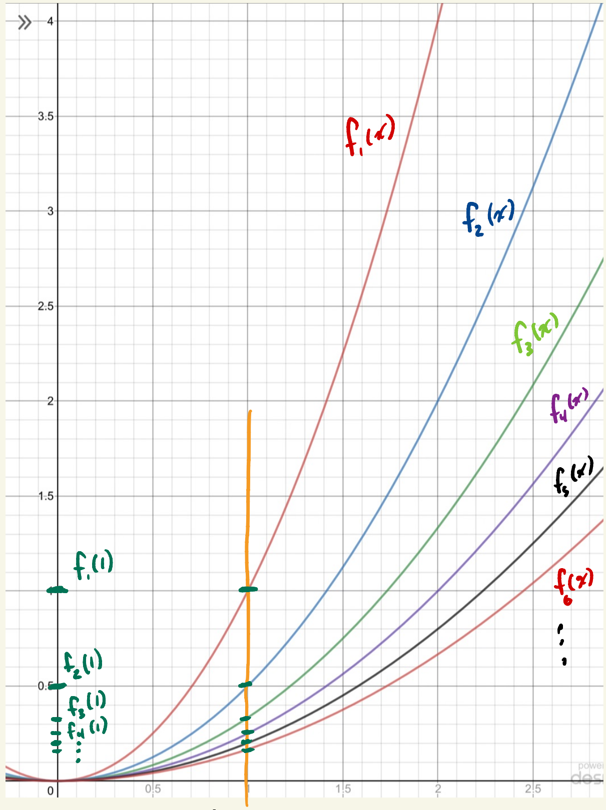
Look at the graphs of these functions.



Now fix some x value, say $x=1$, and notice that for any n, $f_n(1)$ is just a number. Then we can consider the sequence $\{f_n(1)\}_{n=1}^{\infty}$ which is given by $f_n(1) = \frac{(1)^2}{n} = \frac{1}{n}$, so

$$\{f_n(1)\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}.$$

This sequence can be visualized as follows:



Since $\{f_n(1)\}_{n=1}^{\infty}$ is just a sequence of numbers, it is perfectly reasonable to talk about it's convergence & divergence properties. In this case,

$$\lim_{n\to\infty} f_n(1) = \lim_{n\to\infty} \frac{1}{n} = 0.$$

In fact, for any fixed value, a, we can look at the numerical sequence $\{f_n(a)\}_{n=1}^{\infty}$. Here,

$$\lim_{n\to\infty} f_n(a) = \lim_{n\to\infty} \frac{a^2}{n} = a^2 \lim_{n\to\infty} \frac{1}{n} = 0,$$

since a^2 is just a fixed constant. Now things are going to get a bit weird. Since for any fixed value, a, $\{f_n(a)\}_{n=1}^{\infty}$ is a convergent sequence, it is reasonable to construct a function, g, where we define the output of g at a point a by the limit, we say $g(a) = \lim_{n\to\infty} f_n(a)$.

For our sequence, we would have $g(x)=0$ for every x value. However, it would be reasonable to define such a "limit function" whenever we have a sequence of functions where the numerical sequences at fixed input values converge. Our idea of function convergence will be based on this idea.

Def: Given a sequence of functions, $\{f_n\}$, a function, g, and a domain, D, we say that $\{f_n\}$ converges to g on D or $f_n \to g$ on D if for any a in the domain D, when we consider the numerical sequence $\{f_n(a)\}$, and the number $g(a)$, we have

$$\lim_{n\to\infty} f_n(a) = g(a).$$

One important thing to notice is that the "speed of convergence" does not have to be the same in any sense for different x values, in fact for our previous example, no matter how large our n value is, we can choose x values where $f_n(x)$ is rather far away from 0. It is only when we fix an x value that we are garunteed convergence.

Illustration: For $\{f_n(x)\}_{n=1}^{\infty}$ defined by $f_n(x)=\frac{x^2}{n}$, for any n, no matter how large, when we look at x values where $x \geq \sqrt{n}$, we will have that $f_n(x) \geq 1$. The key here is that to get big numbers for large n values, we have to keep changing which x values we look at.

Now that we have introduced this definition, I would like to explicitly state and reiterate an idea we have been using:

When given a sequence of functions, $\{f_n(x)\}$, two ways of thinking about this sequence are

1. An actual list of functions
2. A collection of numerical sequences, where you plug in some real number, a, and obtain the sequence $\{f_n(a)\}$.

It is good to be able to switch between these two ways of thinking, but the second one tends to be more useful for what we will be doing in calc 2.

Now that we have defined what we will mean by "a sequence of functions converging", we get series of functions for free by considering limits of partial sums.

Def: Given a sequence of functions, $\{f_n\}_{n=1}^{\infty}$, a function, g, and a domain D, we say $\sum_{n=1}^{\infty} f_n = g$ on D if

$$\lim_{N\to\infty} \sum_{n=1}^{N} f_n = g,$$

where the second limit is interpreted as the limit given by our most recent definition.

I would like to say that the notion of function convergence will not be tested directly on the midterms or final, and I will not ask particularly hard questions about it on quizzes, but it is what is going on behind the scenes with Taylor polynomials and Taylor series. Taking a moment or two to understand this idea goes a long way towards making this content easier and motivating what is going on in many questions. I will also use this language in developing the concepts that follow.

Check Your Understanding: (Limits of functions)

- For the following sequences of functions, what is the largest domain on which the sequences converge, and what function do they converge to?
  - $\{f_n\}_{n=1}^{\infty}$ where $f_n(x) = \frac{\cos(x)}{n}$
  - $\{f_n\}_{n=1}^{\infty}$ where $f_n(x) = \frac{nx^2}{nx^2+1}$
  - $\{f_n\}_{n=1}^{\infty}$ where $f_n(x) = x^{(\frac{1}{n})}$ and we consider only a domain of positive numbers.
  - $\{f_n\}_{n=1}^{\infty}$ where $f_n(x) = x\cos(\frac{x}{n}) + \sin(\frac{x}{n})$
  - $\{f_n\}_{n=1}^{\infty}$ where $f_n(x) = x^n$
  - $\{f_n\}_{n=1}^{\infty}$ where $f_n(x) = \frac{x}{1+x^n}$
- What is the largest domain that $\sum_{n=1}^{\infty} x^n$ converges on, and what function does the series converge to on that domain?

-(3) Taylor Series

Now that we have developed some terminology for sequences of functions, we return to Taylor polynomials. For a function f, and a point a, we have

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i,$$

and we will consider the sequence $\{T_n\}_{n=0}^{\infty}$.

Notice that

$$T_n(x) = \frac{f^{(n)}(a)}{n!}(x-a)^n + \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!}(x-a)^i = T_{n-1}(x) + \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

This means that we can consider $\lim_{n\to\infty} T_n$ as a series:

$$\lim_{n\to\infty} T_n = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

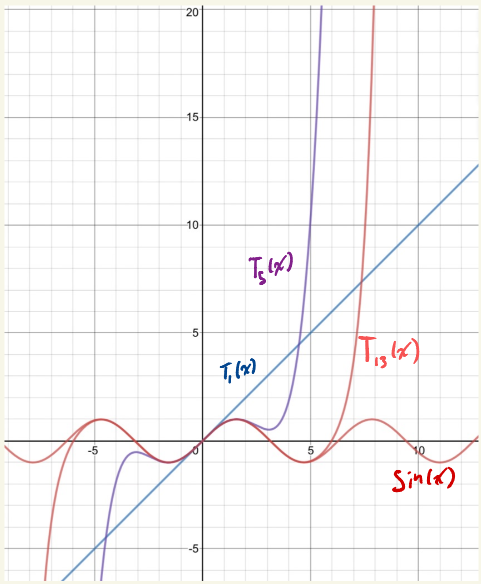
This view is useful because it allows us to use the tools we have developed for series. This also leads us to a new definition.

Def: Given a function f which is differentiable arbitrarily many times at a point, a, we call

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

the Taylor series for f, centered at a.

Now let's consider the example from before, Taylor polynomial approximations for $\sin(x)$ centered at 0.



It appears that at least close to zero, $T_n(x) \to \sin(x)$. A big question we will tackle in this section of the course is "when do Taylor polynomials converge to their function?"

We now conclude this rather short section in order to develop some tools to tackle this question.

☆ Check your understanding: (Taylor Series)

- Write the Taylor series for each of the following functions, centered at the specified point.
  - e^x centered at 0
  - $\sin^x$ centered at 0
  - $3x^3+2x+4$ centered at 0
  - $4x^7+3x^5+4x^2+x+7$ centered at 7
  - $\frac{1}{x}$ centered at 1
- Do you think a power series will always converge to its function for all real numbers? Why or how could this fail? Make some conjectures. (This probably isn't obvious, but it is good to think about. Don't worry if you don't have definite answers)

-(4) Power Series

We will now look at one of the most simple nontrivial kind of series of functions.

Def: A **power series** is a series of functions of the form

$$\sum_{n=0}^{\infty} c_n x^n$$

where $\{c_n\}$ is a sequence of numbers. Here we interpret x^0 as 1 for all values including $x=0$.

Notice that a Taylor series centered at 0,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n,$$

is a power series where $\{c_n\}_{n=0}^{\infty}$ is defined by $c_n = \frac{f^{(n)}(0)}{n!}$. Also notice that any result about a power series can be applied to a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

by translating x. If it is not very clear why this is true you should take a minute to think about it and then ask me.

This implies that any results we obtain about power series will tell us something about Taylor polynomials.

Note/Def: Given a series of functions, $\sum f_n(x)$, we will say that $\sum f_n(x)$ and a number, a, we will say **$\sum f_n(x)$ converges at a** if $\sum f_n(a)$ is a convergent series. This language also applies to sequences of functions.

☆ Stop and think:

- Every power series must converge at $x=0$. Why is this true?
- Can you think of a power series which converges only at $x=0$ and not for any other x value?
- Can you think of a nontrivial (not just $c_n=0$) power series which converges at every x value?

We can obtain a very powerful result dealing with the domains on which a power series can converge. We will present this theorem for series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$ to make it clear how the theorem applies to Taylor series.

Thm: Suppose we have a number, a, a sequence of numbers, $\{c_n\}_{n=0}^{\infty}$, and from these, the series,

$$\sum_{n=0}^{\infty} c_n(x-a)^n.$$

Then exactly one of the following statements is true:

- The series converges only when $x=a$
- The series converges for any value of x
- There is some number, R, such that the series converges when $|x-a|<R$ and the series diverges when $|x-a|>R$.

Note: The proof for this theorem is actually very straight forward and instructive. I will add it as an appendix at the end of this document.

Notice that in the third case for this theorem, it is not specified what happens when $x=a+R$ or $a-R$. The result of this theorem allows us to define a very useful object for discussing power series.

Def: Considering the previous theorem, we will define the radius of convergence of a power series. If the first case holds we say the radius of convergence is 0, if the second case holds we say ∞, and if the third case holds we say R. The above theorem also implies that the largest domain the power series converges on is an interval of one of the forms,

$(a-R, a+R)$, $[a-R, a+R)$, $(a-R, a+R]$, or $[a-R, a+R]$.

Whichever is the largest interval, we call the interval of convergence.

Check your understanding: (Power Series)

- If $\{c_n\}$ is a sequence with only positive terms, and a is some positive number such that

$$\sum_{n=0}^{\infty} c_n x^n$$

converges at a, we can actually get that the series converges at every b satisfying $0<b\leq a$ without using the theorem by using one of our series tests. Which one?