Name: ____

Recently in class we have been exploring ways to simplify calculations of derivatives. We have looked at how to find derivatives of special classes of functions (polynomials and exponentials in particular). We have also looked at how to take derivatives of certain combinations of functions. So far we have explored limits of:

- sums of functions
- constant scalings of functions
- differences of functions
- compositions of functions

Today we will finish off looking at the basic algebraic operations by looking into what the derivatives of products and quotients of functions might look like.

1. Warm Up I: State the chain rule

2. Warm Up II: Compute the following derivatives (let $g(x)$ be some differentiable function)
   - $\frac{d}{dx}(x)$
   - $\frac{d}{dx}(x^2)$
   - $\frac{d}{dx}\left(e^{(g(x)^2)}\right)$
   - $\frac{d}{dx}\left(\frac{1}{e^x}\right)$
   - $\frac{d}{dx}\left(\frac{e^{2x}-1}{e^x+1}\right)$

3. Now lets look at derivatives of products of functions. We know that $(f(x) + g(x))' = f'(x) + g'(x)$, and that this comes from a limit law. We have a corresponding limit law for multiplication so the first intuition may be to think that $(f(x) \cdot g(x))' = f'(x) \cdot g'(x)$. This is NOT true, and we will see that what is going on is more complicated than in the case of sums.

   Find a counterexample in order to show it is not always true that $(f(x) \cdot g(x))' = f'(x) \cdot g'(x)$. (Hint: look at the derivatives you calculated for the warm up)

4. Now that we know what isn't happening, let's see if we can figure out what is going on. So let's do the best thing we can do when we're confused and look at some examples.

   Given a differentiable function f, and a function $g(x) = c$ for all x, for some real number c, what is $(f \cdot g)'$?

5. Now use the limit definition to compute $\frac{d}{dx}(x \cdot f(x))$.

6. Now take a general equation for a line $g(x) = ax + b$ and a function f, and compute $\frac{d}{dx}(g(x) \cdot f(x))$,

7. Given the same f and g from the last question, can you find a formula for $\frac{d}{dx}(g(x) \cdot f(x))$ in terms of f,g,f', and g'?

8. Now, there is really only one meaningful formula for the above derivative if we are only considering taking it as a function of f,g,f', and g'. We hope that these are the only pieces of information we need to compute the derivative of the product, so let's try to prove this with the limit definition to see if it is true.

   Hints:

   - Start with the messier side
   - For continuous functions h, k, the product limit law states $\lim_{x\to a}(h(x) \cdot k(x)) = \left(\lim_{x\to a} h(x)\right) \cdot \left(\lim_{x\to a} k(x)\right)$
   - For a continuous function h, $\lim_{h\to 0} h(x+h) = h(x)$.

9. Now let's state the product rule.

   Product Rule If f and g are functions that are differentiable at a, then

   $$(f(a)+g(a))' =$$

10. Now suppose we have differentiable functions f and g with $g(x) \neq 0$ for all x. Use the product rule, power law, and chain rule to find an expression for

    $$\frac{d}{dx}\Big(\frac{f(x)}{g(x)}\Big)$$

    using $f(x)$,$g(x)$, $f'(x)$, and $g'(x)$. Simplify until you have a single fraction in lowest terms.

11. Now let's do some exercises to make sure we are comfortable using these tools. Let f and g be some differentiable functions such that

    - $f(4) = 3$
    - $g(4) = 2$
    - $f'(4) = 5$
    - $g'(4) = 4$
    - $g''(4) = 6$

    Answer the following questions:

    - $\frac{d}{dx}\left(x^2 e^x\right) =$
    - $\frac{d}{dx}\left(\frac{d}{dx}\left(e^{x^2}\right)\right) =$
    - Let $h = e^{f(x)}$, then $h'(4) =$
    - Let $h(x) = \frac{d}{dx}\Big(f(x) \cdot g(x)\Big)$, then $h'(4) =$
    - Let $h(x) = \frac{d}{dx}\Big(f(g(x)^2)\Big)$, then $h'(4) =$
    - Let $h(x) = \frac{d}{dx}\Big(f(x) \cdot g(g'(x))\Big)$, then $h'(4) =$
    - Take n and m as arbitrary nonzero real numbers, compute $\frac{d}{dx}\Big(x^n \cdot x^m\Big)$ using both the power rule and the chain rule to show that the answers obtained from either method are equal.