EXPLICIT FORMULAE FOR CHARACTERISTIC CLASSES IN NONCOMMUTATIVE GEOMETRY

DISSERTATION

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ABSTRACT

In this thesis we construct explicit formulae for characteristic classes in Noncommutative geometry. The general framework for the construction of characteristic classes in Noncommutative geometry is provided by Connes' theory of cycles and their characters. We first consider the case of generalized cycles with " curvature" and define characters for them. We prove that our definition agrees with Connes' original definition.

We then proceed to apply our construction to several geometric situations. We treat the case of vector bundles on manifolds, equivariant with respect to the action of discrete group, and the case of holonomy equivariant vector bundle on a foliated manifold, and discuss relation of our construction to Connes' construction of the Godbillon-Vey cocycle.

We also derive formulae for the Chern character of finitely summable Fredholm module, as well as transgression formulae. Finally we discuss a different approach to the characteristic classes, using the cyclic analogue of the Paschke-Voiculescu duality.

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CHAPTER 1

INTRODUCTION.

Noncommutative geometry was founded by A. Connes. Since then it has been steadily evolving into a powerful mathematical framework. It has spectacular applications ranging from geometry and topology to number theory and quantum physics.

This thesis is devoted to the explicit formulae for the characteristic classes in noncommutative geometry.

Our approach to the problem of construction of explicit characteristic cocycles is based on the Connes' theory of cycles and their characters. In the situations arising naturally one often has to work with the objects more general than the Connes' cycles, which we call generalized cycles. A generalized cycle of degree n over a unital algebra $\mathcal{A}$ is defined by the following data: graded algebra Ω^*; degree 1 graded derivation $\nabla : \Omega^* \to \Omega^{*+1}$ and element $\theta \in \Omega^2$ such that $\nabla^2\omega = [\theta, \omega]$; and the graded trace $⨍$ on Ω^n such that $⨍ \nabla\omega = 0$; and homomorphism $\rho : \mathcal{A} \to \Omega^0$. Such objects appeared in Connes' work, and he provided a canonical construction allowing to associate canonically a cycle with a generalized cycle. The character of the generalized cycle is then defined as the character of the associated cycle.

We associate with the generalized cycle $\mathcal{C}^n$ of degree n certain canonical n-cocycle in the Connes' (b, B) bicomplex by the formula resembling the character formula of

A. Jaffe, A. Lesniewski and K. Osterwalder. Namely, we put ($a_i \in \mathcal{A}$)

$$\mathrm{Ch}^k(\mathcal{C}^n)(a_0, a_1, \ldots a_k) = \frac{(-1)^{\frac{n-k}{2}}}{(\frac{n+k}{2})!} \sum_{i_0+i_1+\cdots+i_k=\frac{n-k}{2}} \int\!\!\!\!\!- \rho(a_0)\theta^{i_0}\nabla(\rho(a_1))\theta^{i_1}\ldots\nabla(\rho(a_k))\theta^{i_k}$$

$k = n, n-2, \ldots$.

We show then that the class of this cocycle in the cyclic cohomology coincides with the character of the generalized cycle. This provides us with an explicit formula for the character of the generalized cycle. This formula also continues to make sense for generalized chains. It defines in this case not a cocycle anymore, but a cochain. If for the generalized chain $\mathcal{C}$ $\partial\mathcal{C}$ denotes the boundary of $\mathcal{C}$, we have the following relation:

$$(B+b)\,\mathrm{Ch}(\mathcal{C}^n) = S\ \mathrm{Ch}(\partial(\mathcal{C}^n))$$

Here S is the usual periodicity shift in the cyclic bicomplex. This allows one to use cobordisms of generalized cycles to obtain transgression formulae for the characters. We remark that all of the above constructions continue to work for the nonunital algebras after suitable modifications.

We then proceed to apply this formalism in the situations arising in geometry and functional analysis. Main cases considered are:

1. Let V be a orientable manifold on which a discrete group Γ acts by orientation preserving diffeomorphisms, and let F be a Γ equivariant vector bundle over V. We construct an equivariant Chern character of this bundle, which takes values in the cyclic cohomology $HC^*(C_0^\infty(V) \rtimes \Gamma)$.

2. Let W be a foliated manifold, and F be a holonomy equivariant vector bundle on W. In this case equivariant Chern character takes values in $HC^*\left(C_0^\infty(\mathcal{G})\right)$, where $C_0^\infty(\mathcal{G})$ is the convolution algebra of the foliation groupoid.

3. Let $(\mathcal{H}, F, \gamma)$ be a bounded, finitely summable Fredholm module over a unital algebra. Connes constructs explicit formula for the character in the invertible case when $F^2 = 1$, and provides a procedure allowing to reduce the general case to the invertible one. We derive explicit formulae in the general case, which give the same cohomology class as the Connes' reduction procedure.

In the case of Γ-equivariant vector bundle a choice of connection allows one to construct naturally a generalized cycle over the cross-product algebra $C_0^\infty(V) \rtimes \Gamma$. Its character defines a class χ in the cyclic cohomology of $C_0^\infty(V) \rtimes \Gamma$. It is given by the following formula $(a_i U_{g_i} \in C_0^\infty(V) \rtimes \Gamma)$:

$$\chi^k(a_0 U_{g_0}, a_1 U_{g_1}, \ldots a_k U_{g_k}) = \int\limits_{\substack{t_0+\cdots+t_k=1 \\ t_i \geq 0 \;\; i=0,1,\ldots k}} \int\limits_V tr\ a_0 e^{-t_0 \theta^{g_0}} \left(da_1 + a_1 \delta(g_1)\right)^{g_0} e^{-t_1 \theta^{g_0 g_1}}$$
$$\left(da_2 + a_2 \delta(g_2)\right)^{g_0 g_1} e^{-t_0 \theta^{g_0 g_1 g_2}} \ldots \left(da_k + a_k \delta(g_k)\right)^{g_0 \cdots g_{k-1}} e^{-t_k \theta^{g_0 \cdots g_k}} dt_1 \ldots dt_k$$

when $g_0 g_1 \ldots g_k = 1$ and 0 otherwise. Here notations are as follows: ∇ denotes the connection on F, $\theta \in \Omega^2(V, \mathrm{End}(F))$ – its curvature, superscript denotes the group action, and $\delta(g)$ is an endomorphism-valued 1-form defined as $\nabla - \nabla^g$. When V is noncompact we obtain a reduced cyclic cocycle on the cross-product algebra with the unit adjoined by requiring that $\chi^0(1) = 0$.

To identify this class we show that under the Connes' canonical imbedding Φ : $H^*(V \times \mathrm{E}\Gamma/\Gamma) \to \mathrm{HPC}^*(\mathrm{C}_0^\infty(\mathrm{V}) \rtimes \Gamma)$ of the cohomology of the homotopy quotient into the periodic cyclic cohomology equivariant Chern character $\mathrm{Ch}_\Gamma(F) \in H^*(V \times \mathrm{E}\Gamma/\Gamma)$ is mapped into our character χ.

In the case of the holonomy equivariant bundle on the foliated manifold we combine the methods from the previous case with the Connes' construction of the transverse fundamental class of the foliation.

We also relate our construction to the Connes' construction of the Godbillon-Vey cocycle.

For the case of Fredholm modules one can construct the generalized cycle following the Connes' construction in the invertible case. Now we can apply the character formula to obtain the following cyclic cocycle:

In the even case, with $n = 2m$ greater than the degree of summability the cocycle $\mathrm{Ch}_{2m}(F)$ has components Ch^k_{2m} for $k = 0,\ 2\ ,\ \dots\ ,\ 2m$:

$$\mathrm{Ch}^k(F)(a_0, a_1, \dots a_k) = \frac{m!}{(m+\frac{k}{2})!} \sum_{i_0+i_1+\cdots+i_k = m-\frac{k}{2}} \mathrm{Tr}\,\gamma a_0(1-F^2)^{i_0}[F, a_1](1-F^2)^{i_1} \dots [F, a_k](1-F^2)^{i_k}$$

In the odd case the corresponding cocycle $\mathrm{Ch}_{2m+1}(F)$ has components Ch^k_{2m+1} for $k = 1,\ 3\ ,\ \dots\ ,\ 2m+1$, given by the formula

$$\mathrm{Ch}^k_{2m+1}(a_0, a_1, \dots, a_k) = \frac{\Gamma(m+\frac{3}{2})\sqrt{2i}}{(m+\frac{k+1}{2})!} \sum_{i_0+i_1+\cdots+i_k = m-\frac{k-1}{2}} \mathrm{Tr}\, a_0(1-F^2)^{i_0}[F, a_1](1-F^2)^{i_1} \dots [F, a_k](1-F^2)^{i_k}$$

We then show, using transgression formulae, that the formulae above compute the character of the Fredholm module, as defined by Connes.

Alternative approach to the characters of Fredholm modules which we study is based on the (co)homological analogue of Paschke-Voiculescu map. We construct a map in cyclic (co)homology which agrees with the Paschke-Voiculescu in K-theory via the Chern character. This allows us to reduce the problem in K-homology to the problem in K-theory. Using the well-known formulae for the Chern character in K-theory we obtain formulae for the characters of Fredholm modules.

The thesis is organized as follows.

Chapter 2 is devoted to the background information from the Noncommutative geometry. All the material there is well-known and taken from other sources.

In the Chapter 3 we define generalized cycles and chains, their characters, and describe their main properties.

We then apply methods of the Chapter 3 to the geometric situation of equivariant vector bundles in the Chapter 4.

In the Chapter 5 we obtain formulae for the characters of Fredholm modules via the theory of cycles as well as via the cyclic analogue of the Paschke-Voiculescu duality [Pas81].

Finally, the Appendix is devoted to the analogue of the well-known characteristic map for the Lie algebras action in the case when the action is twisted by a cocycle.

CHAPTER 2

PRELIMINARIES.

2.1 Cyclic objects and cyclic (co)homology.

In this section we recall the definitions and properties of the cyclic objects and cyclic (co)homology, as defined by Connes. All the material is taken from [Con94], [Lod92].

Definition 1. A cyclic object in a category is given by a sequence of objects X_n, $n \geq 0$ with the simplicial structure given by the face and degeneracy operators $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$, $0 \leq i \leq n$ with identities

$$d_i d_j = d_{j+1} d_i \;\; i < j \tag{2.1}$$

$$s_i s_j = s_{j+1} s_i \;\; i \leq j \tag{2.2}$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j \\ 1 & \text{if } i = j \text{ or } i = j+1 \\ s_j d_{i-1} & \text{if } i > j+1 \end{cases} \tag{2.3}$$

and the cyclic structure given by the cyclic operator $\tau_n : X_n \to X_n$ satisfying the identities

$$d_i\tau_n = \begin{cases} \tau_{n-1}d_{i-1} & \text{if } 1 \leq i \leq n \\ d_n & \text{if } i = 0 \end{cases} \tag{2.4}$$

$$s_i\tau_n = \begin{cases} \tau_{n+1}s_{i-1} & \text{if } 1 \leq i \leq n \\ \tau_{n+1}^2 s_n & \text{if } i = 0 \end{cases} \tag{2.5}$$

$$\tau_n^{n+1} = 1 \tag{2.6}$$

Example 1. With every unital algebra $\mathcal{A}$ over the ring k one associates the cyclic object $\mathcal{A}^\natural$ in the category of the k-modules. One puts $\mathcal{A}^\natural_n = \mathcal{A}^{\otimes(n+1)}$ and defines

$$d_i\ a_0 \otimes a_1 \cdots \otimes a_n = \begin{cases} a_0 \otimes a_1 \ldots a_i a_{i+1} \cdots \otimes a_n & \text{if } 0 \leq i \leq n-1 \\ a_n a_0 \otimes a_1 \cdots \otimes a_{n-1} & \text{if } i = n \end{cases} \tag{2.7}$$

$$s_i\ a_0 \otimes a_1 \cdots \otimes a_n = a_0 \otimes a_1 \ldots a_i \otimes 1 \otimes a_{i+1} \cdots \otimes a_n \tag{2.8}$$

$$\tau\ a_0 \otimes a_1 \cdots \otimes a_n = a_n \otimes a_0 \otimes a_1 \cdots \otimes a_{n-1} \tag{2.9}$$

Example 2. Let Γ be a discrete group. Then the standard resolution of the trivial module has a structure of a cyclic object. Let β_n be the vector space with the basis consisting of symbols $(g_0, g_1, \ldots, g_n)$, $g_i \in \Gamma$. Put

$$d_i(g_0, g_1, \ldots, g_n) = (g_0, g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_n) \tag{2.10}$$

$$d_i(g_0, g_1, \ldots, g_n) = (g_0, g_1, \ldots, g_{i-1}, g_i, g_i, g_{i+1}, \ldots, g_n) \tag{2.11}$$

$$\tau_n(g_0, g_1, \ldots, g_n) = (g_n, g_0, g_1, \ldots, g_{n-1}) \tag{2.12}$$

Example 3. Let $\mathcal{A}$ be an associative algebra and Γ a discrete group acting on $\mathcal{A}$. Then we can consider a cyclic object $(\mathcal{A} \rtimes \Gamma)^\natural_1$. $\left((\mathcal{A} \rtimes \Gamma)^\natural_1\right)_n \subset \left((\mathcal{A} \rtimes \Gamma)^\natural\right)_n$ is generated by the terms of the form $a_0 U_{g_0} \otimes a_1 U_{g_1} \otimes \cdots \otimes a_n U_{g_n}$ with $g_0 g_1 \ldots g_n = 1$. The face, degeneracy and cyclic operators are restrictions of those from $(\mathcal{A} \rtimes \Gamma)^\natural$.

We now define the cyclic homology of a cyclic object in an abelian category. Put $\nu = 1 + \tau_n + \tau_n^2 + \cdots + \tau_n^n$, $s = s_n$ and define operators $b : X_n \to X_{n-1}$ and $B : X_n \to X_{n+1}$ by

$$b = \sum_{i=0}^{n} d_i \tag{2.13}$$

$$B = (1 - \tau) s \nu \tag{2.14}$$

One then verifies that

$$b^2 = 0 \tag{2.15}$$

$$B^2 = 0 \tag{2.16}$$

$$bB + Bb = 0 \tag{2.17}$$

With this we can give the following definition

Definition 2. Put

$$C_{ij}(X) = \begin{cases} X_{j-i} & \text{if } j \geq i \\ 0 & \text{otherwise} \end{cases} \tag{2.18}$$

Then b and B define operators acting from C_{ij} to $C_{i(j-1)}$ and $C_{(i-1)j}$ respectively. The homology of the bicomplex $\mathcal{B}(X) = (C_{ij}, b, B)$ is the cyclic homology of the cyclic object X.

For the unital algebra $\mathcal{A}$ cyclic homology is defined as homology of the cyclic object $\mathcal{A}^\natural$. We will always use slightly different version of the b B bicomplex, which is defined as follows. Put $\bar{C}_{ij}(\mathcal{A}) = C_n(\mathcal{A}) = \mathcal{A} \otimes \bar{\mathcal{A}}^{\otimes n}$, where $\bar{\mathcal{A}} = \mathcal{A}/k1$. One verifies that operators b and B descend to this bicomplex, which we denote $\bar{\mathcal{B}}$, and that the homology of this bicomplex is the same as the homology of the original bicomplex. The cyclic homology of the algebra $\mathcal{A}$ is denoted $HC_*(\mathcal{A})$.

The reduced cyclic homology of the unital algebra $\mathcal{A}$ is defined as the homology of the factor-bicomplex $\bar{\mathcal{B}}(\mathcal{A})/\bar{\mathcal{B}}(k)$.

For the nonunital algebra $\mathcal{A}$ cyclic homology is defined as the reduced cyclic homology of its unitalization $\mathcal{A}^+$.

One can describe the cyclic bicomplex in the following form. Let u be a formal variable of degree -2. Then one can consider complex $(C_*(\mathcal{A}) \otimes k[u], b + uB)$.

The cyclic homology of the algebra of smooth functions on the smooth manifold V, computed by Connes, can be described in the following way. Let $\Omega^*(V)$ denotes the space of the differential forms on the manifold V with the usual grading. Consider the complex $D_* = (\Omega^*(V) \otimes k[u], ud)$, where d is the de Rham differential. Then the answer is given by the following theorem of Connes:

Theorem 1. *The map of complexes*

$$\Phi : (C_*(\mathcal{A}) \otimes k[u], b + uB) \to (\Omega^*(V) \otimes k[u], ud) \tag{2.19}$$

where $\mathcal{A} = C_0^\infty(V)$ defined by u-linearity and

$$\Phi(a_0 \otimes a_1 \otimes \dots a_k) = \frac{1}{k!} a_0 da_1 \dots da_k \tag{2.20}$$

is a quasiisomorphism.

All the notions and results of this section can be dualized to the cohomological context. Cyclic cohomology of the algebra of functions on the smooth manifold are computed now by the complex $D^* = \big((\Omega^*(V))' \otimes k[u], ud\big)$, where $(\Omega^*(V))'$ denotes the space of currents, and u has degree 2.

2.2 Equivariant cohomology and cyclic cohomology.

This section is devoted to the description of the Connes' map Φ, relating cyclic cohomology of cross-product algebras with equivariant cohomology. It generalizes (transposed of) the map Φ from the Theorem 1 to the case in which discrete group acts on the manifold.

Let V be a manifold, and Γ be a discrete group acting on V by diffeomorphisms. For our purposes it is enough to suppose that V is oriented and that Γ acts by orientation preserving diffeomorphisms, though construction works in the general case as well.

$A_\Gamma \subset \mathrm{Hom}(\beta_*, D^*)$ – the space of *totally antisymmetric* functions on Γ^{l+1} with values in D complex of currents on V, satisfying

$$(\gamma)^g((g_0, g_1, \ldots, g_l) = (\gamma(g_0 g, g_1 g, \ldots, g_l g)) \tag{2.21}$$

The differentials are given by the de Rham differential and by the group cohomology differential, i.e.

$$\delta\gamma(g_0, g_1, \ldots, g_{l+1}) = \sum_{i=0}^{k+1} (-1)^i \gamma(g_0, g_1, \ldots, \hat{g}_i, \ldots, g_{l+1}) \tag{2.22}$$

The map Φ is the map from this complex to the cohomological (b, B) complex of the algebra $\mathcal{A} = C_0^\infty(V) \rtimes \Gamma$.

The construction is the following. Consider an auxiliary algebra $\mathcal{B} = \mathcal{A} \otimes \Lambda'\Gamma$, where $\Lambda'\Gamma$ is a free graded-commutative algebra generated by the elements δ_g of degree 1, $g \in \Gamma$, with $\delta_1 = 0$. We equip it with the differential defined by

$$d(\omega \otimes \varepsilon) = d\omega \otimes \varepsilon \tag{2.23}$$

The group Γ acts on $\Lambda'\Gamma$ by $(\delta_g)^h = \delta_{gh^{-1}} - \delta_{h^{-1}}$, and $\mathcal{B}$ is equipped with the product action.

Consider now the algebra $\mathcal{C} = \mathcal{B} \rtimes \Gamma$. It has a structure of a differential graded algebra, with the differential defined by

$$d(bU_g) = (db)U_g + (-1)^{\deg b} bU_g\delta_g \tag{2.24}$$

Let now γ be a current-valued Γ function on Γ^{l+1}. We associate with it a linear form $\tilde{\gamma}$ on $\mathcal{B}$ defined by

$$\tilde{\gamma}(\omega \otimes \delta_{g_1} \dots \delta_{g_l}) = \langle \omega, \gamma(1, g_1, \dots, g_l) \rangle \tag{2.25}$$

and extend it to $\mathcal{C}$ by

$$\tilde{\gamma}(bU_g) = \begin{cases} \tilde{\gamma}(b) & \text{if } g = 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.26}$$

The map Φ is then defined by the following equation, where $\gamma \in \mathrm{Hom}(\beta_l, D^m)$, $k = l + m + 1$, and $x_i \in \mathcal{A}$.

$$\Phi(\gamma)(x_0, x_1, \dots, x_k) = \lambda_{l,k} \sum_{j=0}^{k} \tilde{\gamma}(dx_{j+1} \dots dx_k x_0 dx_1 \dots dx_j) \tag{2.27}$$

where $\lambda_{l,k} = \frac{l!}{(k+1)!}$. Extend the map Φ by u-linearity. One then has the following

Proposition 2. Φ *is a morphism of complexes.*

2.3 Fredholm modules.

An even summable Fredholm module $(\mathcal{H}, F, \gamma)$ over an algebra $\mathcal{A}$ is given by the following data:

A $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}$ with grading γ, $\gamma^2 = 1$ and a representation on it of an algebra $\mathcal{A}$ by even operators, i.e. a homomorphism $\pi : \mathcal{L} \to \mathrm{End}^+(\mathcal{H})$.

An odd operator F, such that

$$\pi(a)(1 - F^2) \in \mathcal{K} \tag{2.28}$$

$$\pi(a)[F, \pi(b)] \in \mathcal{K} \tag{2.29}$$

for any a, $b \in \mathcal{A}$

The set of even Fredholm modules with the proper equivalence relation (cf [Bla98]) and an operation of direct sum becomes the K-homology group $K^0(\mathcal{A})$. For the unital algebra $\mathcal{A}$ (but not necessarily the unital representation) one can replace the Fredholm module by an equivalent one satisfying

$$(1 - F^2) \in \mathcal{K} \tag{2.30}$$

$$[F, \pi(b)] \in \mathcal{K} \tag{2.31}$$

An odd Fredholm module $(\mathcal{H}, F)$ over an algebra $\mathcal{A}$ is given by the following data:

Hilbert space $\mathcal{H}$ and a representation on it of an algebra $\mathcal{A}$, i.e. a homomorphism $\pi : \mathcal{L} \to \mathrm{End}(\mathcal{H})$.

An operator F, such that

$$\pi(a)(1 - F^2) \in \mathcal{K} \tag{2.32}$$

$$\pi(a)[F, \pi(b)] \in \mathcal{K} \tag{2.33}$$

As in the even case, one constructs the group $K^1(\mathcal{A})$.

If one replaces in the above definitions $\mathcal{K}$ by $\mathcal{L}^p$ one obtains the definition of the p-summable Fredholm module.

In the paper [Con85] Connes shows that with every Fredholm module (pre-Fredholm module in Connes terminology) one can canonicaly associate a Fredholm module, satisfying

$$F^2 = 1 \tag{2.34}$$

representing the same K-homology class. If the original Fredholm module was p-summable, the new one will also be p-summable.

With the Fredholm module satisfying (2.34) Connes associates a cyclic cocycle, the character of the Fredholm module. The cocycle τ_n is defined by the following equations, where $\mathrm{Tr}'(T) = 1/2\,\mathrm{Tr}\,(F(FT + TF))$:

for the even Fredholm module

$$\tau_n(a_0, a_1, \ldots, a_n) = \lambda_n \,\mathrm{Tr}'(\gamma a_0[F, a_1] \ldots [F, a_n]) \tag{2.35}$$

where n is even, $n \geq p$, $\lambda_n = (-1)^{n(n-1)/2}\Gamma(n/2 + 1)$;

for the odd Fredholm module

$$\tau_n(a_0, a_1, \ldots, a_n) = \lambda_n \,\mathrm{Tr}'(a_0[F, a_1] \ldots [F, a_n]) \tag{2.36}$$

where n is odd, $n \geq p$, $\lambda_n = \sqrt{2i}(-1)^{n(n-1)/2}\Gamma(n/2 + 1)$;

CHAPTER 3

GENERALIZED CYCLES AND THEIR PROPERTIES.

3.1 Definition of cycles.

In this section we will give all the definitions in the case of the unital generalized cycles over unital algebras. The case of the nonunital cycles will be considered in the section 3.3.

Definition 3. A unital generalized cycle over a unital algebra $\mathcal{A}$ is given by the following data :

1. A $\mathbb{Z}$-graded unital algebra $\Omega = \bigoplus_{m=0}^{\infty} \Omega^m$ and a homomorphism ρ from $\mathcal{A}$ to Ω^0. We require the homomorphism to be unital.

2. A graded derivation $\nabla : \Omega^k \mapsto \Omega^{k+1}$, k =0, 1, ... and $\theta \in \Omega^2$ which satisfy

$$\nabla(\omega\xi) = \nabla(\omega)\xi + (-1)^{\deg \omega}\omega\nabla(\xi) \tag{3.1}$$

$$\nabla^2(\xi) = \theta\xi - \xi\theta \ \forall \xi \in \Omega \tag{3.2}$$

$$\nabla(\theta) = 0 \tag{3.3}$$

   Equation (3.1) is just the graded derivation property. We will sometimes call ∇ connection and θ curvature. Since we consider for the moment only the unital case we require that $\nabla(1) = 0$.

3. A graded trace $\fint$ defined on Ω^n for some n with the properties

$$\fint \omega\xi = (-1)^{\deg w \deg \xi} \fint \xi\omega \text{ for } \deg\omega + \deg\xi = n \tag{3.4}$$

$$\fint \nabla(\xi) = 0 \ \forall \xi \in \Omega^{n-1} \tag{3.5}$$

Here (3.4) is just the graded trace property. We will call n the degree of a cycle.

The definition of cycle is obtained by requiring θ to be 0 .

A simple example of the generalized cycles is the following.

Example 4. Let V be a smooth compact oriented manifold, and let E be a (complex) vector bundle over V. Let $\mathcal{A} = C^\infty(V)$ be the algebra of smooth functions on V. Choose any connection on the vector bundle E. This data provides us with a generalized cycle over an algebra $\mathcal{A}$. Indeed, let $\Omega^k = \Omega^k(V, \operatorname{End} E) = C^\infty\left(V, \Lambda^k T^*V \otimes \operatorname{End} E\right)$ denote the space of differential k-forms with values in the endomorphisms of the bundle E, and let $\Omega = \bigoplus \Omega^k$ be the corresponding graded algebra. We have a natural unital map $\rho : \mathcal{A} \to \Omega^0$ (to each function corresponds the operator of multiplication by this function). Connection and curvature provide us with the graded derivation ∇ and $\theta \in \Omega^2$ satisfying properties (3.1)-(3.3). Finally, we define the graded trace $\fint$ by

$$\fint \omega = \int_V \operatorname{tr} \omega \tag{3.6}$$

where tr is the usual (pointwise) trace of the endomorphism.

In the following example we introduce some operations over generalized cycles.

Example 5. Let $\mathcal{C} = (\Omega, \nabla, \theta, \fint)$ be a generalized cycle over an algebra $\mathcal{A}$. Then

$$-\mathcal{C} = (\Omega, \nabla, \theta, -\fint) \tag{3.7}$$

is also a generalized cycle, with the homomorphism ρ unchanged.

Let $\mathcal{C}_1 = (\Omega_1, \nabla_1, \theta_1, \fint_1)$ and $\mathcal{C}_2 = (\Omega_2, \nabla_2, \theta_2, \fint_2)$ be two generalized cycles of degree n over an algebra $\mathcal{A}$. Then one can construct their disjoint union

$$\mathcal{C}_1 \sqcup \mathcal{C}_2 = (\Omega_1 \oplus \Omega_2, \nabla_1 \oplus \nabla_2, \theta_1 \oplus \theta_2, \fint_1 \oplus \fint_2) \tag{3.8}$$

where $\fint_1 \oplus \fint_2 (\omega_1 \oplus \omega_2) = \fint_1 \omega_1 + \fint_2 \omega_2$, which is again a cycle of degree n, with the map $\rho = \rho_1 \oplus \rho_2 : \mathcal{A} \to \Omega_1 \oplus \Omega_2$.

Let now $\mathcal{C}_1$ and $\mathcal{C}_2$ be generalized cycles over algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ of degrees n_1 and n_2 respectively. Then one can construct their product which is the generalized cycle over the algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$ of degree $n_1 n_2$.

$$\mathcal{C}_1 \times \mathcal{C}_2 = (\Omega_1 \widehat{\otimes} \Omega_2, \nabla_1 \widehat{\otimes} 1 + 1 \widehat{\otimes} \nabla_2, \theta_1 \widehat{\otimes} 1 + 1 \widehat{\otimes} \theta_2, \fint_1 \widehat{\otimes} \fint_2) \tag{3.9}$$

where $\widehat{\otimes}$ denotes the graded tensor product, the homomorphism from $\mathcal{A}_1 \otimes \mathcal{A}_2$ to $(\Omega_1 \otimes \Omega_2)^0$ is given by $\rho_1 \otimes \rho_2$ and

$$\fint_1 \widehat{\otimes} \fint_2 (\omega_1 \widehat{\otimes} \omega_2) = \begin{cases} (-1)^{n_1 n_2} \deg \fint_1 \omega_1 \fint_2 \omega_2 & \text{if } \deg \omega_i = n_i \\ 0 & \text{otherwise} \end{cases}$$

We now introduce morphisms between generalized cycles

Definition 4. A morphism H between two generalized cycles $\mathcal{C}_1 = (\Omega_1, \nabla_1, \theta_1, \fint_1)$ and $\mathcal{C}_2 = (\Omega_2, \nabla_2, \theta_2, \fint_2)$ over algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively, covering the (unital) homomorphism $h : \mathcal{A}_1 \to \mathcal{A}_2$ is a linear grading-preserving unital homomorphism $H : \Omega_1 \to \Omega_2$ intertwining all the data, i.e. satisfying

$$H(\rho_1(a)) = \rho_2(h(a)) \tag{3.10}$$

$$H(\nabla_1(\omega)) = \nabla_2(H(\omega)) \tag{3.11}$$

$$H(\theta_1) = \theta_2 \tag{3.12}$$

$$⨍_1 \omega = ⨍_2 H(\omega) \tag{3.13}$$

When talking about the morphisms of generalized cycles over the same algebra $\mathcal{A}$ we mean, unless otherwise specified, morphism covering the identity homomorphism.

The next notion we will consider is that of the generalized chain. It should be thought of as a generalized cycle with boundary. The definition (in the unital case) is the following

Definition 5. A unital generalized chain over an algebra $\mathcal{A}$ is given by the following data:

1. Graded unital algebras Ω and $\partial\Omega$ with a surjective homomorphism $r : \Omega \to \partial\Omega$ of degree 0 , and a homomorphism $\rho : \mathcal{A} \to \Omega^0$. We require that ρ and r are unital.

2. Graded derivations of degree 1 ∇ on Ω and ∇' on $\partial\Omega$ and element $\theta \in \Omega^2$ such

that

$$\nabla(\omega\xi) = \nabla(\omega)\xi + (-1)^{\deg\omega}\omega\nabla(\xi) \tag{3.14}$$

$$\nabla'(\omega'\xi') = \nabla'(\omega')\xi' + (-1)^{\deg\omega'}\omega'\nabla'(\xi') \tag{3.15}$$

$$\nabla^2(\xi) = \theta\xi - \xi\theta \ \forall\xi \in \Omega \tag{3.16}$$

$$\nabla(\theta) = 0 \tag{3.17}$$

$$r \circ \nabla = \nabla' \circ r \tag{3.18}$$

3. A graded trace $⨍$ on Ω^n for some n (called the degree of the chain) such that

$$⨍\omega\xi = (-1)^{\deg w \deg\xi} ⨍\xi\omega \text{ for } \deg\omega + \deg\xi = n \tag{3.19}$$

$$⨍\nabla(\xi) = 0 \ \forall\xi \in \Omega^{n-1} \text{ such that } r(\xi) = 0 \tag{3.20}$$

One defines morphisms of chains similarly to the Definition 4.

Similarly to the Example 4 one can construct the following example of the generalized chain.

Example 6. Let V now be the compact oriented manifold with boundary ∂V, and E be a bundle over V. Choose a connection on the bundle E. Let, as in the Example 4 $\mathcal{A} = C^\infty(V)$, Ω^* – the algebra of endomorphism valued differential forms, with a graded derivation induced by the connection and θ given by the curvature of the connection. Then let $\partial\Omega$ denote the algebra of the differential forms on ∂V with values in the endomorphisms of the restriction of E to the boundary, r be the restriction map and ∇' be induced by the restriction of ∇ to the boundary. Finally, define the graded trace $⨍$ by the equation (3.6). Then this data defines a generalized chain.

We now define the boundary of the generalized chain and cobordism of generalized cycles.

Definition 6. Let the notations be as in the definition 5. The boundary of the generalized chain is a generalized cycle $(\partial\Omega, \nabla', \theta', \fint')$ of degree $n-1$ over an algebra $\mathcal{A}$ with the following notations: θ' is an element of $\partial\Omega$ defined by

$$\theta' = r(\theta) \tag{3.21}$$

$\fint'$ is the graded trace on $\partial\Omega^{n-1}$ defined by

$$\fint' \omega' = \fint \nabla(\omega) \tag{3.22}$$

where $\omega' \in (\partial\Omega)^{n-1}$ and $\omega \in \Omega^n$ such that $r(\omega) = \omega'$. This definition is unambiguous, as follows from (3.20). Homomorphism $\rho' : \mathcal{A} \to \partial\Omega^0$ is given by

$$\rho' = r \circ \rho \tag{3.23}$$

Notice that for $\omega' \in \partial\Omega$ $(\nabla')^2(\xi) = \theta'\omega' - \omega'\theta'$ and the definition of $\fint'$ is independent of the choice of $\omega \in \Omega^n$. For the generalized chain $\mathcal{C}$ let $\partial\mathcal{C}$ denote its boundary.

Example 7. Similarly to the Example 5, given a generalized chain $\mathcal{C}$ one can construct generalized chain $-\mathcal{C}$, with $\partial(-\mathcal{C}) = -\partial\mathcal{C}$. For the two generalized chains $\mathcal{C}_1$ and $\mathcal{C}_2$ over an algebra $\mathcal{A}$ one can construct $\mathcal{C}_1 \sqcup \mathcal{C}_2$ with $\partial(\mathcal{C}_1 \sqcup \mathcal{C}_2) = \partial\mathcal{C}_1 \sqcup \partial\mathcal{C}_2$. While the product of two generalized chains is not defined, one can construct a product of the generalized cycle $\mathcal{C}_1$ over an algebra $\mathcal{A}_1$ with the generalized chain $\mathcal{C}_2$ over an algebra $\mathcal{A}_2$, obtaining the generalized chain $\mathcal{C}_1 \times \mathcal{C}_2$ over the algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$ with $\partial(\mathcal{C}_1 \times \mathcal{C}_2) = \mathcal{C}_1 \times \partial\mathcal{C}_2$

Definition 7. Two generalized cycles $\mathcal{C}_1$ and $\mathcal{C}_2$ over an algebra $\mathcal{A}$ are called cobordant if there exists a generalized chain $\mathcal{C}$ such that $\partial\mathcal{C} = (-\mathcal{C}_1) \sqcup \mathcal{C}_2$ (with notation as in Example 5).

Example 8. The construction of the generalized cycle in the Example 4 involves a choice of connection on the bundle E. If we choose two different connections ∇_1 and ∇_2 then the corresponding generalized cycles $\mathcal{C}_1$ and $\mathcal{C}_2$ are cobordant.

The cobordism is constructed as follows. Consider the manifold $M = V \times [0,1]$, and let $\pi : M \to V$ be the natural projection. Let $\Omega = \Omega^*(M, \operatorname{End} \pi^* E)$. To each function $f \in C^\infty(V)$ corresponds the element of $\operatorname{End} \pi^* E$ – operator of multiplication by $\pi^* f$. This defines a homomorphism $\rho : C^\infty(V) \to \Omega^0$. We can join ∇_1 and ∇_2 by an affine family of connections ∇^t such that $\nabla^0 = \nabla_1$, $\nabla^1 = \nabla_2$. One then considers connection $\nabla^t + d$ on $\pi^* E$, where d is the de Rham differential on $[0,1]$. $\partial\Omega = \Omega^*(V, \operatorname{End} E) \oplus \Omega^*(V, \operatorname{End} E)$. The restriction map $r = i_1^* \oplus i_2^*$, where i_1 (resp. i_2) is the imbedding of V into M as $V \times 0$ (resp. $V \times 1$). Finally the graded trace $⨍$ is defined by

$$⨍\omega = \int\limits_M \operatorname{tr} \omega$$

This data provides one with a generalized chain $\mathcal{C}$. One can easily identify $\partial\mathcal{C}$ with $(-\mathcal{C}_1) \sqcup \mathcal{C}_2$.

3.2 Characters of cycles.

With every generalized chain $\mathcal{C}^n$ of degree n one can associate a canonical n-cochain $\mathrm{Ch}(\mathcal{C}^n)$ in the $b+B$ bicomplex of the algebra $\mathcal{A}$, which we call a character of the generalized chain.

Definition 8. The character of the (unital) generalized chain $\mathcal{C}^n$ is the cyclic cochain $\mathrm{Ch}(\mathcal{C}^n)$, which has components $\mathrm{Ch}^k(\mathcal{C}^n)$ of degree k for $k = n,\ n-2,\ \dots$. given by the following formula.

$$\mathrm{Ch}^k(\mathcal{C}^n)(a_0, a_1, \dots a_k) = \frac{(-1)^{\frac{n-k}{2}}}{(\frac{n+k}{2})!} \sum_{i_0+i_1+\dots+i_k=\frac{n-k}{2}} ⨍ \rho(a_0)\theta^{i_0}\nabla(\rho(a_1))\theta^{i_1}\dots\nabla(\rho(a_k))\theta^{i_k} \tag{3.24}$$

This formula is closely related to the Jaffe-Lesniewski-Osterwalder formula [JLO88].

For the generalized chain $\mathcal{C}$ let $\partial\mathcal{C}$ denote the boundary of $\mathcal{C}$.

Theorem 3. *Let $\mathcal{C}^n$ be a unital chain, and $\partial(\mathcal{C}^n)$ be its boundary. Then*

$$(B+b)\,\mathrm{Ch}(\mathcal{C}^n) = S\ \mathrm{Ch}(\partial(\mathcal{C}^n)) \tag{3.25}$$

Here S is the usual periodicity shift in the cyclic bicomplex.

Proof. We need to prove that $b\ \mathrm{Ch}^n(\mathcal{C}^n) = 0$ and that $b\ \mathrm{Ch}^{k-1}(\mathcal{C}^n) + B\ \mathrm{Ch}^{k+1}(\mathcal{C}^n) = \mathrm{Ch}^k\,(\partial(\mathcal{C}^n))$ for $k = 1,\ 3\ ,\ 5\ ,\ \dots\ ,\ n-1$. Since

$$\mathrm{Ch}^n(\mathcal{C}^n) = \mathrm{const} ⨍ \rho(a_0)\nabla(\rho(a_1))\nabla(\rho(a_2))\dots\nabla(\rho(a_n))$$

verification of the first equality is a standard calculation. The rest of the proof is devoted to the verification of the second equality. To simplify the formulae we introduce the notation

$$c_k^n = \frac{(-1)^{\frac{n-k}{2}}}{(\frac{n+k}{2})!} \tag{3.26}$$

We now compute (here the parity of k is opposite to the parity of n)

$$\begin{aligned}
&B\ \mathrm{Ch}^{k+1}(\mathcal{C}^n)(a_0, a_1, \dots, a_k) = \\
&\quad c_{k+1}^n \sum_{\substack{i_0+i_1+\cdots+i_{k+1}=\frac{n-k-1}{2} \\ \lambda-\text{cyclic permutation}}} \mathrm{sgn}(\lambda) \fint \nabla(\rho(a_{\lambda(0)})\theta^{i_1} \dots \nabla(\rho(a_{\lambda(k)}))\theta^{i_{k+1}+i_0} = \\
&\quad c_{k+1}^n \sum_{\substack{i_0+i_1+\cdots+i_{k+1}=\frac{n-k-1}{2} \\ \lambda-\text{cyclic permutation}}} \fint \nabla(\rho(a_0))\theta^{i_{\lambda(1)}} \dots \nabla(\rho(a_j))\theta^{i_{\lambda(j+1)}} \dots \\
&\qquad \nabla(\rho(a_{\lambda^{-1}(k)}))\theta^{i_{k+1}+i_0} \dots \nabla(\rho(a_k))\theta^{i_{\lambda(k+1)}}
\end{aligned} \tag{3.27}$$

Clearly, the result is the sum of the terms of the form

$$\fint \nabla(\rho(a_0))\theta^{j_0}\nabla(\rho(a_1))\theta^{j_1} \dots \nabla(\rho(a_k))\theta^{j_k} \tag{3.28}$$

with $j_0 + j_1 + \cdots + j_k = \frac{n-k-1}{2}$. Let us compute coefficient with which the term (3.28) enters the sum in the (3.27). If we consider the summands, coming from the one fixed cyclic permutation λ the term (3.28) will appear as many times as are there i_0, i_1, ... ,i_{k+1} with

$$j_l = \begin{cases} i_0 + i_{k+1} & \text{for } l = \lambda^{-1}(k) \\ i_{\lambda(l)} & \text{otherwise} \end{cases}$$

i.e. $j_{\lambda^{-1}(k)}$ times. Summing over all the cyclic permutations λ we obtain that the term (3.28) enters into the sum in (3.27) $(j_0+1)+(j_1+1)+\cdots+(j_k+1) = \frac{n-k-1}{2}+k+1 = \frac{n+k+1}{2}$. Hence we obtain that the sum in (3.27) equals

$$(\frac{n+k+1}{2}) \sum_{j_0+j_1+\cdots+j_k=\frac{n-k-1}{2}} \fint \nabla(\rho(a_0))\theta^{j_0}\nabla(\rho(a_1))\theta^{j_1}\ldots\nabla(\rho(a_k))\theta^{j_k}$$

Since $(\frac{n+k+1}{2})c^n_{k+1} = c^{n-1}_k$ we can write the final result as

$$B\ \mathrm{Ch}^{k+1}(\mathcal{C}^n)(a_0, a_1, \ldots, a_k) = c^{n-1}_k \sum_{j_0+j_1+\cdots+j_k=\frac{n-k-1}{2}} \fint \nabla(\rho(a_0))\theta^{j_0}\nabla(\rho(a_1))\theta^{j_1}\ldots\nabla(\rho(a_k))\theta^{j_k} \quad (3.29)$$

Now we compute $b\ \mathrm{Ch}^{k-1}(\mathcal{C}^n)(a_0, a_1, \ldots, a_k)$.We obtain:

$$\begin{aligned} b\ \mathrm{Ch}^{k-1}(\mathcal{C}^n)(a_0, a_1, \ldots, a_k) = \\ c^n_{k-1} \sum_{i_0+i_1+\cdots+i_{k-1}=\frac{n-k+1}{2}} \Big(\fint \rho(a_0a_1)\theta^{i_0}\nabla(\rho(a_2))\theta^{i_1}\ldots\nabla(\rho(a_k))\theta^{i_{k-1}} + \\ \sum_{l=1}^{k-1}(-1)^l \fint \rho(a_0)\theta^{i_0}\ldots\nabla(\rho(a_la_{l+1}))\theta^{i_l}\ldots\nabla(\rho(a_k))\theta^{i_{k-1}} + \\ (-1)^k \fint \rho(a_ka_0)\theta^{i_0}\nabla(\rho(a_1))\theta^{i_1}\ldots\nabla(\rho(a_k))\theta^{i_{k-1}}\Big) \quad (3.30) \end{aligned}$$

Using the derivation property of ∇ (and the trace property of $\fint$) one can easily see that

$$\begin{aligned}
&⨍\rho(a_0a_1)\theta^{i_0}\nabla(\rho(a_2))\theta^{i_1}\dots\nabla(\rho(a_k))\theta^{i_{k-1}} + \\
&\qquad\sum_{l=1}^{k-1}(-1)^l ⨍\rho(a_0)\theta^{i_0}\dots\nabla(\rho(a_l a_{l+1}))\theta^{i_l}\dots\nabla(\rho(a_k))\theta^{i_{k-1}} + \\
&\qquad\quad(-1)^k ⨍\rho(a_k a_0)\theta^{i_0}\nabla(\rho(a_1))\theta^{i_1}\dots\nabla(\rho(a_k))\theta^{i_{k-1}} = \\
&\qquad\sum_{l=0}^{k-1}(-1)^l ⨍\rho(a_0)\theta^{i_0}\dots\nabla(\rho(a_l))[\rho(a_{l+1}),\theta^{i_l}]\nabla(\rho(a_{l+2}))\theta^{i_{l+1}} \\
&\qquad\qquad\dots\nabla(\rho(a_k))\theta^{i_{k-1}} \qquad (3.31)
\end{aligned}$$

Since

$$[\rho(a_{l+1}),\theta^{i_l}] = \sum_{p+q=i_l-1}\theta^p[\rho(a_{l+1}),\theta]\theta^q = -\sum_{p+q=i_l-1}\theta^p\nabla^2(\rho(a_{l+1}))\theta^q$$

and $\frac{n-k+1}{2}-1=\frac{n-k-1}{2}$ the expression in (3.30) can be rewritten as

$$\begin{aligned}
&-c^n_{k-1}\sum_{j_0+j_1+\dots+j_k=\frac{n-k-1}{2}}\sum_{l=0}^{k-1}(-1)^l ⨍\rho(a_0)\theta^{j_0}\dots \\
&\qquad\nabla(\rho(a_l))\theta^{j_l}\nabla^2(\rho(a_{l+1}))\theta^{j_{l+1}}\nabla(\rho(a_{l+2}))\theta^{j_{l+2}}\dots\nabla(\rho(a_k))\theta^{j_k} \qquad (3.32)
\end{aligned}$$

Notice that $-c^n_{k-1}=c^{n-1}_k$. Also

$$\begin{aligned}
&⨍\nabla(\rho(a_0))\theta^{j_0}\nabla(\rho(a_1))\theta^{j_1}\dots\nabla(\rho(a_k))\theta^{j_k} + \\
&\qquad\sum_{l=0}^{k-1}(-1)^l ⨍\rho(a_0)\theta^{j_0}\dots\nabla(\rho(a_l))\theta^{j_l}\nabla^2(\rho(a_{l+1}))\theta^{j_{l+1}}\dots\nabla(\rho(a_k))\theta^{j_k} = \\
&\qquad\qquad ⨍\nabla\big(\rho(a_0)\theta^{j_0}\nabla(\rho(a_1))\theta^{j_1}\dots\nabla(\rho(a_k))\theta^{j_k}\big) \qquad (3.33)
\end{aligned}$$

Using the definitions of θ', ρ' and $\displaystyle\fint'$ from (3.21), (3.23), (3.22) we can rewrite the last expression in the following way:

$$\fint \nabla\big(\rho(a_0)\theta^{j_0}\nabla(\rho(a_1))\theta^{j_1}\dots\nabla(\rho(a_k))\theta^{j_k}\big) = \\ \fint' r\big(\rho(a_0)\theta^{j_0}\nabla(\rho(a_1))\theta^{j_1}\dots\nabla(\rho(a_k))\theta^{j_k}\big) = \\ \fint' \rho'(a_0)(\theta')^{j_0}\nabla'(\rho'(a_1))(\theta')^{j_1}\dots\nabla'(\rho'(a_k))(\theta')^{j_k} \quad (3.34)$$

Hence adding expressions (3.29), (3.32), and using (3.33) we obtain:

$$\big(B\ \mathrm{Ch}^{k+1}(\mathcal{C}^n) + b\ \mathrm{Ch}^{k-1}(\mathcal{C}^n)\big)\,(a_0, a_1, \dots, a_k) = \\ c_k^{n-1} \sum_{j_0+j_1+\dots+j_k=\frac{n-k-1}{2}} \fint' \rho'(a_0)(\theta')^{j_0}\nabla'(\rho'(a_1))(\theta')^{j_1}\dots\nabla'(\rho'(a_k))(\theta')^{j_k} = \\ \mathrm{Ch}^k(\partial(\mathcal{C}^n))(a_0, a_1, \dots a_k) \quad (3.35)$$

□

Remark 4. A natural framework for such identities in cyclic cohomology is provided by the theory of operations on cyclic cohomology of Nest and Tsygan, cf. [NT95a, NT95b]

Corollary 5. *If $\mathcal{C}^n$ is a unital generalized cycle then* $\mathrm{Ch}(\mathcal{C}^n)$ *is an n-cocycle in the cyclic bicomplex of an algebra $\mathcal{A}$, and hence defines a class in $HC^n(\mathcal{A})$.*

Example 9. We have in the notations of the Example 5

$$\mathrm{Ch}(-\mathcal{C}) = -\,\mathrm{Ch}(\mathcal{C}) \quad (3.36)$$

$$\mathrm{Ch}(\mathcal{C}_1 \sqcup \mathcal{C}_2) = \mathrm{Ch}(\mathcal{C}_1) + \mathrm{Ch}(\mathcal{C}_2) \quad (3.37)$$

The proof of the identities above is straightforward. It is also true that $[\mathrm{Ch}((\mathcal{C}_1)_X)] \cup [\mathrm{Ch}((\mathcal{C}_2)_X)] = [\mathrm{Ch}((\mathcal{C}_1)_X \times (\mathcal{C}_2)_X)]$, as will be proved in the Corollary 21.

Corollary 6. *Whenever two unital generalized cycles $\mathcal{C}_1^n$ and $\mathcal{C}_2^n$ are cobordant*

$$[S\ \mathrm{Ch}(\mathcal{C}_1^n)] = [S\ \mathrm{Ch}(\mathcal{C}_2^n)]$$

in $HC^{n+2}(\mathcal{A})$ (where $[\ ]$ denotes the class in the cyclic cohomology).

We will also use several times the following straightforward observation:

Proposition 7. *Let $H : \mathcal{C}_1 \to \mathcal{C}_2$ be a (unital) morphism of two (unital) generalized chains covering the homomorphism h of the underlying algebras. Then $h^*\,\mathrm{Ch}(\mathcal{C}_2) = \mathcal{H}(\mathcal{C}_1)$.*

Formula (3.24) can also be written in the different form, closer to the formula from [JLO88]. We will use the following notations. First, $⨍$ can be extended to the whole algebra Ω by setting $⨍\xi = 0$ if $\deg \xi \neq 0$. For $\xi \in \Omega$ e^ξ is defined as $\sum_{j=0}^{\infty} \frac{\xi^j}{j!}$. Then denote Δ^k the k-simplex $\{(t_0, t_1, \ldots, t_k) | t_0 + t_1 + \cdots + t_k = 1\}$ with the measure $dt_1 dt_2 \ldots dt_k$. Finally, α is an arbitrary nonzero real parameter. Then

$$\begin{aligned} &\mathrm{Ch}^k(\mathcal{C}^n)(a_0, a_1, \ldots a_k) = \\ &\qquad \alpha^{\frac{k-n}{2}} \int\limits_{\Delta^k} \left(⨍\rho(a_0) e^{-\alpha t_0 \theta} \nabla(\rho(a_1)) e^{-\alpha t_1 \theta} \ldots \nabla(\rho(a_k)) e^{-\alpha t_k \theta} \right) dt_1 dt_2 \ldots dt_k \end{aligned} \tag{3.38}$$

where k is of the same parity as n. Indeed,

$$\begin{aligned} &⨍\rho(a_0) e^{-\alpha t_0 \theta} \nabla(\rho(a_1)) e^{-\alpha t_1 \theta} \ldots \nabla(\rho(a_k)) e^{-\alpha t_k \theta} = \\ &\qquad (-\alpha)^{\frac{n-k}{2}} \sum_{i_0 + i_1 + \cdots + i_k = \frac{n-k}{2}} \frac{t_0^{i_0} t_1^{i_1} \ldots t_k^{i_k}}{i_0! i_1! \ldots i_k!} ⨍\rho(a_0) \theta^{i_0} \nabla(\rho(a_1)) \theta^{i_1} \ldots \nabla(\rho(a_k)) \theta^{i_k} \end{aligned} \tag{3.39}$$

and our assertion follows from the equality

$$\int\limits_{\Delta^n} t_0^{i_0} t_1^{i_1} \dots t_k^{i_k}\, dt_1 dt_2 \dots dt_k = \frac{i_0! i_1! \dots i_k!}{(i_0 + i_1 + \dots + i_k + k)!}$$

Note that if $\mathcal{C}^n$ is a (non-generalized) cycle $\mathrm{Ch}(\mathcal{C}^n)$ coincides (up to a constant) with the character of $\mathcal{C}^n$ as defined by Connes.

3.3 The nonunital case.

We worked above only in the context of unital algebras and maps. Here we will treat the general case. In this section we explain how given a nonunital generalized chain over a nonunital algebra one can canonically construct a unital generalized chain over the algebra with adjoined unit.

First we need to define a nonunital generalized cycle. The definition of the generalized cycle in the nonunital case differ from the definition in the unital case only in two aspects: first, we do not require algebras and morphisms to be unital, second, we do not require any more that the curvature θ is an element of Ω^2; rather we require it to be a multiplier of the algebra Ω. We follow [Nis97].

Definition 9. A (nonunital) generalized cycle over an algebra $\mathcal{A}$ is given by the following data :

1. A $\mathbb{Z}$-graded unital algebra $\Omega = \bigoplus_{m=0}^{\infty} \Omega^m$ and a homomorphism ρ from $\mathcal{A}$ to Ω^0.

2. A graded derivation $\nabla : \Omega^k \mapsto \Omega^{k+1}$, k =0, 1, … and a multiplier θ of degree 2 of the algebra Ω (i.e. $\deg \theta\omega = \deg \omega\theta = \deg \omega + 2$) which satisfy

$$\nabla(\omega\xi) = \nabla(\omega)\xi + (-1)^{\deg \omega}\omega\nabla(\xi) \tag{3.40}$$

$$\nabla^2(\omega) = \theta\omega - \omega\theta \ \forall\omega \in \Omega \tag{3.41}$$

$$\nabla(\theta\omega) = \theta\nabla\omega \ \nabla(\omega\theta) = \nabla(\omega)\theta \tag{3.42}$$

3. A graded trace $⨍$ defined on Ω^n for some n with the properties

$$⨍\omega\xi = (-1)^{\deg w \deg \xi}⨍\xi\omega \text{ for } \deg \omega + \deg \xi = n \tag{3.43}$$

$$⨍\nabla(\omega) = 0 \ \forall\omega \in \Omega^{n-1} \tag{3.44}$$

$$⨍\theta\omega = ⨍\omega\theta \tag{3.45}$$

Example 10. In the Example 4 we can now remove the condition that V is compact. The algebra $\mathcal{A}$ we consider now is $C_0^\infty(V)$, and the graded algebra $\Omega = \Omega_0^*(V, \operatorname{End} E) = C_0^\infty(V, \Lambda^* T^* V \otimes \operatorname{End} E)$. With all the other data defined exactly as in the Example 4 we obtain a nonunital generalized cycle.

Example 11. We can define negative of a generalized cycle, disjoint union, and product for the nonunital case exactly as in the example 5.

The definition of morphisms in the nonunital case is a suitable reformulation of the Definition 4.

Definition 10. A morphism H between two generalized cycles $\mathcal{C}_1 = (\Omega_1, \nabla_1, \theta_1, ⨍_1)$

and $\mathcal{C}_2 = (\Omega_2, \nabla_2, \theta_2, \fint_2)$ over algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively, covering the homomorphism $h : \mathcal{A}_1 \to \mathcal{A}_2$ is a linear grading-preserving homomorphism $H : \Omega_1 \to \Omega_2$ satisfying

$$H(\rho_1(a)) = \rho_2(h(a)) \tag{3.46}$$

$$H(\nabla_1(\omega)) = \nabla_2(H(\omega)) \tag{3.47}$$

$$H(\theta_1\omega) = \theta_2\omega \;\; H(\omega\theta_1) = \omega\theta_2 \tag{3.48}$$

$$\fint_1 \omega = \fint_2 H(\omega) \tag{3.49}$$

Here again talking about the morphisms of generalized cycles over the same algebra $\mathcal{A}$ we mean, unless otherwise specified, morphism covering the identity homomorphism.

With each nonunital generalized cycle $\mathcal{C} = (\Omega, \nabla, \theta \fint)$ over an algebra $\mathcal{A}$ one can canonically associate a unital generalized cycle $\mathcal{C}^+ = (\Omega^+, \nabla^+, \theta^+, \fint^+)$ over the algebra $\mathcal{A}^+$ – algebra $\mathcal{A}$ with the unit adjoined. The construction is the following.

The graded algebra Ω^+ is obtained from the algebra Ω by adjoining an element θ^+ of degree 2 with relations $\theta^+\omega = \theta\omega$ and $\omega\theta^+ = \omega\theta$ (in the right hand side we have the action of the multiplier θ), and adjoining the unit (of degree 0). The homomorphism $\rho^+ : \mathcal{A}^+ \to (\Omega^+)^0$ is the unique unital extension of the homomorphism $\rho : \mathcal{A} \to \Omega^0$.

The graded derivation ∇^+ is the unique unital (i.e. $\nabla^+(1) = 0$) extension of ∇ to Ω^+ satisfying $\nabla^+(\theta^+) = 0$. Conditions (3.40)-(3.42) assure that such an extension really exists.

Finally, the graded trace $\fint^+$ is an extension of the $\fint$. If the degree of the cycle n is odd, $(\Omega^+)^n = \Omega^n$, and $\fint^+$ coincides with $\fint$. If n is even, there is one new element in $(\Omega^+)^n$, namely $(\theta^+)^{n/2}$ if $n > 0$ and 1 if $n = 0$. We put $\displaystyle\fint^+ (\theta^+)^{n/2} = 0$ if $n > 0$ and $\displaystyle\fint^+ 1 = 0$ if $n = 0$.

One can easily check that $\mathcal{C}^+$ is indeed a unital generalized cycle.

Example 12. We have $(-\mathcal{C})^+ = -(\mathcal{C}^+)$ and $(\mathcal{C}_1 \sqcup \mathcal{C}_2)^+ = \mathcal{C}_1^+ \sqcup \mathcal{C}_2^+$. It is not true that $(\mathcal{C}_1 \times \mathcal{C}_2)^+ = \mathcal{C}_1^+ \times \mathcal{C}_2^+$, but we have a natural morphism $(\mathcal{C}_1 \times \mathcal{C}_2)^+ \to \mathcal{C}_1^+ \times \mathcal{C}_2^+$

If we have a morphism $H : \mathcal{C}_1 \to \mathcal{C}_2$ covering the homomorphism $h : \mathcal{A}_1 \to \mathcal{A}_2$, one defines a morphism $H^+ : \mathcal{C}_1^+ \to \mathcal{C}_2^+$ covering $h^+ : \mathcal{C}_1^+ \to \mathcal{C}_2^+$, which is a unital extension of H, satisfying $H(\theta_1^+) = \theta_2^+$

We now give the definition of the nonunital generalized chain.

Definition 11. A (nonunital) generalized chain over an algebra $\mathcal{A}$ is given by the following data:

1. Graded unital algebras Ω and $\partial\Omega$ with a surjective homomorphism $r : \Omega \to \partial\Omega$ of degree 0 , and a homomorphism $\rho : \mathcal{A} \to \Omega^0$.

2. Graded derivations of degree 1 ∇ on Ω and ∇' on $\partial\Omega$ and multiplier θ of degree

2 of the algebra Ω such that

$$\nabla(\omega\xi) = \nabla(\omega)\xi + (-1)^{\deg\omega}\omega\nabla(\xi) \tag{3.50}$$

$$\nabla'(\omega'\xi') = \nabla'(\omega')\xi' + (-1)^{\deg\omega'}\omega'\nabla'(\xi') \tag{3.51}$$

$$\nabla^2(\omega) = \theta\omega - \omega\theta \ \forall\omega\in\Omega \tag{3.52}$$

$$\nabla(\theta\omega) = \theta\nabla\omega \ \ \nabla(\omega\theta) = \nabla(\omega)\theta \tag{3.53}$$

$$r\circ\nabla = \nabla'\circ r \tag{3.54}$$

$$r(\theta\omega) = r(\omega\theta) = 0 \text{ if } r(\omega) = 0 \tag{3.55}$$

3. A graded trace $⨍$ on Ω^n for some n (called the degree of the chain) such that

$$⨍\omega\xi = (-1)^{\deg w\deg\xi}⨍\xi\omega \text{ for } \deg\omega + \deg\xi = n \tag{3.56}$$

$$⨍\nabla(\omega) = 0 \ \forall\omega\in\Omega^{n-1} \text{ such that } r(\omega) = 0 \tag{3.57}$$

$$⨍\theta\omega = ⨍\omega\theta \tag{3.58}$$

Notice that we can define a multiplier θ' of $\partial\Omega$ by $\theta'\omega' = r(\theta\omega)$, $\omega'\theta' = r(\omega'\theta)$ where ω is such that $r(\omega) = \omega'$. This definition is unambiguous, as follows from (3.55). With this one can define the boundary of the (nonunital) generalized chain $\mathcal{C}$ as the (nonunital) generalized cycle $\partial\mathcal{C} = (\partial\Omega, \nabla', \theta', ⨍')$, where $⨍'$, as in the unital case is defined by $⨍'\omega' = ⨍\nabla\omega$, with $r(\omega) = \omega'$.

With a nonunital generalized chain over an algebra $\mathcal{A}$ one can, similarly to the case of generalized cycle, canonically associate a unital generalized chain. The details are the following.

The graded algebra Ω^+ is obtained from the algebra Ω by adjoining an element θ^+ of degree 2 with relations $\theta^+\omega = \theta\omega$ and $\omega\theta^+ = \omega\theta$ (in the right hand side we have the action of the multiplier θ), and adjoining the unit (of degree 0). The homomorphism $\rho^+ : \mathcal{A}^+ \to (\Omega^+)^0$ is the unique unital extension of the homomorphism $\rho : \mathcal{A} \to \Omega^0$. The algebra $\partial\Omega^+$ is similarly obtained from the algebra $\partial\Omega$ by adjoining elements $(\theta')^+$ and 1. The homomorphism r is extended as a unital homomorphism with $r(\theta^+) = (\theta')^+$. It is easily seen to remain surjective.

The graded derivation ∇^+, $(\nabla')^+$ are unique unital extensions of ∇ and ∇' to Ω^+ and ∂W^+ respectively, satisfying $\nabla^+(\theta^+) = 0$ and $(\nabla')^+ ((\theta')^+) = 0$.

Finally, the graded trace $\fint^+$ is an extension of the $\fint$. If the degree of the cycle n is odd, $(\Omega^+)^n = \Omega^n$, and $\fint^+$ coincides with $\fint$. If n is even, there is one new element in $(\Omega^+)^n$, namely $(\theta^+)^{n/2}$. We put $\fint^+ (\theta^+)^{n/2} = 0$.

Proposition 8. *For a generalized chain $\mathcal{C}$ $\partial(\mathcal{C}^+) = (\partial\mathcal{C})^+$, i.e. unitalization and taking boundaries commute.*

Proof. The only thing not immediate from the definitions is that graded traces $\left(\fint'\right)^+$ and $\left(\fint^+\right)'$ coincide. They clearly coincide on the elements of $\partial\Omega$. Hence the only thing we need to check is that if the degree n of the cycle is odd,

$$\left(\fint^+\right)' ((\theta')^+)^{(n-1)/2} = 0$$

(or $\left(\fint^+\right)' 1 = 0$ if $n = 0$). But since $r(\theta^+) = (\theta')^+$, $\left(\fint^+\right)' ((\theta')^+)^{(n-1)/2} = \fint^+ \nabla(\theta^+)^{(n-1)/2} = 0$, and similarly in the case $n = 0$. ∎

We now turn to the characters in the nonunital case. For every nonunital generalized chain $\mathcal{C}$ over an algebra $\mathcal{A}$ we can construct its unitalization $\mathcal{C}^+$ which is a generalized chain over $\mathcal{A}^+$, and consider its character $\mathrm{Ch}(\mathcal{C}^+)$ (defined in the Definition 8) which is a cyclic cocycle in the (b, B) bicomplex of the algebra $\mathcal{A}^+$.

Proposition 9. *The cochain* $\mathrm{Ch}(\mathcal{C}^+)$ *defines a cochain in the reduced* (b, B) *bicomplex of the algebra* $\mathcal{A}^+$.

Proof. If the degree of n $\mathcal{C}$ is odd, there is nothing to prove. If the degree of $\mathcal{C}$ is even, we need to verify that $\mathrm{Ch}^0(\mathcal{C}^+)(1) = 0$. But

$$\mathrm{Ch}^0(\mathcal{C}^+)(1) = \fint^{+} (\theta^+)^{n/2} = 0$$

by the definition of $\fint^{+}$ □

Definition 12. The character of the (nonunital) generalized chain $\mathcal{C}$ over an algebra $\mathcal{A}$ is the cochain $\mathrm{Ch}(\mathcal{C})$ of the reduced (b, B) bicomplex of the algebra $\mathcal{A}^+$.

We have now two different definitions for the character of the unital chain $\mathcal{C}$ over the unital algebra $\mathcal{A}$. One is a cochain of the (b, B) bicomplex of the algebra $\mathcal{A}$, and the other is a cochain of the reduced (b, B) bicomplex of the algebra $\mathcal{A}^+$. The next theorem shows that these two definitions agree.

Theorem 10. *Let* $\mathcal{C}$ *be the unital chain over the unital algebra* $\mathcal{A}$. *Let* h *be the canonical quasiisomorphism* $\mathcal{B}(\mathcal{A}) \to \mathcal{B}_{\mathrm{red}}(\mathcal{A}^+)$ *(cf. [Lod92]. Then*

$$h(\mathrm{Ch}(\mathcal{C})) = \mathrm{Ch}(\mathcal{C}^+)$$

Proof. This is a direct consequence of the explicit formulae for h and the characters.

□

It is clear that the following generalization of the Theorem 3, where we no longer require unitality holds:

Theorem 11. *Let $\mathcal{C}$ be a chain, and $\partial(\mathcal{C})$ be its boundary. Then*

$$(B+b)\,\mathrm{Ch}(\mathcal{C}) = S\ \mathrm{Ch}(\partial\mathcal{C}) \tag{3.59}$$

Here S is the usual periodicity shift in the cyclic bicomplex.

This theorem together with the Theorem 10 has the following corollaries.

Corollary 12. *If $\mathcal{C}^n$ is a generalized cycle then $\mathrm{Ch}(\mathcal{C}^n)$ is an n-cocycle in the reduced cyclic bicomplex of an algebra $\mathcal{A}$, and hence defines a class in $HC^n_{\mathrm{red}}(\mathcal{A}^+) = HC^n(\mathcal{A})$. The definition of the class in $HC^n(\mathcal{A})$ agrees with the previous definition in the case of the unital cycle.*

Corollary 13. *Whenever two generalized cycles $\mathcal{C}^n_1$ and $\mathcal{C}^n_2$ are cobordant*

$$[S\ \mathrm{Ch}(\mathcal{C}^n_1)] = [S\ \mathrm{Ch}(\mathcal{C}^n_2)]$$

in $HC^{n+2}(\mathcal{A})$.

We also have

Proposition 14. *Let $H : \mathcal{C}_1 \to \mathcal{C}_2$ be a morphism of two generalized chains covering the homomorphism h of the underlying algebras. Then $h^*\,\mathrm{Ch}(\mathcal{C}_2) = \mathrm{Ch}(\mathcal{C}_1)$.*

3.4 Variation of connection.

Proposition 15. *Let* $\mathcal{C}_0 = (\Omega, \partial\Omega, r\nabla, \nabla'_0, \theta, \fint)$ *be a unital generalized chain of degree* n *over an algebra* $\mathcal{A}$

For some element $\eta \in \Omega^1$ *put*

$$\nabla_\eta = \nabla + \operatorname{ad}\eta \tag{3.60}$$

$$\theta_\eta = \theta + \nabla\eta + \eta^2 \tag{3.61}$$

Then $\mathcal{C}_\eta = (\Omega, \partial\Omega, r\nabla_\eta, \nabla'_1, \theta_\eta, \fint)$ *is also a generalized chain. If* $\mathcal{C}_0$ *is a cycle, then so is* $\mathcal{C}_\eta$*.*

Proof. All the conditions of the Definition 5 are immediate, except possibly (3.16), (3.17). We will now verify (3.16).

$$(\nabla_\eta)^2\omega = (\nabla + \operatorname{ad}\eta)^2\omega = \nabla^2\omega + \nabla([\eta, \omega]) + [\eta, \nabla\omega] + [\eta, [\eta, \omega]] \tag{3.62}$$

Using the identities $[\eta, [\eta, \omega]] = [\eta^2, \omega]$ and $\nabla([\eta, \omega]) + [\eta, \nabla(\omega)] = [\nabla(\eta), \omega]$ we rewrite the result of the previous computation as $[\theta + \nabla\eta + \eta^2, \omega]$.

We will now verify (3.17).

$$\nabla_\eta\theta_\eta =$$

$$(\nabla + \operatorname{ad}\eta)\left(\theta + \nabla\eta + \eta^2\right) = +\nabla^2\eta + \nabla(\eta)\eta - \eta\nabla\eta) + [\eta, \theta] + [\eta, \nabla(\eta)] + [\eta, \eta^2] = 0 \tag{3.63}$$

□

Theorem 16. *Let* $\mathcal{C} = (\Omega, \nabla, \theta, \fint)$ *be a generalized cycle of degree* n *over an algebra* $\mathcal{A}$*. For some element* $\eta \in \Omega^1$ *put*

$$\nabla_\eta = \nabla + \operatorname{ad}\eta \tag{3.64}$$

$$\theta_\eta = \theta + \nabla\eta + \eta^2 \tag{3.65}$$

Then $\mathcal{C}_\eta$ *is a generalized cycle by Proposition 15, and*

$$[\mathrm{Ch}(\mathcal{C})] = [\mathrm{Ch}(\mathcal{C}_\eta)] \tag{3.66}$$

Proof. We start by constructing a cobordism between cycles $\mathcal{C}$ and $\mathcal{C}_\eta$. This is analogous to a construction from [Nis97]. The cobordism is provided by the chain $\mathcal{C}^c = (\Omega^c, \partial\Omega^c, r^c, \nabla^c, (\nabla^c)', \theta^c, \fint^c)$ with $\partial\mathcal{C}^c = -\mathcal{C}_\eta \sqcup \mathcal{C}_2$ defined as follows.

The graded algebra Ω^c is defined as $\Omega^*([0,1])\widehat{\otimes}\Omega$, where $\widehat{\otimes}$ denotes the graded tensor product, and $\Omega^*([0,1])$ is the algebra of the differential forms on the segment $[0,1]$. The map $\rho^c : \mathcal{A} \to \Omega^c$ is given by

$$\rho^c(a) = 1\widehat{\otimes}\rho(a) \tag{3.67}$$

We denote by t the variable on the segment $[0,1]$.

The graded derivation ∇^c is defined by

$$\nabla^c(\alpha\widehat{\otimes}\omega) = d\alpha\widehat{\otimes}\omega + (-1)^{\deg\alpha}\alpha\widehat{\otimes}\nabla\omega + (-1)^{\deg\alpha}t\alpha\widehat{\otimes}[\eta,\omega] \tag{3.68}$$

Here d is the de Rham differential on $[0,1]$.

The curvature θ^c is defined to be

$$1\widehat{\otimes}\theta + t\widehat{\otimes}\nabla\eta + t^2\widehat{\otimes}\eta^2 + dt\widehat{\otimes}\eta \tag{3.69}$$

As expected, the algebra $\partial\Omega^c$ is defined to be $\Omega \oplus \Omega$. The restriction map r^c : $\Omega^c \to \Omega \oplus \Omega$ is defined by

$$r^c(\alpha\widehat{\otimes}\omega) = \begin{cases} \alpha(0)\omega \oplus \alpha(1)\omega & \text{if } \deg\alpha = 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.70}$$

The connection ∇' on $\Omega \oplus \Omega$ is given by $\nabla \oplus \nabla_\eta$. All the conditions (3.14)-(3.18) are clear, except possibly (3.16), (3.17). To verify them notice that in the case $\eta = 0$ they are clear, and the general case follows from the computations from the proof of the Proposition 15. Indeed, connections ∇^c differs from the corresponding connection in the case $\eta = 0$ by $\mathrm{ad}\, t\widehat{\otimes}\eta$.

The graded trace $\displaystyle⨍^c$ on $(\Omega^c)^{n+1}$ is given by the formula

$$⨍^c \alpha\widehat{\otimes}\omega = \begin{cases} \int\limits_{[0,1]} \alpha ⨍\omega & \text{if } \deg\omega = n \text{ and } \deg\alpha = 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.71}$$

It is easy to see that

$$⨍^c \nabla^c(\alpha\widehat{\otimes}\omega) = \begin{cases} (\alpha(1) - \alpha(0))⨍\omega & \text{if } \deg\omega = n \text{ and } \deg\alpha = 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.72}$$

This computation both checks the condition (3.57) and shows that the "boundary" trace $(⨍^c)'$ induced on $\Omega \oplus \Omega$ equals $-⨍ \oplus ⨍$. Conditions (3.56) and (3.58) are clear.

Hence we constructed the generalized chain $\mathcal{C}^c$, providing the cobordism between $\mathcal{C}$ and $\mathcal{C}_\eta$. The Corollary 6 implies that $[S\ \mathrm{Ch}(\mathcal{C})] = [S\ \mathrm{Ch}(\mathcal{C}_\eta)]$. To obtain the more precise statement (3.66) and finish the proof of the Theorem, we need to examine the character $\mathrm{Ch}(\mathcal{C}^c)$, since by the Theorem 3 $S\ \mathrm{Ch}(\mathcal{C}) - S\ \mathrm{Ch}(\mathcal{C}_\eta) = (b + B)\,\mathrm{Ch}(\mathcal{C}^c)$.

$\mathrm{Ch}(\mathcal{C}^c)$ has components $\mathrm{Ch}^k(\mathcal{C}^c)$ for $k = n+1,\ n-1,\ \dots$. Its top component $\mathrm{Ch}^{n+1}(\mathcal{C}^c)$ is given by the formula

$$\mathrm{Ch}^{n+1}(\mathcal{C}^c)(a_0, a_1, \dots, a_{n+1}) = \frac{1}{(n+1)!} \fint^c \rho^c(a_0)\nabla^c(\rho^c(a_1))\dots\nabla^c(\rho^c(a_{n+1})) \quad (3.73)$$

where $a_i \in \mathcal{A}$. But the expression under $\fint^c$ is easily seen to be of the form $\alpha\widehat{\otimes}\omega$, with α of degree 0. Hence the expression (3.73) is identically 0, by the definition (3.71) of $\fint^c$. It follows that $\mathrm{Ch}(\mathcal{C}^c)$ is in the image of the map S, and this implies that $[\mathrm{Ch}(\mathcal{C})] = [\mathrm{Ch}(\mathcal{C}_\eta)]$. □

From the proof of the Theorem 16 one obtains also an explicit formula for the $n-1$ cochain T in the (b, B) bicomplex such that $\mathrm{Ch}(\mathcal{C}) - \mathrm{Ch}(\mathcal{C}_\eta) = (b+B)T$. To simplify the formulae we introduce the notations

$$\nabla_t = \nabla + t\,\mathrm{ad}\,\eta \quad (3.74)$$

$$\theta_t = \theta + t\nabla\eta + t^2\eta^2 \quad (3.75)$$

Corollary 17. *In the notations of the Theorem 16 Proposition 15 we have:*

$$\mathrm{Ch}(\mathcal{C}) - \mathrm{Ch}(\mathcal{C}_\eta) = (b+B)T \quad (3.76)$$

where T is the cochain in the (b, B) bicomplex of the algebra $\mathcal{A}$, with components given by

$$T^k(a_0, a_1, \dots, a_k) = \mathrm{Ch}(\mathcal{C}^c)^k(a_0, a_1, \dots, a_k) =$$
$$\frac{(-1)^{\frac{n-1-k}{2}}}{(\frac{n+1+k}{2})!} \sum_{i_0+i_1+\dots+i_k=\frac{n-1-k}{2}} \sum_{j=0}^{k} (-1)^j \int_0^1 \fint \rho(a_0)\theta_t^{i_0}\nabla_t(\rho(a_1))\theta_t^{i_1}\dots$$
$$\nabla_t(\rho(a_j))\theta_t^{i_j}\eta\nabla_t(\rho(a_{j+1}))\theta_t^{i_{j+1}}\dots\nabla_t(\rho(a_k))\theta_t^{i_k} \quad (3.77)$$

We now proceed to the nonunital case. First, we establish the analogue of the Proposition 15.

Proposition 18. *Let* $\mathcal{C} = (\Omega, \partial\Omega, r\nabla, \nabla'_0, \theta, ⨍)$ *be a nonunital generalized chain of degree* n *over an algebra* $\mathcal{A}$*, and let* η *be a degree 1 multiplier of* Ω*, such that*

$$⨍\eta\omega = (-1)^{(n-1)/2} ⨍\omega\eta \tag{3.78}$$

$$r(\eta\omega) = r(\omega\eta) = 0 \text{ if } r(\omega) = 0 \tag{3.79}$$

We can then define a multiplier of Ω $\nabla\eta$ *by*

$$(\nabla\eta)\omega = \nabla(\eta\omega) + \eta\nabla\omega;\ \omega\nabla\eta = (-1)^{\deg\omega}(\nabla(\omega\eta) - \nabla(w)\eta) \tag{3.80}$$

and a multiplier $r(\eta)$ *of* $\partial\Omega$ *by* $r(\eta)\omega' = r(\eta\omega)$*, where* $r(\omega) = \omega'$*. Put*

$$\nabla_\eta = \nabla + \operatorname{ad}\eta \tag{3.81}$$

$$\nabla'_1 = \nabla'_0 + \operatorname{ad} r(\eta) \tag{3.82}$$

$$\theta_\eta = \theta + \nabla\eta + \eta^2 \tag{3.83}$$

Then $\mathcal{C}_\eta = (\Omega, \partial\Omega, r\nabla_\eta, \nabla'_1, \theta_\eta, ⨍)$ *is also a generalized chain. If* $\mathcal{C}$ *is a cycle, then so is* $\mathcal{C}_\eta$*.*

Proof. The fact that $\nabla\eta$ and $r(\eta)$ are multipliers of the respective algebras follows from the definitions. All the conditions of the Definition 11 are immediate, except possibly (3.52), (3.53). The proof of (3.52) is identical to the similar proof in the

Proposition 15. We will now verify first identity of (3.17) (the other identity is similar).

$$\nabla_\eta(\theta_\eta\omega) =$$

$$(\nabla + \mathrm{ad}\,\eta)\big(\theta\omega + \nabla(\eta)\omega + \eta^2\omega\big) = \theta\nabla\omega + \nabla^2(\eta)\omega + \nabla(\eta)\nabla\omega +$$

$$(\nabla(\eta)\eta - \eta\nabla\eta)\omega + \eta^2\nabla\omega + [\eta,\theta\omega] + [\eta,\nabla(\eta)\omega] + [\eta,\eta^2\omega] =$$

$$\theta\nabla\omega + \nabla(\eta)\nabla\omega + \eta^2\nabla(\omega) + \theta[\eta,\omega] + \nabla(\eta)[\eta,\omega] + \eta^2[\eta,\omega] =$$

$$\theta_\eta\nabla_\eta\omega \quad (3.84)$$

□

We now proceed to the nonunital analogue of the Theorem 16.

Theorem 19. *Let $\mathcal{C} = (\Omega, \nabla, \theta, \fint)$ be a generalized cycle of degree n over an algebra $\mathcal{A}$, let η be a degree 1 multiplier of Ω, such that $\fint \eta\omega = (-1)^{(n-1)/2}\fint \omega\eta$. Put*

$$\nabla_\eta = \nabla + \mathrm{ad}\,\eta \tag{3.85}$$

$$\theta_\eta = \theta + \nabla\eta + \eta^2 \tag{3.86}$$

where $\nabla\eta$ is defined in (3.80)*. Then $\mathcal{C}_\eta$ is a generalized cycle by Proposition 18, and*

$$[\mathrm{Ch}(\mathcal{C})] = [\mathrm{Ch}(\mathcal{C}_\eta)] \tag{3.87}$$

Proof. We start by enlarging the algebra Ω similarly to the construction of the cycle $\mathcal{C}^+$. More precisely, consider the following generalized cycle $\mathcal{C}^p = (\Omega^p, \nabla^p, \theta^p, \fint^p)$ over the algebra $\mathcal{A}^+$. The algebra Ω^p is obtained from the algebra Ω by adjoining unit of degree 0, element η^p of degree 1, and elements θ^p and $(\nabla\eta)^p$ of degree 2. The relations of this algebra are the following, where $\omega \in \Omega$

$$\theta^p\omega = \theta\omega;\ \omega\theta^p = \omega\theta \tag{3.88}$$

$$\eta^p\omega = \eta\omega;\ \omega\eta^p = \omega\eta \tag{3.89}$$

$$(\nabla\eta)^p\omega = \nabla\eta\omega;\ \omega(\nabla\eta)^p = \omega\nabla\eta \tag{3.90}$$

and the usual relations with respect to the unit. The ∇^p extends ∇, satisfies

$$\nabla^p 1 = 0 \tag{3.91}$$

$$\nabla^p\theta^p = 0 \tag{3.92}$$

$$\nabla^p\eta^p = (\nabla\eta)^p \tag{3.93}$$

$$\nabla^p(\nabla\eta)^p = \theta^p\eta^p - \eta^p\theta^p \tag{3.94}$$

and is extended to the whole Ω^p as a derivation. It follows that $(\nabla^p)^2 = \operatorname{ad}\theta^p$. Indeed, on the both sides of this equality are derivations, and they coincide since they coincide on generators.

The graded trace $⨍^p$ is the extension of $⨍$ satisfying $⨍^p P(\theta^p, \eta^p, (\nabla\eta)^p) = 0$ for any polynomial P.

Notice that there is a natural unital morphism $H : \mathcal{C}^+ \to \mathcal{C}^p$ defined by $H(\theta^+) = \theta^p$, and hence $\mathrm{Ch}(\mathcal{C}^p) = \mathrm{Ch}(\mathcal{C}^+)$. Notice also that $\mathrm{Ch}(\mathcal{C}^p)$ is clearly a reduced cocycle.

Consider now the cycles $(\mathcal{C}_\eta)^+$ and $\mathcal{C}^p_\eta$. We can construct a unital morphism F between them defined by $F\left((\theta_\eta)^+\right) = (\theta^p)_\eta$. The only thing in need of verification is that if $n = \deg\mathcal{C}$ is even then

$$⨍^p \left((\theta^p)_\eta\right)^{n/2} = 0 \tag{3.95}$$

But

$$\int\!\!\!\!\!-^{p} ((\theta^p)_\eta)^{n/2} = \int\!\!\!\!\!-^{p} (\theta^p + \nabla^p \eta^p + (\eta^p)^2)^{n/2} = 0 \tag{3.96}$$

by the definition of $\int\!\!\!\!\!-^{p}$. Hence $\mathrm{Ch}((\mathcal{C}_\eta)^+) = \mathrm{Ch}(\mathcal{C}^p_\eta)$.

Then Corollary 17 shows that $\mathrm{Ch}(\mathcal{C}^p)_\eta - \mathrm{Ch}(\mathcal{C}^p) = (b+B)\,\mathrm{Ch}((\mathcal{C}^p)^c)$, and hence

$$\mathrm{Ch}((\mathcal{C}_\eta)^+) - \mathrm{Ch}(\mathcal{C}^+) = (b+B)\,\mathrm{Ch}((\mathcal{C}^p)^c)$$

To finish the proof it remains to remark that $\mathrm{Ch}((\mathcal{C}^p)^c)$ is also a reduced cocycle, as follows from the explicit formula for its components and a computation similar to (3.96). □

3.5 Relation with Connes' construction.

With every generalized cycle $\mathcal{C} = (\Omega, \nabla, \theta, \int\!\!\!\!\!-)$ over an algebra $\mathcal{A}$ Connes shows how to associate canonically a cycle $\mathcal{C}_X = (\Omega_X, d_X, \int\!\!\!\!\!-_X)$. Then definition of the character of a cycle allows one to associate Connes character with it. The main goal of this section is to prove that the class of this character in the cyclic cohomology coincides with the class of the character of the generalized cycle.

We start by recalling Connes' construction. The algebra Ω_X consists of elements of the form $\omega_{11} + \omega_{12}X + X\omega_{21} + X\omega_{22}X$, where X is a formal symbol of degree 1, and multiplication is defined formally by relations $X^2 = \theta$ and $\omega_1 X \omega_2 = 0$. In other words, graded algebra Ω_X as a vector space can be identified with the space of 2 by 2 matrices over an algebra Ω, with the grading given by the following:

$$\begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix} \in \Omega_X^k \text{ if } \omega_{11} \in \Omega^k \; \omega_{12}, \omega_{21} \in \Omega^{k-1} \text{ and } \omega_{22} \in \Omega^{k-2}$$

and the product of the two elements $\omega = \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix}$ and $\omega' = \begin{bmatrix} \omega'_{11} & \omega'_{12} \\ \omega'_{21} & \omega'_{22} \end{bmatrix}$ is given by

$$\omega * \omega' = \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \theta \end{bmatrix} \begin{bmatrix} \omega'_{11} & \omega'_{12} \\ \omega'_{21} & \omega'_{22} \end{bmatrix} \tag{3.97}$$

Here we identify the matrix $\omega = \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix}$ with the element $\omega_{11} + \omega_{12}X + X\omega_{21} + X\omega_{22}X$.

The homomorphism $\rho_X : \mathcal{A} \to \Omega_X$ is a composition of the homomorphism $\rho : \mathcal{A} \to \Omega$ with the natural inclusion $\Omega \hookrightarrow \Omega_X$. In terms of matrices it is given by

$$\rho_X(a) = \begin{bmatrix} \rho(a) & 0 \\ 0 & 0 \end{bmatrix} \tag{3.98}$$

We can construct an extension ∇_X of ∇ to Ω_X by formally requiring that $\nabla_X X = 0$, or in terms of matrices

$$\nabla_X \left(\begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix} \right) = \begin{bmatrix} \nabla(\omega_{11}) & \nabla(\omega_{12}) \\ -\nabla(\omega_{21}) & -\nabla(\omega_{22}) \end{bmatrix} \tag{3.99}$$

We have

$$\nabla^2_X \left(\begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix} \right) = \begin{bmatrix} \nabla^2(\omega_{11}) & \nabla^2(\omega_{12}) \\ \nabla^2(\omega_{21}) & \nabla^2(\omega_{22}) \end{bmatrix} =$$

$$\begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix} \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix} - \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix} \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix} = \begin{bmatrix} \theta & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix} - \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix} * \begin{bmatrix} \theta & 0 \\ 0 & 1 \end{bmatrix} \tag{3.100}$$

When $\mathcal{C}$ is nonunital we treat $\begin{bmatrix} \theta & 0 \\ 0 & 1 \end{bmatrix}$ as a multiplier of the algebra Ω_X.

Define now the graded derivation d_X on Ω_X formally by

$$d_X(X) = 0 \tag{3.101}$$

$$d_X(\omega) = \nabla\omega + X\omega + (-1)^{\deg \omega}\omega X \tag{3.102}$$

Then $d_X^2 = 0$. In terms of the matrices d_X is written as

$$d_X \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix} = \begin{bmatrix} \nabla(\omega_{11}) & \nabla(\omega_{12}) \\ -\nabla(\omega_{21}) & -\nabla(\omega_{22}) \end{bmatrix} + \begin{bmatrix} 0 & -\theta \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \omega_{11} & \omega_{12} \\ w_{21} & \omega_{22} \end{bmatrix}$$
$$+ (-1)^{\deg \omega} \begin{bmatrix} \omega_{11} & \omega_{12} \\ w_{21} & \omega_{22} \end{bmatrix} \begin{bmatrix} 0 & -\theta \\ 1 & 0 \end{bmatrix} \tag{3.103}$$

Finally, the graded trace $⨍_X$ is defined by $⨍_X \omega_{11} + \omega_{12}X + X\omega_{21} + X\omega_{22}X = ⨍\omega_{11} - (-1)^{\deg \omega} ⨍_X \omega_{22}\theta$.

The main result of this section is the following

Theorem 20. *Let $\mathcal{C}$ be the generalized cycle and $\mathcal{C}_X$ be the associated cycle. Then*

$$[\mathrm{Ch}(\mathcal{C})] = [\mathrm{Ch}(\mathcal{C}_X)] \tag{3.104}$$

Proof. Consider the cycle $\mathcal{C}'$ over the algebra $\mathcal{A}$ given by $(\Omega_X, \nabla_X, \theta_X, \fint_X)$, where $\theta_X = \begin{bmatrix} \theta & 0 \\ 0 & 1 \end{bmatrix}$. We have seen already that $\nabla_X^2 = \mathrm{ad}\,\theta_X$, and it is easy to see that all the required properties of the trace are satisfied. Notice that $\mathcal{C}'$ is never a unital cycle. An easy computation shows that $\mathrm{Ch}(\mathcal{C}') = \mathrm{Ch}(\mathcal{C}^+)$, as cocycles over $\mathcal{A}^+$. Consider now a degree 1 multiplier of the algebra Ω^c given by the matrix

$$\mathcal{X} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{3.105}$$

In other words, for $\omega = \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix}$,

$$\mathcal{X} * \omega = \begin{bmatrix} 0 & \theta \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix} = \begin{bmatrix} \theta\omega_{21} & \theta\omega_{22} \\ -\omega_{11} & -\omega_{12} \end{bmatrix} \tag{3.106}$$

$$\omega * \mathcal{X} = \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -\theta & 0 \end{bmatrix} = \begin{bmatrix} -\omega_{12}\theta & \omega_{11} \\ -\omega_{22}\theta & \omega_{21} \end{bmatrix} \tag{3.107}$$

We can then construct, as in the Proposition 18, the generalized cycle $\mathcal{C}'_{\mathcal{X}}$ by perturbing connection and curvature. Indeed, all the conditions of the Proposition are easily verified.

We claim that the generalized cycle $\mathcal{C}'_{\mathcal{X}}$ coincides with the cycle $\mathcal{C}_X$. Indeed, the identities 3.106 show that $\nabla_X + \mathrm{ad}\,\mathcal{X} = d_X$. It remains to verify that $\theta_X + \nabla_X \mathcal{X} + \mathcal{X}^2 = 0$. But it is easy to see that $\nabla_X \mathcal{X} = 0$ and

$$\mathcal{X}^2 = \mathcal{X} * \mathcal{X} = \begin{bmatrix} -\theta & 0 \\ 0 & -1 \end{bmatrix} = -\theta_X$$

The Theorem 19 now implies that

$$[\mathrm{Ch}(\mathcal{C}^+)] = [\mathrm{Ch}(\mathcal{C}')] = [\mathrm{Ch}(\mathcal{C}'_{\mathcal{X}})] = [\mathrm{Ch}(\mathcal{C}_X)]$$

in the reduced cyclic cohomology of $\mathcal{A}^+$, and the Theorem follows. $\square$

Corollary 21. *In the Examples 5 and 11 we defined for two generalized cycles* $\mathcal{C}_1^n = (\Omega_1, \nabla_1, \theta_1, \fint_1)$ *and* $\mathcal{C}_2^m = (\Omega_2, \nabla_2, \theta_2, \fint_2)$ *their product as* $\mathcal{C}_1 \times \mathcal{C}_2 = (\Omega_1 \widehat{\otimes} \Omega_2, \nabla_1 \widehat{\otimes} 1 + 1 \widehat{\otimes} \nabla_2, \theta_1 \widehat{\otimes} 1 + 1 \widehat{\otimes} \theta_2, \fint_1 \widehat{\otimes} \fint_2)$. *Then* $[\mathrm{Ch}(\mathcal{C}_1 \times \mathcal{C}_2)] = [\mathrm{Ch}(\mathcal{C}_1)] \cup [\mathrm{Ch}(\mathcal{C}_2)]$.

Proof. For the (non-generalized) cycles this follows from Connes' definition of the cup-product. In the general case this implies that $[\mathrm{Ch}((\mathcal{C}_1)_X)] \cup [\mathrm{Ch}((\mathcal{C}_2)_X)] = [\mathrm{Ch}((\mathcal{C}_1)_X \times (\mathcal{C}_2)_X)]$. Now the statement follows from the Theorem 20 and the existence of the morphism $(\mathcal{C}_1 \times \mathcal{C}_2)_X \to (\mathcal{C}_1)_X \times (\mathcal{C}_2)_X$. If we denote by X_1, X_2, X_{12} the formal elements, corresponding to $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_1 \times \mathcal{C}_2$ respectively, the morphism mentioned above is the extension of the identity map $\Omega_1 \widehat{\otimes} \Omega_2 \to \Omega_1 \widehat{\otimes} \Omega_2$ defined (again formally) by $X_{12} \mapsto (X_1 \widehat{\otimes} 1 + 1 \widehat{\otimes} X_2)$.

$\square$

CHAPTER 4

EQUIVARIANT CHERN CHARACTER.

4.1 Construction of the equivariant character in cyclic cohomology.

This section concerns vector bundles equivariant with respect to discreet group actions. We show that there is a generalized cycle associated naturally to such a bundle with (not necessarily invariant) connection. The character of this generalized cycle turns out to be related (see Theorem 22) to the equivariant Chern character. The formulae thus obtained are related to the formulae for the character of the K-homology class constructed in [CM98]. The classes in cyclic cohomology thus obtained are all concentrated at the identity of the group. There should be the corresponding formulae for the other conjugacy classes, related to the cohomology of the loop spaces, cf. [Bur85].

Let V be an orientable smooth manifold of dimension n, E a complex vector bundle over V, and $\mathcal{A} = \mathrm{End}(E)$ – algebra of endomorphisms with compact support. One can construct a generalized cycle over an algebra $\mathcal{A}$ in the following way. The algebra $\Omega = \Omega^*(V, \mathrm{End}(E))$ – the algebra of endomorphism-valued differential forms. Any connection ∇ on the bundle E defines a connection for the generalized cycle, with the curvature $\theta \in \Omega^2(V, \mathrm{End}(E))$ – the usual curvature of the connection. On the $\Omega^n(V, \mathrm{End}(E))$ one defines a graded trace $\fint$ by the formula $\fint \omega = \int\limits_V \mathrm{tr}\,\omega$, where in the right hand side we have a usual matrix trace and a usual integration over a manifold. Note that when V is noncompact, this cycle is nonunital. The formula (3.24) then defines a cyclic n-cocycle $\{\mathrm{Ch}^k\}$ on the algebra $\mathcal{A}$, given by the formula

$$\mathrm{Ch}^k(a_0, a_1, \dots a_k) =$$
$$\int\limits_{\Delta^k} \left(\int\limits_V \mathrm{tr}\, a_0 e^{-t_0\theta} \nabla(a_1) e^{-t_1\theta} \dots \nabla(a_k) e^{-t_k\theta} \right) dt_1 dt_2 \dots dt_k \quad (4.1)$$

Hence we recover the formula of Quillen from [Qui88]. (Recall that for noncompact V these expressions should be viewed as defining reduced cocycle over the algebra $\mathcal{A}$ with unit adjoined, with Ch^0 extended by $\mathrm{Ch}^0(1) = 0$).

One can restrict this cocycle to the subalgebra of functions $C^\infty(V) \subset \mathrm{End}(E)$. As a result one obtains an n-cocycle on the algebra $C^\infty(V)$, which we still denote by $\{\mathrm{Ch}^k\}$, given by the formula

$$\mathrm{Ch}^k(a_0, a_1, \dots a_k) = \frac{1}{k!} \int\limits_V a_0 da_1 \dots da_k \ \mathrm{tr}\, e^{-\theta} \quad (4.2)$$

To this cocycle corresponds a current on V, defined by the form $\mathrm{tr}\, e^{-\theta}$. Hence in this case we recover the Chern character of the bundle E. Note that we use normalization of the Chern character from [BGV92].

Let now an orientable manifold V of dimension n be equipped with an action of the discrete group Γ of orientation preserving transformations, and E be a Γ-invariant bundle. In this situation, one can again construct a cycle of degree n over the algebra

$\mathcal{A} = \mathrm{End}(E) \rtimes \Gamma$. Our notations are the following : the algebra $\mathcal{A}$ is generated by the elements of the form aU_g, $a \in \mathrm{End}(E)$, $g \in \Gamma$, and U_g is a formal symbol. The product is $(a'U_{g'})(aU_g) = a'a^{g'}U_{gg'}$. The superscript here denotes the action of the group.

The graded algebra Ω is defined as $\Omega^*(V, \mathrm{End}(E)) \rtimes \Gamma$. Elements of Ω clearly act on the forms with values in E, and any connection ∇ in the bundle E defines a connection for the algebra Ω, which we also denote by ∇, by the identity (here $\omega \in \Omega$, and $s \in \Omega^*(V, E)$)

$$\nabla(\omega s) = \nabla(\omega)s + (-1)^{\deg \omega}\omega\nabla(s) \tag{4.3}$$

One checks that the above formula indeed defines a degree 1 derivation, which can be described by the action on the elements of the form αU_g where $\alpha \in \Omega^*(V, \mathrm{End}(E))$, $g \in \Gamma$, by the equation

$$\nabla(\alpha U_g) = (\nabla(\alpha) + \alpha \wedge \delta(g))\, U_g \tag{4.4}$$

where δ is $\Omega^1(V, \mathrm{End}(E))$-valued group cocycle, defined by

$$\delta(g) = \nabla - g \circ \nabla \circ g^{-1} \tag{4.5}$$

One defines a curvature as an element θU_1, where 1 is the unit of the group, and θ is the (usual) curvature of ∇ . The graded trace $⨍$ on Ω^n is given by

$$⨍ \alpha U_g = \begin{cases} \int\limits_V \alpha & \text{if } g = 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.6}$$

One can associate with this cycle a cyclic n-cocycle over an algebra $\mathcal{A}$, by the equation (3.24). By restricting it to the subalgebra $C_0^\infty(V) \rtimes \Gamma$ one obtains an n-cocycle $\{\chi^k\}$ on this algebra. Its k-th component is given by the formula

$$\chi^k(a_0 U_{g_0}, a_1 U_{g_1}, \dots a_k U_{g_k}) = \sum_{1 \le i_1 < i_2 < \dots < i_l \le k} \int_V a_0 da_1^{\gamma_1} da_2^{\gamma_2} \dots da_{i_1-1}^{\gamma_{i_1-1}} a_{i_1}^{\gamma_{i_1}} da_{i_1+1}^{\gamma_{i_1+1}} \dots \Theta_{i_1,i_2,\dots,i_l}(\gamma_1, \dots, \gamma_k) \quad (4.7)$$

for $g_0 g_1 \dots g_k = 1$ and 0 otherwise. Here the summation is over all the subsets of $\{1, 2, \dots k\}$ and the following notations are used: γ_j are group elements defined by $\gamma_j = g_0 g_1 \dots g_{j-1}$. $\Theta_{i_1,i_2,\dots,i_l}(\gamma_1, \dots, \gamma_k)$ is the form (depending on $g_0, g_1 \dots$) defined by the formula

$$\Theta_{i_1,i_2,\dots,i_l}(\gamma_1, \dots, \gamma_k) = \int_{\Delta^k} tr e^{-t_0 \theta^{\gamma_1}} e^{-t_1 \theta^{\gamma_2}} \dots e^{-t_{i_1-1} \theta^{\gamma_{i_1}}} \delta(g_{i_1})^{\gamma_{i_1}} e^{-t_{i_1} \theta^{\gamma_{i_1+1}}} \dots e^{-t_{i_2-1} \theta^{\gamma_{i_2}}} \delta(g_{i_2})^{\gamma_{i_2}} \dots e^{-t_k \theta} dt_1 \dots dt_k \quad (4.8)$$

The change of connection does not change the class in the cyclic cohomology, as can be seen by constructing a cobordism between corresponding cycles. This formula gives a class in the cyclic cohomology, which represents the equivariant Chern character, cf. Bott [Bot78]. More precisely, let $\Phi : H^*(V \times_\Gamma \mathrm{E}\Gamma) \to \mathrm{HP}^*(\mathrm{C}_0^\infty(\mathrm{V}) \rtimes \Gamma)$ be the canonical imbedding, constructed by Connes, cf. [Con94].Then the following theorem holds:

Theorem 22. *Let* $\mathrm{Ch}_\Gamma(E) \in H^*(V \times_\Gamma \mathrm{E}\Gamma)$ *be the equivariant Chern character. Then*

$$\Phi\left(\mathrm{Ch}_\Gamma(E)\right) = [\chi]$$

where χ *is the cocycle given by* (4.7).

Here E pulls back to an equivariant bundle on $V \times \mathrm{E}\Gamma$, and then drops down to $V \times_\Gamma \mathrm{E}\Gamma$, and the equivariant Chern character $\mathrm{Ch}_\Gamma(E)$ is the Chern character of the resulting bundle. We recall that we use normalization from [BGV92]. The proof of this Theorem will be given in the Section 4.3

4.2 Some properties of Connes' map Φ.

Let the group Γ act on the manifold V, as in the previous section. In the cyclic complex of the algebra $C_0^\infty \rtimes \Gamma$ one can consider the subcomplex of the cochains satisfying $\phi(f_0 U_{g_0}, f_1 U_{g_1}, \ldots, f_k U_{g_k}) = 0$ unless $g_0 g_1 \ldots g_k = 1$ (it is easy to see that the differentials b and B preserve this condition). We say that cocycles satisfying this condition are concentrated at the identity. The main goal of this section is to show that the image of the Connes map Φ consists of the part of cyclic cohomology concentrated at the identity.

We will use the following well-known statement:

Lemma 23. *Let* Γ *be a group, and let* (X_i^*, d_i) $i = 1, 2$, *be two complexes of projective* Γ *modules, with differentials commuting with the action of* Γ. *Denote by* $\left(X_i^{\mathrm{inv}}\right)^*$ *the corresponding complexes of* Γ*-invariants. Let* $\Psi : X_1^* \to X_2^*$ *be a* Γ*-equivariant quasiisomorphism of complexes. Then the induced map* $\Psi^{\mathrm{inv}} : \left(X_1^{\mathrm{inv}}\right)^* \to \left(X_2^{\mathrm{inv}}\right)^*$ *is a quasiisomorphism.*

Proof. Let $(\beta_j(\Gamma), \delta)$ be the standard resolution. Consider the bicomplex $C_i^{p,q} = \mathrm{Hom}_\Gamma(\beta_p, X_i^q)$, with the horizontal differential induced by δ and vertical by d_i. Since the X_i^q is a projective Γ-module, all the rows of this bicomplex are exact, and its cohomology coincides with the cohomology of the complex $\mathrm{Ker}\,\delta_0 = \mathrm{Hom}_\Gamma(\mathbb{C}, X_i^*) = \left(X_i^{\mathrm{inv}}\right)^*$.

Ψ induces map $C_1^{*,*} \to C_2^{*,*}$, and the corresponding map in the cohomology is Ψ^{inv}. By our hypothesis, and since β_p is free, Ψ induces a quasiisomorphism in every column, and hence quasiisomorphism of bicomplexes, and the statement of the Lemma follows.

□

To describe the part of the cyclic cohomology of cross-products concentrated at the identity, V. Nistor considers the following cyclic object N in the category of vector spaces (which we consider only in the case of algebras of smooth functions). Its m-th component is $C_0^\infty(V)^\natural \otimes \beta_m(\Gamma)$, with the face, degeneracy and cyclic operators coming from factors. We denote generators of this tensor product by symbols like $(f_0, f_1, \ldots, f_m; g_0, g_1, \ldots, g_m)$, where f_i are functions and g_i are elements of the group. This cyclic object is a free Γ module with the action given by

$$(f_0, f_1, \ldots f_k; g_0, g_1, \ldots g_k)^g = (f_0^{g^{-1}}, f_1^{g^{-1}}, \ldots f_k^{g^{-1}}; g_0 g, g_1 g, \ldots g_k g) \qquad (4.9)$$

It is easy to see that the group action commutes with all the face, degeneracy and cyclic operators. We now define a Γ-invariant map of the cyclic objects $p : N \to$

$(C_0^\infty(V) \rtimes \Gamma)_1^\natural$. The definition is the following:

$$p\left((f_0, f_1, \ldots, f_m; g_0, g_1, \ldots, g_m)\right) = f_0^{g_0} U_{g_0 g_1^{-1}} \otimes f_1^{g_1} U_{g_1 g_2^{-1}} \ldots f_j^{g_j} U_{g_j g_{j+1}^{-1}} \ldots f_m^{g_m} U_{g_m g_0^{-1}} \tag{4.10}$$

It is easy to see that the map p is Γ-invariant, and that the map p^* identifies $\mathcal{B}(N)^{\mathrm{inv}}$ with $\mathcal{B}(C_0^\infty(V) \rtimes \Gamma)$.

Consider also the bicomplex $A \subset \mathrm{Hom}(\beta_*, D^*)$ – the space of *totally antisymmetric* functions on Γ^{l+1} with values in D complex of currents on V. The differentials are given by the de Rham differential and by the group cohomology differential, i.e.

$$\delta\gamma(g_0, g_1, \ldots, g_{l+1}) = \sum_{i=0}^{k+1} (-1)^i \gamma(g_0, g_1, \ldots, \hat{g}_i, \ldots, g_{l+1}) \tag{4.11}$$

The group Γ acts on this complex by

$$g(\gamma)((g_0, g_1, \ldots, g_k) = (\gamma(g_0 g, g_1 g, \ldots, g_k g))^{g^{-1}} \tag{4.12}$$

We will construct now a Γ-equivariant map of complexes $\Psi : A^{*,*} \to \mathcal{B}(N)$. The construction is the following. Let $\Lambda^*\Gamma$ be the graded algebra generated by the anticommuting variables of degree 1 labeled by the elements of Γ. Each $\gamma \in A^{*,*}$ defines a functional on the algebra $\Omega(V)\widehat{\otimes}\Lambda^*\Gamma$ by

$$\langle \gamma, \omega \widehat{\otimes} g_0 \wedge g_1 \wedge \ldots g_l \rangle = < \gamma(g_0, g_1, \ldots, g_l), \omega > \tag{4.13}$$

With the right action of Γ on $\Omega(V)\widehat{\otimes}\Lambda^*\Gamma$ defined by

$$(\omega \widehat{\otimes} g_0 \wedge g_1 \wedge \ldots g_l)^g = \omega^{g^{-1}} \widehat{\otimes} g_0 g \wedge g_1 g \wedge \ldots g_l g \tag{4.14}$$

this pairing satisfies

$$\langle\gamma,(\omega\widehat{\otimes}g_0\wedge g_1\wedge\dots g_l)^g\rangle=\langle g(\gamma),\omega\widehat{\otimes}g_0\wedge g_1\wedge\dots g_l\rangle \tag{4.15}$$

Put now

$$\Psi(\gamma)((f_0,f_1,\dots f_k;g_0,g_1,\dots g_k))=$$
$$(-1)^m\lambda_{l,k}\langle\gamma,f_0(\sum_{j=0}^{k}g_j)(df_1+f_1(g_1-g_2))(df_2+f_2(g_2-g_3))\dots(df_k+f_k(g_k-g_0))\rangle \tag{4.16}$$

where γ is of the type l, m, and $k=l+m+1$, and

$$\lambda_{l,k}=\frac{l!}{(k+1)!} \tag{4.17}$$

Proposition 24. *The map Ψ is Γ-equivariant.*

Proof. Indeed, we have

$$\Psi(\gamma)((f_0,f_1,\dots f_k;g_0,g_1,\dots g_k)^g)=\Psi(\gamma)((f_0^{g^{-1}},f_1^{g^{-1}},\dots f_k^{g^{-1}};g_0g,g_1g,\dots g_kg))=$$
$$(-1)^m\lambda_{l,k}\langle\gamma,f_0^{g^{-1}}(\sum_{j=0}^{l}g_jg)(df_1^{g^{-1}}+f_1^{g^{-1}}(g_1g-g_2g))(df_2^{g^{-1}}+f_2^{g^{-1}}(g_2g-g_3g))\dots$$
$$(df_k^{g^{-1}}+f_k^{g^{-1}}(g_kg-g_0g))\rangle=$$
$$\Psi(g(\gamma))((f_0,f_1,\dots f_k;g_0,g_1,\dots g_k)) \tag{4.18}$$

□

Proposition 25. *Restricted to the invariants, the map Ψ coincides with Connes' map Φ, after the identification of $\mathcal{B}(N)^{\mathrm{inv}}$ with $\mathcal{B}(C_0^\infty(V)\rtimes\Gamma)$.*

Proof. We have $p(f_0,\dots f_k;g_0,\dots g_k)=x_0\otimes x_1\cdots\otimes x_k$, where $x_i=f_i^{g_i}U_{g_ig_{i+1}^{-1}}$. we will now show that

$$
\begin{aligned}
&(-1)^{(j-1)(k-j+1)}(\widetilde{\gamma})(dx_j\dots dx_kx_0dx_1\dots dx_{j-1})=\\
&\quad(-1)^m\langle g_j^{-1}(\gamma),f_0g_j(df_1+f_1(g_1-g_2))(df_2+f_2(g_2-g_3))\dots(df_k+f_k(g_k-g_0))\rangle
\end{aligned}
\tag{4.19}
$$

Indeed,

$$
\begin{aligned}
dx_j\dots dx_kx_0dx_1\dots dx_{j-1}=&\left(df_j^{g_j}U_{g_jg_{j+1}^{-1}}+f_j^{g_j}U_{g_jg_{j+1}^{-1}}\delta_{g_jg_{j+1}^{-1}}\right)\dots\\
&\left(df_k^{g_k}U_{g_kg_{k+1}^{-1}}+f_k^{g_k}U_{g_kg_0^{-1}}\delta_{g_kg_0^{-1}}\right)f_0U_{g_0g_1^{-1}}\\
&\left(df_1^{g_1}U_{g_1g_2^{-1}}+f_1^{g_1}U_{g_1g_2^{-1}}\delta_{g_1g_2^{-1}}\right)\dots\left(df_{j-1}^{g_{j-1}}U_{g_{j-1}g_j^{-1}}+f_{j-1}^{g_{j-1}}U_{g_{j-1}g_j^{-1}}\delta_{g_{j-1}g_j^{-1}}\right)=\\
&\left(df_j^{g_j}+f_j^{g_j}(\delta_{g_jg_j^{-1}}-\delta_{g_{j+1}g_j^{-1}})\right)\dots\left(df_k^{g_j}+f_k^{g_j}(\delta_{g_kg_j^{-1}}-\delta_{g_0g_j^{-1}})\right)f_0^{g_j}\\
&\left(df_1^{g_j}+f_1^{g_j}(\delta_{g_1g_j^{-1}}-\delta_{g_2g_j^{-1}})\right)\dots\left(df_{j-1}^{g_j}+f_{j-1}^{g_j}(\delta_{g_{j-1}g_j^{-1}}-\delta_{g_jg_j^{-1}})\right)=\\
&(-1)^{(j-1)(k-j+1)}f_0^{g_j}\left(df_1^{g_j}+f_1^{g_j}(\delta_{g_1g_j^{-1}}-\delta_{g_2g_j^{-1}})\right)\dots\left(df_k^{g_j}+f_k^{g_j}(\delta_{g_kg_j^{-1}}-\delta_{g_0g_j^{-1}})\right)
\end{aligned}
\tag{4.20}
$$

But since

$$
\begin{aligned}
&<\gamma(1,g_{i_1}g_j^{-1},g_{i_2}g_j^{-1},\dots,g_{i_l}g_j^{-1}),\omega^{g_j}>=\\
&\quad<\gamma(1,g_{i_1}g_j^{-1},g_{i_2}g_j^{-1},\dots,g_{i_l}g_j^{-1})^{g_j},\omega>=<g_j^{-1}(\gamma)(g_j,g_{i_1},\dots g_{i_l}),\omega>
\end{aligned}
\tag{4.21}
$$

we have

$$
\begin{aligned}
&(-1)^{(j-1)(k-j+1)}\widetilde{\gamma}(dx_j\dots dx_kx_0dx_1\dots dx_{j-1})=\\
&\quad(-1)^m\langle g_j^{-1}(\gamma),f_0g_j(df_1+f_1(g_1-g_2))(df_2+f_2(g_2-g_3))\dots(df_k+f_k(g_k-g_0))\rangle
\end{aligned}
\tag{4.22}
$$

Let now γ be Γ-invariant. Then $g_j^{-1}(\gamma) = \gamma$ and the statement then follows by summation of (4.19) over j. $\square$

Proposition 26. Ψ *is a map of complexes.*

Proof. We need to check two identities:

$$B\Psi(\gamma) = \Psi(d\gamma) \tag{4.23}$$

$$b\Psi(\gamma) = \Psi(\partial\gamma) \tag{4.24}$$

The first one is proved as follows. We have

$$B\Psi(\gamma)(f_1, \dots, f_k; g_1, \dots, g_k) =$$
$$(-1)^m(k+1)\lambda_{l,k}\langle\gamma, (\sum_{j=1}^{k} g_j)(df_1 + f_1(g_1 - g_2))(df_2 + f_2(g_2 - g_3)) \dots (df_k + f_k(g_k - g_1))\rangle \tag{4.25}$$

But from the proof of the identity (4.19) we have

$$(-1)^{(j-1)(k-j+1)}(\widetilde{g_j(\gamma)}(dx_j \dots dx_k dx_1 \dots dx_{j-1}) =$$
$$(-1)^m\langle\gamma, g_j(df_1 + f_1(g_1 - g_2))(df_2 + f_2(g_2 - g_3)) \dots (df_k + f_k(g_k - g_0))\rangle \tag{4.26}$$

The identity

$$(\widetilde{\gamma})(dx_j \dots dx_k dx_1 \dots dx_{j-1}) = (-1)^{k-j+1}(d\widetilde{\gamma})(dx_j \dots dx_k x_1 \dots dx_{j-1}) \tag{4.27}$$

then immediately implies that

$$\langle\gamma, g_j(df_1 + f_1(g_1 - g_2))(df_2 + f_2(g_2 - g_3)) \dots (df_k + f_k(g_k - g_0))\rangle =$$
$$- \langle d\gamma, g_j f_1(df_2 + f_2(g_2 - g_3)) \dots (df_k + f_k(g_k - g_0))\rangle \tag{4.28}$$

Summing this equality over j and using $(k+1)\lambda_{l,k} = \lambda_{l,(k-1)}$ we obtain the desired equality.

The second identity will follow from the following lemmata:

Lemma 27. *Extend γ to the group algebra by linearity. Then*

$$\partial\gamma(h, g_1^0 - g_1^1, g_2^0 - g_2^1, \dots g_k^0 - g_k^1) = (-1)^{m-1}\gamma(g_1^0 - g_1^1, g_2^0 - g_2^1, \dots g_k^0 - g_k^1) \quad (4.29)$$

Proof.

$$\partial\gamma(h, g_1^0 - g_1^1, g_2^0 - g_2^1, \dots g_l^0 - g_l^1) = \sum(-1)^{\sigma_1+\sigma_2+\dots\sigma_l}\partial\gamma(h, g_1^{\sigma_1}, g_2^{\sigma_2}, \dots, g_l^{\sigma_l}) =$$

$$(-1)^{m-1}\gamma(g_1^0 - g_1^1, g_2^0 - g_2^1, \dots g_l^0 - g_l^1) +$$

$$(-1)^{m-1}\sum_{i=1}^{l}\sum(-1)^{\sigma_1+\sigma_2+\dots\sigma_l}\gamma(h, g_1^{\sigma_1}, g_2^{\sigma_2}, \dots, \widehat{g_i^{\sigma_i}} \dots g_l^{\sigma_l}) \quad (4.30)$$

where each σ_i is either 0 or 1. But the last sum is 0. Indeed, each term of the form $\gamma(h, g_1^{\sigma_1}, g_2^{\sigma_2}, \dots, \widehat{g_i^{\sigma_i}} \dots g_l^{\sigma_l})$ enters the sum twice, for $\sigma_i = 0$ and 1, with opposite signs. □

Lemma 28.

$$b\Psi(\gamma)((f_0, \dots, f_{k+1}; g_0, \dots, g_{k+1})) =$$

$$(-1)^{m-1}\lambda_{l+1,k}\langle\gamma, f_0(df_1 + f_1(g_1 - g_2))(df_2 + f_2(g_2 - g_3)) \dots (df_{k+1} + f_{k+1}(g_{k+1} - g_0))\rangle$$

$$(4.31)$$

Proof. This is proved by the following computation:

$$b\Psi(\gamma)((f_0,\dots,f_{k+1};g_0,\dots,g_{k+1})) =$$
$$\sum_{i=0}^{k}(-1)^{m+i}\lambda_{l,k}\langle\gamma, f_0(\sum_{j\neq i+1} g_j)(df_1+f_1(g_1-g_2))(df_2+f_2(g_2-g_3))\dots$$
$$(d(f_if_{i+1})+f_if_{i+1}(g_i-g_{i+2}))\dots(df_{k+1}+f_{k+1}(g_{k+1}-g_0))\rangle +$$
$$(-1)^{m+k+1}\lambda_{l,k}\langle\gamma, f_{k+1}f_0(\sum_{j\neq 0} g_j)(df_1+f_1(g_1-g_2))(df_2+f_2(g_2-g_3))\dots\rangle =$$
$$\sum_{i=1}^{k}(-1)^{m-1}\lambda_{l,k}\langle\gamma, f_0(df_1+f_1(g_1-g_2))(df_2+f_2(g_2-g_3))\dots(f_i(g_i-g_{i+1}))\dots$$
$$= (-1)^{m-1}(l+1)\lambda_{l,k}\langle\gamma, f_0(df_1+f_1(g_1-g_2))(df_2+f_2(g_2-g_3))\dots(df_i+f_i(g_i-g_{i+1}))\dots$$
$$(df_{k+1}+f_{k+1}(g_{k+1}-g_0))\rangle \quad (4.32)$$

□

We now have by the Lemma 27

$$\Psi(\partial\gamma)((f_0,\dots,f_{k+1};g_0,\dots,g_{k+1})) = (-1)^{m-1}(k+2)\lambda_{k+1,l+1}$$
$$\langle\gamma, f_0(df_1+f_1(g_1-g_2))(df_2+f_2(g_2-g_3))\dots(df_{k+1}+f_{k+1}(g_{k+1}-g_0))\rangle \quad (4.33)$$

and comparison with the result of the Lemma 28 finishes the proof, since $(k+2)\lambda_{k+1,l+1} = \lambda_{k,l+1}$. □

Proposition 29. Ψ *is a quasiisomorphism of complexes.*

Proof. In [Nis90] it is proved that the following map $\psi : D^* \to \mathcal{B}^*(N)$ is a quasiisomorphism:

$$\psi(C)((f_0,\dots,f_k;g_0,\dots,g_k)) = \frac{1}{k!} < C, f_0df_1\dots df_k > \quad (4.34)$$

Here C is a current, and the map is u-linear.

There is a natural inclusion $i : D^* \hookrightarrow A^{*,*}$, each current considered as a constant function on the group Γ. It is easy to see that

$$\Psi \circ i = \psi \tag{4.35}$$

Hence to prove the Proposition it is enough to show that i is a quasiisomorphism.

To do this we introduce the projection $p : A^{*,*} \to D^*$ defined by

$$p(\gamma) = \begin{cases} \gamma(1) & \text{if } l = 0 \\ 0 & \text{if } l > 0 \end{cases} \tag{4.36}$$

It is clear that

$$p \circ i = id \tag{4.37}$$

Define now an operator $h : A^{l,m} \to A^{l-1,m}$ by

$$(h\gamma)(g_1, \ldots, g_l) = \begin{cases} 0 & \text{if } l = 0 \\ (-1)^{m-1}\gamma(1, g_1, \ldots, g_l) & \text{if } l > 0 \end{cases} \tag{4.38}$$

Then it is easy to see that

$$i \circ p = 1 - \partial \circ h - h \circ \partial \tag{4.39}$$

and hence i and p define inverse quasiisomorphism $\square$

The following Theorem is then an immediate corollary of the Lemma 23:

Theorem 30. *The image of the map Φ is the part of cyclic cohomology concentrated at the identity of the group.*

Remark 31. We also obtain another proof of the injectivity of the map Φ, different from the Connes' proof from [Con94].

Remark 32. The Theorem and the proof remain true for the case of the convolution algebra of an etale groupoid.

4.3 Relation with Bott's characteristic classes.

This section is devoted to the proof of the Theorem 22.

To prove the theorem we need some preliminary constructions and facts. For a Γ-manifold Y by Y_Γ we denote the homotopy quotient $Y \times_\Gamma \mathrm{E}\Gamma$.

Suppose we are given Γ-manifolds V and X, X oriented. We construct then a map $I : HC^j(C_0^\infty(V) \rtimes \Gamma) \to HC^{j+\dim X}(C_0^\infty(V \times X) \rtimes \Gamma)$. The construction is the following: in $HC^{\dim X}(C_0^\infty(X) \rtimes \Gamma)$ there is a class represented by the cocycle

$$\tau(f_0U_{g_0}, f_1U_{g_1}, \ldots, f_kU_{g_k}) = \begin{cases} \int_X f_0 df_1^{g_0} \ldots df_k^{g_0g_1\ldots g_{k-1}} & \text{if } g_0g_1\ldots g_k = 1 \\ 0 & \text{otherwise} \end{cases}$$

One then constructs the map I from the following diagram:

$$\begin{aligned} HC^j(C_0^\infty(V) \rtimes \Gamma) &\xrightarrow{\cup\tau} HC^{j+\dim X}\left(\left(C_0^\infty(V) \rtimes \Gamma\right) \otimes \left(C_0^\infty(X) \rtimes \Gamma\right)\right) \\ &\xrightarrow{\Delta^*} HC^{*+\dim X}(C_0^\infty(V \times X) \rtimes \Gamma) \end{aligned} \tag{4.40}$$

Here the last arrow is induced by the natural map

$$\begin{aligned} \Delta : C_0^\infty(V \times X) \rtimes \Gamma &= \left(C_0^\infty(V) \otimes C_0^\infty(X)\right) \rtimes \Gamma \\ &\to \left(C_0^\infty(V) \rtimes \Gamma\right) \otimes \left(C_0^\infty(X) \rtimes \Gamma\right) \end{aligned} \tag{4.41}$$

defined by $\Delta\left((f \otimes f')U_g\right) = (fU_g) \otimes (f'U_g)$.

Suppose now that V is also oriented.

Proposition 33. *The following diagram is commutative:*

$$\begin{tikzcd} HP^*(C_0^\infty(V \times X) \rtimes \Gamma) & H^*\left((V \times X)_\Gamma\right) \arrow[l, "\Phi"'] \\ HP^*(C_0^\infty(V) \rtimes \Gamma) \arrow[u, "I"] & H^*\left(V_\Gamma\right) \arrow[l, "\Phi"'] \arrow[u, "\pi^*"'] \end{tikzcd} \tag{4.42}$$

Here $\pi : (V \times X)_\Gamma \to V_\Gamma$ *is induced by the (* Γ*-equivariant) projection* $V \times X \times \mathrm{E}\Gamma \to \mathrm{V} \times \mathrm{E}\Gamma$.

Proof. We can consider $V \times X$ with action of $\Gamma \times \Gamma$. We start with showing that the following diagram is commutative:

$$\begin{tikzcd} HP^*(C_0^\infty(V \times X) \rtimes (\Gamma \times \Gamma)) & H^*\left((V \times X)_{(\Gamma \times \Gamma)}\right) \arrow[l, "\Phi"'] \\ HP^*(C_0^\infty(V) \rtimes \Gamma) \arrow[u, "\cup\tau"] & H^*\left(V_\Gamma\right) \arrow[l, "\Phi"'] \arrow[u, "\pi^*"'] \end{tikzcd} \tag{4.43}$$

Here we identify $C_0^\infty(V \times X) \rtimes (\Gamma \times \Gamma)$ with $(C_0^\infty(V) \rtimes \Gamma) \otimes (C_0^\infty(X) \rtimes \Gamma)$ and $(V \times X)_{(\Gamma \times \Gamma)}$ with $X_\Gamma \times V_\Gamma$. This is verified by the direct computation, using the Eilenberg-Silber theorem and the shuffle map in cyclic cohomology, cf. [Lod92].

Now we note that the commutativity of the following diagram is clear:

$$\begin{tikzcd} HP^*(C_0^\infty(V \times X) \rtimes (\Gamma \times \Gamma)) \arrow[d] & H^*\left((V \times X)_{(\Gamma \times \Gamma)}\right) \arrow[l, "\Phi"'] \arrow[d] \\ HP^*(C_0^\infty(V \times X) \rtimes \Gamma) & H^*\left((V \times X)_\Gamma\right) \arrow[l, "\Phi"'] \end{tikzcd} \tag{4.44}$$

where the vertical arrows are induced by the diagonal maps $\Gamma \to \Gamma \times \Gamma$ and $\mathrm{E}\Gamma \to \mathrm{E}\Gamma \times \mathrm{E}\Gamma$. This ends the proof. $\square$

Proposition 34. *Let E be an equivariant vector bundle on V with connection ∇. Let $\chi \in HC^n(C_0^\infty(V) \rtimes \Gamma)$ be the character of the associated cycle, and let $\chi' \in HC^{n+k}(C_0^\infty(V \times X) \rtimes \Gamma)$ be the character of the cycle constructed with the bundle $pr_V^* E$ and connection $pr_V^* \nabla$, where $pr_V : X \times V \to V$. Then $I(\chi) = \chi'$ (Here n and k are dimensions of V and X respectively)*

Proof. Let $\mathcal{C}$ denote the corresponding cycle over $C_0^\infty(V) \rtimes \Gamma$, and T -transverse fundamental cycle of X. Then $\mathcal{C} \times T$ is a cycle over $(C_0^\infty(V) \rtimes \Gamma) \otimes (C_0^\infty(X) \rtimes \Gamma)$, and

$$\mathrm{Ch}(\mathcal{C} \times T) = \mathrm{Ch}(\mathcal{C}) \cup \tau$$

by the Corollary 21. If by $pr^*\mathcal{C}$ we denote the corresponding cycle over $C_0^\infty(V \times X) \rtimes \Gamma$, we have

$$\mathrm{Ch}(pr^*\mathcal{C}) = \Delta^* \left(\mathrm{Ch}(\mathcal{C} \otimes T)\right) = \Delta^* \left(\mathrm{Ch}(\mathcal{C}) \cup \tau\right) = I(\mathrm{Ch}(\mathcal{C}))$$

□

Lemma 35. *Suppose in addition to the conditions of the Theorem 22 that Γ acts freely and properly on V. Then the statement of the Theorem holds.*

Proof. Since the group acts freely and properly, one can find a connection on E which is Γ-invariant. For the class of the cocycle χ written with the invariant connection the result follows easily from the definition of the map Φ. □

Lemma 36. *Let X be a topological space, and $\beta \in H^*(X; \mathbb{Q})$ be a cohomology class of X. Suppose that for any smooth oriented manifold M and continuous map $f : M \to X$ we have $f^*\beta = 0$. Then $\beta = 0$.*

Proof of the Theorem 22. Comparison of the construction from [Nis90] with the definition of the map Φ implies that (class of) χ is in the image of Φ, $[\chi] = \Phi(\xi)$ for some (necessarily unique) $\xi \in H^*(V \times_\Gamma \mathrm{E}\Gamma)$. We need to verify that $\xi = \mathrm{Ch}_\Gamma(E)$. We do this by showing that for any oriented manifold W and any map continuous $f : W \to V_\Gamma$ $f^*\xi = f^* \mathrm{Ch}_\Gamma(E)$.

Let $\widetilde{W}$ be the principal Γ-bundle obtained by pullback of the bundle $V \times \mathrm{E}\Gamma \to \mathrm{V}_\Gamma$, so that the following diagram is commutative, and $\widetilde{f}$ is Γ-equivariant:

$$\begin{array}{ccc} \widetilde{W} & \xrightarrow{\tilde{f}} & V \times \mathrm{E}\Gamma \\ \downarrow & & \downarrow \\ W & \xrightarrow{f} & V_\Gamma \end{array} \tag{4.45}$$

We can write $\widetilde{f}$ as a composition of two Γ-equivariant maps $\widetilde{f_1} : \widetilde{W} \to \widetilde{W} \times V \times \mathrm{E}\Gamma$, which embeds $\widetilde{W}$ as the graph of $\widetilde{f}$ and $pr : \widetilde{W} \times V \times \mathrm{E}\Gamma \to \mathrm{V} \times \mathrm{E}\Gamma$, projection. Let $\pi : (\widetilde{W} \times V)_\Gamma \to V_\Gamma$ and $f_1 : W \to (\widetilde{W} \times V)_\Gamma$ be the induced maps. We have $f = \pi f_1$.

Construct now the class $\chi' \in HP^n(C_0^\infty(\widetilde{W} \times V) \rtimes \Gamma)$ using the bundle pr^*E with connection $pr^*\nabla$. By the Proposition 34 $\chi' = I(\chi)$, where $I : HP^*(C_0^\infty(V) \rtimes \Gamma) \to HP^{*+\dim W}(C_0^\infty(\widetilde{W} \times V) \rtimes \Gamma)$. By the Proposition 33 $\chi' = I(\chi) = I(\Phi(\xi)) = \Phi(\pi^*\xi)$. By the Lemma 35, since $\widetilde{W} \times V$ is acted by Γ freely and properly, $\chi' = \Phi(\mathrm{Ch}(pr^*E))$. But since $\mathrm{Ch}(pr^*(E)) = \pi^* \mathrm{Ch}(E)$, and using injectivity of Φ we conclude that

$$\pi^* \mathrm{Ch}(E) = \pi^*\xi$$

Hence

$$f^* \mathrm{Ch}(E) = f_1^*\pi^* \mathrm{Ch}(E) = f_1^*\pi^*\xi = f^*\xi$$

□

4.4 Relation with the Godbillon-Vey cyclic cocycle.

In the paper [Con86] Connes considers (in particular) the case of the circle S^1 acted by the group of its diffeomorphisms $\mathrm{Diff}(S^1)$. Here we present the Connes construction in the multidimensional case and indicate some relations with our construction of cyclic cocycles representing equivariant classes.

In the situation of the Section 4.1 take the bundle E to be $\Lambda^n T^* X$. This is a 1 -dimensional trivial bundle, naturally equipped with the action of the group $\Gamma = \mathrm{Diff}(X)$. Let ϕ be a nowhere 0 section of this bundle, i.e. a volume form. Define a flat connection ∇ on E by

$$\nabla(f\phi) = df\phi,\ \phi \in C^\infty(X) \tag{4.46}$$

We can thus define the cycle $\mathcal{C}$ over the algebra $C^\infty(X)$.

Let now $\delta(g)$ be defined as above, and put

$$\mu(g) = \frac{\phi^g}{\phi} \in C^\infty(X) \tag{4.47}$$

Then μ is a cocycle, i.e.

$$\mu(gh) = \mu(h)^g \mu(g) \tag{4.48}$$

We also have

$$\delta(g) = d \log \mu(g) \tag{4.49}$$

Indeed,

$$\delta(g)\phi^g = \nabla\phi^g - (\nabla(\phi))^g = \nabla(\mu(g)\phi) = d\mu(g)\phi$$

and

$$\delta(g) = \frac{d\mu(g)\phi}{\phi^g} = \frac{d\mu(g)}{\mu(g)} = dlog(\mu(g))$$

For every t we define a homomorphism $\rho_t : C^\infty(X) \rtimes \Gamma \to \mathrm{End}(E) \rtimes \Gamma$ by

$$\rho_t(aU_g) = a(\mu(g))^t U_g \tag{4.50}$$

This is a homomorphism due to the cocycle property of μ, which according to [Con86] is the Tomita-Takesaki flow associated with the state given by the volume form ϕ.

Consider now the transverse fundamental cycle Φ over the algebra $\mathcal{A} = C^\infty(X)$ defined by the following data:

the differential graded algebra $\Omega^*(X) \rtimes \Gamma$ with the differential $d(\omega U_g) = (d\omega)U_g$

the graded trace $⨍$ on $\Omega^*(X) \rtimes \Gamma$ defined by

$$⨍\omega U_g = \begin{cases} \int\limits_X \omega & \text{if } g = 1 \\ 0 & \text{otherwise} \end{cases}$$

the homomorphism $\rho = \rho_0 = id$ from $\mathcal{A} = C^\infty(X) \rtimes \Gamma$ to $C^\infty(X) \rtimes \Gamma$. The flow (4.50) acts on the cycle Φ, by replacing ρ_0 by ρ_t. We call the cycle thus obtained Φ_t. Using the identities

$$d(\rho_t(aU_g)) = \big(da + ta\ dlog\mu(g)\big)\mu(g)^t U_g = \big(da + ta\ \delta(g)\big)\mu(g)^t U_g \tag{4.51}$$

and

$$\mu(g_0)\mu(g_1)^{g_0}\mu(g_2)^{g_0 g_1} \dots \mu(g_k)^{g_0 g_1 \cdots g_{k-1}} = \mu(g_0 g_1 \dots g_k) \tag{4.52}$$

we can explicitly compute $\mathrm{Ch}(\Phi_t)$. This is the cyclic n-cocycle with the only component of degree n The result is:

$$\mathrm{Ch}(\Phi_t) = \sum_{j=0}^{n} t^j p_j \tag{4.53}$$

where p_j is the cyclic cocycle given by

$$p_j(a_0 U_{g_0}, a_1 U_{g_1}, \dots, a_n U_{g_n}) = \frac{1}{n!} \sum_{1 \le i_1 < i_2 < \dots < i_j \le n} \int_X a_0 da_1^{\gamma_1} da_2^{\gamma_2} \dots da_{i_1-1}^{\gamma_{i_1-1}} a_{i_1}^{\gamma_{i_1}} da_{i_1+1}^{\gamma_{i_1+1}} \dots \Theta_{i_1, i_2, \dots, i_j}(\gamma_1, \dots, \gamma_k) \tag{4.54}$$

for $g_0 g_1 \dots g_k = 1$ and 0 otherwise, where we define as before $\gamma_j = g_0 g_1 \dots g_{j-1}$, and the j-form $\Theta_{i_1, i_2, \dots, i_j}(\gamma_1, \dots, \gamma_k)$ is given by

$$\Theta_{i_1, i_2, \dots, i_j}(\gamma_1, \dots, \gamma_k) = \delta(g_{i_1})^{\gamma_{i_1}} \delta(g_{i_2})^{\gamma_{i_2}} \dots \delta(g_{i_j})^{\gamma_{i_j}} \tag{4.55}$$

In particular, p_0 is the transverse fundamental class. Comparing these formulae with the formulae from the Section 4.1 we obtain

Proposition 37. *Let Φ_1 be the image of the transverse fundamental cycle Φ under the action of the Tomita-Takesaki flow for the time 1 . Let $\mathcal{C}$ be the cycle over $C^\infty(X) \rtimes \Gamma$ associated to the equivariant bundle $\Lambda^n T^* X$ with the connection from* (4.46)*. Then, on the level of cocycles* $\mathrm{Ch}(\Phi_1) = \mathrm{Ch}(\mathcal{C})$.

We now sketch a construction of a family of chains Ψ_s providing the cobordism between Φ_0 and Φ_s, $s \in \mathbb{R}$. The algebra $\Omega^* = \Omega * ([0, s]) \widehat{\otimes} \Omega^*(X) \rtimes \Gamma$. The homomorphism from $\mathcal{A}$ to Ω^0 maps $aU_g \in \Omega^*(X) \rtimes \Gamma$ to $a\mu(g)^t U_g$, where t is the variable on

$[0,s]$. The connection is given by $1\widehat{\otimes}\nabla + d\widehat{\otimes}1$ where d is the de Rham differential, and the curvature is 0 . The restriction map is given by the restriction to the endpoints of the interval and the graded trace is given by

$$\fint \alpha\widehat{\otimes}(\omega U_g) = (-1)^{\deg\omega} \int_{[0,s]} \alpha \int_X \omega$$

if $\deg\alpha = 1$ and $g = 1$ and 0 otherwise. This chain provides a cobordism between Φ_0 and Φ_s. Its character is given by the formula

$$\mathrm{Ch}(\Psi_s) = \sum_{j=1}^{n+1} s^j q_j \tag{4.56}$$

where q_j is the cyclic cochain given by

$$q_j(a_0U_{g_0}, a_1U_{g_1}, \ldots, a_nU_{g_n}) = \\ \frac{1}{n!} \sum_{1\le i_1<i_2<\cdots<i_j\le n} \int_X a_0 da_1^{\gamma_1} da_2^{\gamma_2} \ldots da_{i_1-1}^{\gamma_{i_1-1}} a_{i_1}^{\gamma_{i_1}} da_{i_1+1}^{\gamma_{i_1+1}} \ldots \\ \Xi_{i_1,i_2,\ldots,i_j}(\gamma_1,\ldots,\gamma_k) \tag{4.57}$$

for $g_0g_1\ldots g_k = 1$ and 0 otherwise, where we define as before $\gamma_j = g_0g_1\ldots g_{j-1}$, and the $j-1$-form $\Xi_{i_1,i_2,\ldots,i_j}(\gamma_1,\ldots,\gamma_k)$ is given by

$$\Xi_{i_1,i_2,\ldots,i_j}(\gamma_1,\ldots,\gamma_k) = \\ \frac{1}{j}\sum_{l=1}^{j}(-1)^l \delta(g_{i_1})^{\gamma_{i_1}}\delta(g_{i_2})^{\gamma_{i_2}} \ldots \log\mu(g_{i_l})^{\gamma_{i_l}} \ldots \delta(g_{i_j})^{\gamma_{i_j}} \tag{4.58}$$

Comparing this formula with (4.53) we obtain:

Proposition 38. *Let p_j, $j = 1,\ldots,$ n be the chains defined in* (4.53), (4.54), *and q_j, $j = 1,\ldots,$ $n+1$ be from* (4.56), (4.57). *Then for $j = 1,\ldots,n$ we have*

$$B\,q_j = p_j \text{ and } b\,q_j = 0 \tag{4.59}$$

Also

$$B\,q_{n+1} = 0 \ \textit{and}\ b\,q_{n+1} = 0 \tag{4.60}$$

In particular all p_j define trivial classes in periodic cyclic cohomology, and q_{n+1} is a cyclic cocycle.

The cocycle q_{n+1} should represent (up to a constant) the Godbillon-Vey class in the cyclic cohomology (i.e. class defined by $h_1 c_1^n$, while p_j and q_j represent forms c_1^j and $h_1 c_1^j$, $j = 1$,...,n, see [Bot78].

4.5 Transverse fundamental class of the foliation.

The construction of the equivariant characteristic classes works equally well in the case of a foliation. The new ingredient required here is the Connes' construction of the transverse fundamental (generalized) cycle. We now will write a simple formula for the character of this cycle.

We start by briefly recalling Connes' construction from [Con94]. Details can be found in [Con94]. Let (V, F) be a transversely oriented foliated manifold, F being an integrable subbundle of TV. The graph of the foliation $\mathcal{G}$ is a groupoid, the objects being the points of V and morphisms being the equivalence classes of paths in the leaves, with equivalence given by holonomy. Equipped with a suitable topology it becomes a smooth (possibly non-Hausdorff) manifold. By r and s we denote the range and source maps $\mathcal{G} \to V$. By $\Omega_F^{1/2}$ we denote the line bundle on V of the half-densities in the direction of F. Let $\mathcal{A} = C_0^\infty\left(\mathcal{G}, s^*(\Omega_F^{1/2}) \otimes r^*(\Omega_F^{1/2})\right)$

be the convolution algebra of $\mathcal{G}$. We define a nonunital generalized cycle over the algebra $\mathcal{A}$ as follows. The k-th component of the graded algebra Ω^* is given by $C_0^\infty\left(\mathcal{G}, s^*(\Omega_F^{1/2}) \otimes r^*(\Omega_F^{1/2}) \otimes r^*(\Lambda^k \tau^*)\right)$. Here $\tau = TV/F$ is the normal bundle, and the product $\Omega^k \otimes \Omega^l \to \Omega^{k+l}$ is induced by the convolution and exterior product.

The definition of the transverse differentiation (connection) requires a choice of a subbundle $H \subset TV$, complementary to F. The choice of the complementary subbundle H allows one to construct an isomorphism

$$j : C^\infty(V, \Lambda^* F^* \otimes \Lambda^* \tau^*) \to C^\infty(V, \Lambda^* TV^*) \tag{4.61}$$

and identify $C^\infty(V, \Lambda^* TV^*)$ with $C^\infty(V, \Lambda^* F^* \otimes \Lambda^* \tau^*)$ via this isomorphism.

We say that form $\omega \in C^\infty(V, \Lambda^* TV^*)$ is of the type (r, s) if it is in $C^\infty(V, \Lambda^r F^* \otimes \Lambda^s \tau^*)$ under this identification. For such a form we have

$$d\omega = d_V \omega + d_H \omega + \sigma\omega \tag{4.62}$$

where $d_V\omega$, $d_H\omega$, $\sigma\omega$ are defined to be components of $d\omega$ of the types $(r+1, s)$, $(r, s+1)$, $(r-1, s+2)$ respectively (our notations are slightly different from those of [Con94]).

We will now derive more explicit formula for d_H. If X is a section of the bundle τ, complementary subbundle allows to lift it to the horizontal vector field, which we denote X^H. Let ω be of the type (q, p), and let $X_1, X_2, \ldots, X_{p+1} \in C^\infty(V, \tau)$, $Y_1, Y_2, \ldots, Y_q \in C^\infty(V, F)$. Let also π_H be the canonical projection $TV \to TV/F$, and π_V be the projection $TV \to F$, defined by the choice of H. Then

$$\begin{aligned}
&d_H\omega(X_1, X_2, \ldots, X_{p+1}, Y_1, Y_2, \ldots, Y_q) = \\
&\qquad d(j\omega)(X_1^H, X_2^H, \ldots, X_{p+1}^H, Y_1, Y_2, \ldots, Y_q) = \\
&\sum(-1)^{i-1} X_i^H (j\omega)(X_1^H, \ldots \widehat{X_i^H}, \ldots, X_{p+1}^H, Y_1, \ldots, Y_q) \\
&+ \sum(-1)^{i+p+1} Y_i (j\omega)(X_1^H, \ldots, X_{p+1}^H, Y_1, \ldots, \widehat{Y_i}, \ldots, Y_q) \\
&+ \sum_{i<k}(-1)^{i+k} (j\omega)([X_i^H, X_k^H], X_1^H, \ldots, \widehat{X_i^H}, \ldots, \widehat{X_k^H}, Y_1, \ldots, Y_q) \\
&+ \sum(-1)^{i+k+p+1} (j\omega)([X_i^H, Y_k], X_1^H, \ldots, \widehat{X_i^H}, \ldots, Y_1, \ldots, \widehat{Y_k}, \ldots, Y_q) \\
&+ \sum_{i<k}(-1)^{i+k} (j\omega)([Y_i, Y_k], X_1^H, \ldots, X_{p+1}^H, Y_1, \ldots, \widehat{Y_i}, \ldots, \widehat{Y_k}, \ldots, Y_q) \qquad (4.63)
\end{aligned}$$

Since ω is a form of the type (q, p), we have

$$(j\omega)([Y_i, Y_k], X_1^H, \ldots, X_{p+1}^H, Y_1, \ldots, \widehat{Y_i}, \ldots, \widehat{Y_k}, \ldots, Y_q) = 0$$

and

$$(j\omega)(X_1^H, \ldots, X_{p+1}^H, Y_1, \ldots, \widehat{Y_i}, \ldots, Y_q) = 0$$

Also

$$(j\omega)(X_1^H, \ldots \widehat{X_i^H}, \ldots, X_{p+1}^H, Y_1, \ldots, Y_q) = \omega(X_1, \ldots \widehat{X_i}, \ldots, X_{p+1}, Y_1, \ldots, Y_q)$$

$$\begin{aligned}
&(j\omega)([X_i^H, X_k^H], X_1^H, \ldots, \widehat{X_i^H}, \ldots, \widehat{X_k^H}, Y_1, \ldots, Y_q) = \\
&\qquad \omega(\pi_H([X_i^H, X_k^H]), X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_k} \ldots, X_{p+1}, Y_1, \ldots, Y_q)
\end{aligned}$$

$$\begin{aligned}
&(j\omega)([X_i^H, Y_k], X_1^H, \ldots, \widehat{X_i^H}, \ldots, Y_1, \ldots, \widehat{Y_k}, \ldots, Y_q) = \\
&\qquad (-1)^p \omega(X_1, \ldots, \widehat{X_i}, \ldots, \pi_V([X_i^H, Y_k]), Y_1, \ldots, \widehat{Y_k}, \ldots, Y_q)
\end{aligned}$$

So, finally we obtain

$$d_H\omega(X_1, X_2, \dots, X_{p+1}, Y_1, Y_2, \dots, Y_q) =$$
$$\sum(-1)^{i-1} X_i^H \omega(X_1, \dots \widehat{X_i}, \dots, X_{p+1}, Y_1, \dots, Y_q) +$$
$$\sum_{i<k}(-1)^{i+k}\omega(\pi_H([X_i^H, X_k^H]), X_1, \dots, \widehat{X_i}, \dots, \widehat{X_k} \dots, X_{p+1}, Y_1, \dots, Y_q) +$$
$$\sum(-1)^{i+k+1}\omega(X_1, \dots, \widehat{X_i}, \dots, \pi_V([X_i^H, Y_k]), Y_1, \dots, \widehat{Y_k}, \dots, Y_q) \quad (4.64)$$

We now extend d_H to half-densities. Writing locally $\rho \in C^\infty(V, \Omega_F^{1/2})$ as $\rho = f|\omega|^{1/2}$, $f \in C^\infty(V)$, $\omega \in C^\infty(V, \Lambda^{\dim F} F^*)$ we define

$$d_H\rho = (d_H f)|\omega|^{1/2} + f|\omega|^{1/2}\frac{d_H\omega}{2\omega} \quad (4.65)$$

Finally, d_H can be extended uniquely as a graded derivation of the graded algebra $C_0^\infty\left(\mathcal{G}, s^*(\Omega_F^{1/2}) \otimes r^*(\Omega_F^{1/2}) \otimes r^*(\Lambda^*\tau^*)\right)$ so that the following identities are satisfied:

$$d_H\left(r^*(\rho_1) f s^*(\rho_2)\right) =$$
$$r^*(d_H\rho_1) f s^*(\rho_2) + r^*(\rho_1) d_H f s^*(\rho_2) + r^*(\rho_1) f s^*(d_H\rho_2)$$
$$\text{for } \rho_1, \rho_2 \in C^\infty(V, \Omega_F^{1/2}), f \in C_0^\infty(\mathcal{G}) \quad (4.66)$$

and

$$d_H(\phi r^*(\omega)) = d_H(\phi) r^*(\omega) + \phi r^*(d_H\omega)$$
$$\text{for } \phi \in C_0^\infty\left(\mathcal{G}, s^*(\Omega_F^{1/2}) \otimes r^*(\Omega_F^{1/2})\right), \omega \in C^\infty(V, \Lambda^*\tau^*) \quad (4.67)$$

Now, for the form ω $d_H^2\omega = -(d_V\sigma + \sigma d_V)\omega$. The operator $\theta = -(d_V\sigma + \sigma d_V)$ contains only longitudinal Lie derivatives, and hence defines a multiplier (of degree 2) of the algebra $C_0^\infty\left(\mathcal{G}, s^*(\Omega_F^{1/2}) \otimes r^*(\Omega_F^{1/2}) \otimes r^*(\Lambda^*\tau^*)\right)$.

Finally, the graded trace on $C_0^\infty\left(\mathcal{G}, s^*(\Omega_F^{1/2}) \otimes r^*(\Omega_F^{1/2}) \otimes r^*(\Lambda^q \tau^*)\right)$, $q = \operatorname{codim} F$ is given by $\displaystyle⨍\omega = \int_V \omega$, where we pull ω back to V via the natural map $V \to \mathcal{G}$.

Lemma 39. *([Con94])* $\left(C_0^\infty\left(\mathcal{G}, s^*(\Omega_F^{1/2}) \otimes r^*(\Omega_F^{1/2}) \otimes r^*(\Lambda^q \tau^*)\right), d_H, \theta, ⨍\right)$ *is a generalized cycle of degree q over the algebra $\mathcal{A}$.*

We can now write an explicit formula for the character of this cycle.

Proposition 40. *The following formula defines a reduced cyclic cocycle χ in the (b, B)-bicomplex of the algebra $\mathcal{A}$ (with adjoined unit).*

$$\chi^k(\phi_0, \phi_1, \dots, \phi_k) = \frac{(-1)^{\frac{q-k}{2}}}{\left(\frac{q+k}{2}\right)!} \sum_{i_0+\dots+i_k=\frac{q-k}{2}} \int_V \phi_0 \theta^{i_0} d_H(\phi_1) \dots d_H(\phi_k) \theta^{i_k} \tag{4.68}$$

Here $k = q$, $q-2$, ..., and ϕ_j, $j \geq 1$ are elements of $\mathcal{A}$, while ϕ_0 is an element of $\mathcal{A}$ with unit adjoined.

Recall, that for q even to define the cocycle over $\mathcal{A}$ with the unit adjoined we extend $⨍$ by requiring that $⨍\theta^{q/2} = 0$.

The results of the Chapter 3 imply that the class of the cocycle χ is the transverse fundamental class of the foliation as defined in [Con94].

The cocycle thus constructed depends on the choice of the complementary subbundle H, but the class in the cyclic cohomology does not. This result of Connes (cf. [Con94]) can be deduced from Theorem 16. Indeed, let H' be another complementary subbundle. For X – section of the bundle τ let X^H denote its lifting to the horizontal vector field corresponding to H, and $X^{H'}$ – corresponding to H'. Let

$$\alpha(X) = X^H - X^{H'} \tag{4.69}$$

be the corresponding vertical vector field. Denote the Lie derivative along $\alpha(\cdot)$ by $\mathcal{L}_\alpha$, and consider it as a function of a section of τ. Also, we define a linear operator $T(X) : F \to F$, depending on a section of τ by

$$T(X)Y = -\alpha(\pi_H([X^H, Y])) \tag{4.70}$$

Remark that $T(X)Y = -\alpha(D_Y X)$, where D is the Bott connection (cf. [Bot78]), but we will not use this fact. Then $\mathcal{L}_\alpha + \frac{1}{2}\operatorname{tr} T$ defines a multiplier of the algebra $C_0^\infty\left(\mathcal{G}, s^*(\Omega_F^{1/2}) \otimes r^*(\Omega_F^{1/2})\right)$, as follows from the proof of the Lemma below.

Lemma 41.

$$d_H - d_{H'} = \operatorname{ad}\left(\mathcal{L}_\alpha + \frac{1}{2}\operatorname{tr} T\right) \tag{4.71}$$

Proof. First, we compute $d_H - d_{H'}$ on forms of the type (q, p), using the formula (4.64). Note that

$$\begin{aligned}
&\pi_H([X_i^H, X_k^H]) - \pi_H([X_i^{H'}, X_k^{H'}]) = \\
&\qquad \pi_H([X_i^H, X_k^H]) - \pi_H([X_i^H - \alpha(X_i), X_k^H - \alpha(X_k)]) = \\
&\qquad\qquad \pi_H([X_i^H, \alpha(X_k)]) + \pi_H([\alpha(X_i), X_k^H])
\end{aligned}$$

and

$$\begin{aligned}
&\pi_V([X_i^H, Y_k]) - \pi_V'([X_i^{H'}, Y_k]) = \pi_V([X_i^H, Y_k]) - \pi_V'([X_i^H, Y_k]) + [\alpha(X_i), Y_k] = \\
&\qquad - \alpha(\pi_H([X_i^H, Y_k])) + [\alpha(X_i), Y_k] = T(X_i)Y_k + [\alpha(X_i), Y_k]
\end{aligned}$$

Then we obtain

$$
\begin{aligned}
&(d_H - d_{H'})\omega(X_1, X_2, \ldots, X_{p+1}, Y_1, Y_2, \ldots, Y_q) = \\
&\qquad \sum (-1)^i \alpha(X_i)\omega(X_1, \ldots \widehat{X_i}, \ldots, X_{p+1}, Y_1, \ldots, Y_q) + \\
&\qquad \sum (-1)^i \omega(X_1, \ldots, \pi_H([\alpha(X_i), X_k^H]), \ldots, \widehat{X_i}, \ldots, X_{p+1}, Y_1, \ldots, Y_q) - \\
&\qquad \sum (-1)^i \omega(X_1, \ldots, \widehat{X_i}, \ldots, Y_1, \ldots, [\alpha(X_i), Y_k], \ldots, Y_q) + \\
&\qquad \sum (-1)^i \omega(X_1, \ldots, \widehat{X_i}, \ldots, Y_1, \ldots, T(X_i)Y_k, \ldots, Y_q) = \\
&\qquad \sum (-1)^i \Big(\mathcal{L}_{\alpha(X_i)}\omega)(X_1, \ldots \widehat{X_i}, \ldots, X_{p+1}, Y_1, \ldots, Y_q) + \\
&\qquad\qquad \sum_{k=1}^{q} \omega(X_1, \ldots, \widehat{X_i}, \ldots, Y_1, \ldots, T(X_i)Y_k, \ldots, Y_q) \Big)
\end{aligned} \tag{4.72}
$$

In particular, for the form of the type $(0, p)$ we have

$$
\begin{aligned}
&(d_H - d_{H'})\omega(X_1, X_2, \ldots, X_{p+1}) = \\
&\qquad \sum (-1)^{i-1} \mathcal{L}_{\alpha(X_i)}\omega)(X_1, \ldots \widehat{X_i}, \ldots, X_{p+1})
\end{aligned} \tag{4.73}
$$

and for the form of the type $(q, 0)$

$$
\begin{aligned}
&(d_H - d_{H'})\omega(X, Y_1, \ldots, Y_q) = \\
&\qquad (\mathcal{L}_{\alpha(X)}\omega)(Y_1, \ldots, Y_q) + \operatorname{tr} T(X)\omega(Y_1, \ldots, Y_q)
\end{aligned} \tag{4.74}
$$

Hence for the half-density ω

$$
(d_H - d_{H'})\omega(X) = \mathcal{L}_{\alpha(X)}\omega + \frac{1}{2}\operatorname{tr} T(X)\omega \tag{4.75}
$$

and for $\omega \in \Omega_F^{1/2} \otimes \Lambda^* \tau$

$$(d_H - d_{H'})\omega(X_1, \ldots, X_{p+1}) =$$
$$\sum (-1)^{i-1} \Big(\mathcal{L}_{\alpha(X_i)} \omega(X_1, \ldots, \widehat{X_i}, \ldots, X_{p+1}) + \frac{1}{2} \operatorname{tr} T(X_i) \omega(X_1, \ldots, \widehat{X_i}, \ldots, X_{p+1}) \Big) \tag{4.76}$$

and the statement of the Lemma follows. □

The independence of the transverse fundamental class of the choice of H is now immediate. Note also that from the Theorem 16 one also obtains explicit transgression formulae.

4.6 Equivariant Chern character for foliations.

In this section we show how to construct a character of the holonomy equivariant vector bundle on the foliated manifold with values in the cyclic cohomology of the foliation algebra.

Let, in the notations of the previous section, E be a holonomy equivariant bundle on V. This means that for any two points x and y of the same leaf and for any path $\mathcal{G}$ from x to y there is a linear transformation $E_x \to E_y$ depending only on the holonomy class of $\mathcal{G}$. Fix a connection ∇ on E. We will now construct a generalized cycle associated with this data (and a choice of the complementary subbundle H).

The k-th component of the graded algebra Ω are defined as

$$C_0^\infty \Big(\mathcal{G}, s^*(\Omega_F^{1/2}) \otimes r^*(\Omega_F^{1/2}) \otimes r^*(\Lambda^k \tau^*) \otimes \operatorname{End} r^* E \Big)$$

with the product given by convolution. We have a natural inclusion

$$\mathcal{A} = C_0^\infty\left(\mathcal{G}, s^*(\Omega_F^{1/2}) \otimes r^*(\Omega_F^{1/2})\right) \subset C_0^\infty\left(\mathcal{G}, s^*(\Omega_F^{1/2}) \otimes r^*(\Omega_F^{1/2}) \otimes r^*(\Lambda^k \tau^*) \otimes \operatorname{End} r^* E\right)$$

The graded derivation on the algebra Ω is given by the horizontal component of connection, defined as follows. The choice of H again provides us with (depending on H) identification of $C^\infty(V, \Lambda^* F \otimes \Lambda^* \tau \otimes \operatorname{End} E)$ with $C^\infty(V, \Lambda^* T^* V \otimes \operatorname{End} E)$. Using this identification, connection defines an operator

$$\nabla : C^\infty(V, \Lambda^* F \otimes \Lambda^* \tau \otimes \operatorname{End} E) \to C^\infty(V, \Lambda^* F \otimes \Lambda^* \tau \otimes \operatorname{End} E) \tag{4.77}$$

such that for a differential form $\omega \in C^\infty(V \Lambda^r F \otimes \Lambda^s \tau \otimes \operatorname{End} E)$ $\nabla\omega$ has components of the types $(r+1, s)$, $(r, s+1)$, and $(r-1, s+2)$. We denote these components as $\nabla_V \omega$, $\nabla_H \omega$ and $\sigma\omega$ respectively; it is easy to see that $\sigma\omega$ is independent of the choice of connection ∇ (but clearly depends on the choice of H). We have

$$\nabla = \nabla_V + \nabla_H + \sigma \tag{4.78}$$

∇_H then can be extended to half-densities by $f \in C^\infty(V)$

$$\nabla_H f|\omega|^{1/2} = (\nabla_H f)|\omega|^{1/2} + f|\omega|^{1/2} \frac{d_H \omega}{2\omega} \tag{4.79}$$

where now $f \in C^\infty(V, E)$. We now extend ∇_H to the algebra Ω exactly in the same manner as d_H was extended in the previous section. Let Θ be the curvature of the connection ∇. We then have the following identity for action of ∇_H on forms:

$$\nabla_H^2 = \Theta^{0,2} - (\nabla_V \sigma + \sigma \nabla_V) \tag{4.80}$$

where $\Theta^{0,2}$ is the $0,2$ component of Θ. The right hand side defines a multiplier of the algebra Ω, and we put $\theta = \Theta^{0,2} - (\nabla_V \sigma + \sigma \nabla_V)$. The graded trace $\fint$ is defined by

$$\fint \omega = \int_V \operatorname{tr} \omega \tag{4.81}$$

for ω of degree $q = \operatorname{codim} F$, where tr is the usual matrix trace, and ω is again pulled back to V via the natural map $V \to \mathcal{G}$. We then have the following

Theorem 42. *1)* $\left(\Omega, \nabla_H, \theta, \fint\right)$ *define a generalized cycle over an algebra* $\mathcal{A}$.

2) The class of the character of this generalized cycle in cyclic cohomology is independent of the choice of H.

Proof. The first part of the Theorem is clear. The proof of the second part uses the analogue of the Lemma 41. Let H' be another complementary subbundle, and let $\nabla_{H'}$ be the coresponding derivation. Since the bundle E is holonomy equivariant we have well defined Lie derivative of its sections along the vertical vector fields (which is exactly covariant derivative with respect to the Bott connection). Let $\alpha(X)$ be defined by the equation (4.69). Then we can again define a degree one multiplier $\nabla_\alpha + \frac{1}{2} \operatorname{tr} T$, where we define for the vertical field Y, $s \in C^\infty(V, E)$, and $\omega \in C^\infty(V, \Lambda^* \tau \otimes \Omega_F^{1/2})$

$$\nabla_Y (s\omega) = (\nabla_Y s) + s \mathcal{L}_Y \omega \tag{4.82}$$

$$\nabla_H - \nabla_{H'} = \operatorname{ad}\left(\nabla_\alpha + \frac{1}{2} \operatorname{tr} T\right) \tag{4.83}$$

The statement of the Theorem now follows from the Theorem 19. ∎

CHAPTER 5

CHARACTERS OF FREDHOLM MODULES AND DUALITY MAP IN CYCLIC THEORY.

5.1 Fredholm modules.

In this section we write formulae for the character of the generalized cycle associated with a finitely summable bounded Fredholm module (cf. [Con85]) . In other words we obtain a formula for the character of a Fredholm module. We show that this definition agrees with Connes' definition [Con85].

Let $(\mathcal{H}, F, \gamma)$ be an even finitely summable bounded Fredholm module over the algebra $\mathcal{A}$. Here $\mathcal{H}$ is a Hilbert space, on which the algebra $\mathcal{A}$ acts, γ is a $\mathbb{Z}_2$-grading on $\mathcal{H}$, and F is an odd selfadjoint operator on $\mathcal{H}$. We assume that $\mathcal{A}$ is represented by the even operators in $\mathcal{H}$, and since we almost always consider only one representation of $\mathcal{A}$, we drop this representation from our notations, and do not distinguish elements of the algebra and corresponding operators. We also suppose that the algebra $\mathcal{A}$ is unital, and the representation of it on the Hilbert space $\mathcal{H}$ is unital. Let p be a number such that $[F, a] \in \mathcal{L}^p$ and $(F^2 - 1) \in \mathcal{L}^{\frac{p}{2}}$. We remark that for any p summable Fredholm module one can achieve these summability conditions by altering the operator F and keeping all the other data intact. We associate with the Fredholm module a generalized cycle similarly to [Con85] where it is done in the case when $F^2 = 1$. Consider a $\mathbb{Z}$-graded algebra $\Omega = \bigoplus_{m=0}^{\infty} \Omega^m$ generated by the symbols $a \in \mathcal{A}$ of degree 0, $[F, a]$, $a \in \mathcal{A}$ of degree 1, and symbol $(F^2 - 1)$ of degree 2, with a relation $[F, ab] = a[F, b] + [F, a]b$. This algebra can be naturally represented on the Hilbert space $\mathcal{H}$, and we will not distinguish in our notations between elements of the algebra and the corresponding operators. Ω is equipped with a natural connection ∇, given by the formula $\nabla(\xi) = [F, \xi]$ (graded commutator) in terms of the representation of Ω, or on generators by the formulae

$$\nabla(a) = [F, a] \tag{5.1}$$

$$\nabla([F, a]) = \big((F^2 - 1)a - a(F^2 - 1)\big) = [(F^2 - 1), a] \tag{5.2}$$

$$\nabla\left((F^2 - 1)\right) = 0 \tag{5.3}$$

Notice that $\nabla^2(\xi) = [(F^2 - 1), \xi]$ for $\xi \in \Omega$. Hence we define the curvature θ to be $(F^2 - 1)$. Clearly, $\xi \in \Omega^n$ is of trace class if $n \geq p$. Here we need to choose n to be even, $n = 2m$. Hence we can define the graded trace on Ω^n by $⨍ \xi = m! Tr\, \gamma \xi$. The equality $Tr\, \gamma \nabla(\xi) = 0$ for $\xi \in \Omega^{n-1}$ follows from the relation

$$Tr\, \gamma \omega = \frac{1}{2} Tr\, \gamma F \nabla(\omega) - Tr\, \gamma (F^2 - 1) \omega$$

which holds for ω of trace class). Indeed, for $\xi \in \Omega^{n-1}$, $\nabla(\xi)$ is of trace class and

$$\begin{aligned} Tr\, \gamma \nabla(\xi) &= \frac{1}{2} Tr\, \gamma F \nabla^2(\xi) + Tr\, \gamma \nabla(\xi) \\ &= \frac{1}{2} Tr\, \gamma F [(F^2 - 1), \xi] - Tr\, \gamma (F^2 - 1)[F, \xi] = 0 \end{aligned} \tag{5.4}$$

Now we can apply the formula (3.24) to obtain a cyclic cocycle $\mathrm{Ch}_{2m}(F)$ in the cyclic bicomplex of the algebra $\mathcal{A}$. Its components $\mathrm{Ch}^k_{2m}(F)$ $k = 0, 2, 4, \ldots, 2m$ are given by the formula

$$\mathrm{Ch}^k(F)(a_0, a_1, \ldots a_k) =$$
$$\frac{m!}{(m+\frac{k}{2})!} \sum_{i_0+i_1+\cdots+i_k=m-\frac{k}{2}} Tr\,\gamma a_0(1-F^2)^{i_0}[F, a_1](1-F^2)^{i_1}\ldots[F, a_k](1-F^2)^{i_k} \quad (5.5)$$

Note that for the case when $F^2 = 1$ we get the formula from [Con85], normalized as in [Con94].

We will now associate a generalized chain with homotopy between Fredholm modules. If the two Fredholm modules $(\mathcal{H}, F_0, \gamma)$ and $(\mathcal{H}, F_1, \gamma)$ are connected by a smooth operator homotopy (meaning that there exists a C^1 family F_t of operators with $[F_t, a] \in \mathcal{L}^p$ and $(F_t^2 - 1) \in \mathcal{L}^{\frac{p}{2}}$, $t \in [0, 1]$ with $F_t|_{t=0} = F_0$, $F_t|_{t=1} = F_1$), this generalized chain will provide cobordism between cycles corresponding to the modules.

We start by constructing, exactly as before, an algebra Ω_t generated by the elements a, $[F_t, a]$, (F_t^2-1), with the connection ∇_t and the curvature $\theta_t = (F_t^2-1)$. For each $t \in [0, 1]$ one constructs a natural representation π_t of this algebra on the Hilbert space $\mathcal{H}$. Let $\Omega^*([0, 1])$ be the DGA of the differential forms on the interval $[0, 1]$ with the usual differential d. We can form a graded tensor product $\Omega^*([0, 1])\widehat{\otimes}\Omega_t$. Choose an odd number $n = 2m+1$ so that $n \geq p+2$; if in addition we suppose that $\frac{dF_t}{dt} \in \mathcal{L}^p$, we can choose $n \geq p + 1$. In order to define the connection and the curvature we will have to adjoin to our algebra an element of degree 2 $dt\widehat{\otimes}\frac{dF_t}{dt}$ and an element of degree 3 $dt\widehat{\otimes}(F_t\frac{dF_t}{dt} + \frac{dF_t}{dt}F_t)$. The algebra with the adjoined elements will be denoted

Ω_c. The homomorphism $\rho_c : \mathcal{A} \to \Omega_c$ is given by $\rho_c(a) = 1\widehat{\otimes}a$. We define the connection ∇_c as $\frac{d}{dt} \wedge dt + \nabla_t$, i.e. on the generators the definition is the following (here $\beta \in \Omega^*([0,1])$):

$$\nabla_c(\beta\widehat{\otimes}a) = d\beta\widehat{\otimes}a + (-1)^{\deg(\beta)}\beta\widehat{\otimes}[F_t, a] \tag{5.6}$$

$$\nabla_c(\beta\widehat{\otimes}[F_t, a]) = d\beta\widehat{\otimes}[F_t, a] + (-1)^{\deg(\beta)}\beta\widehat{\otimes}[(F_t^2 - 1), a] + \beta \wedge dt\widehat{\otimes}[\frac{dF_t}{dt}, a] \tag{5.7}$$

$$\nabla_c(\beta\widehat{\otimes}(F_t^2 - 1)) = d\beta\widehat{\otimes}(F_t^2 - 1) + \beta \wedge dt\widehat{\otimes}(F_t\frac{dF_t}{dt} + \frac{dF_t}{dt}F_t) \tag{5.8}$$

$$\nabla_c(dt\widehat{\otimes}\frac{dF_t}{dt}) = -dt\widehat{\otimes}(F_t\frac{dF_t}{dt} + \frac{dF_t}{dt}F_t) \tag{5.9}$$

$$\nabla_c(dt\widehat{\otimes}(F_t\frac{dF_t}{dt} + \frac{dF_t}{dt}F_t)) = dt\widehat{\otimes}[(F_t^2 - 1), \frac{dF_t}{dt}] \tag{5.10}$$

The curvature θ_c of this connection is defined as

$$\theta_c = 1\widehat{\otimes}(F_t^2 - 1) + dt\widehat{\otimes}\frac{dF_t}{dt} \tag{5.11}$$

and the identity $(\nabla_c)^2\cdot = [\theta_c, \cdot]$ is verified by computation. One then defines the graded trace $\displaystyle⨍_c$ on $(\Omega^*([0,1])\widehat{\otimes}\Omega_t)^n$ by the formula

$$⨍_c \beta\widehat{\otimes}\xi = \begin{cases} (-1)^{\deg(\xi)} m! \int\limits_{[0,1]} \big(\beta\, Tr\, \gamma\pi_t(\xi)\big) & \text{if } \beta \in \Omega^1([0,1]) \\ 0 & \text{if } \beta \in \Omega^0([0,1]) \end{cases}$$

The restriction maps $r_0 : \Omega_c \to \Omega_0$ and $r_1 : \Omega_c \to \Omega_1$ are defined as follows. $r_0(\beta\widehat{\otimes}\xi))$ is 0 if β is of degree 1 , and $\beta(0)\xi_0$ where ξ_0 is obtained from ξ by replacing F_t by F_0 if β is of degree 0 , and similarly for r_1. One can check that the map $r_1 \oplus r_0$ identifies $\partial\Omega_c$ with $\Omega^1 \oplus \widetilde{\Omega^0}$ and provides required cobordism.

Now we can use the Theorem 3 to study the properties of $\mathrm{Ch}(F)$ with respect to the operator homotopy.

Theorem 43. *Suppose $(\mathcal{H}, F_0, \gamma)$ and $(\mathcal{H}, F_1, \gamma)$ are two finitely summable Fredholm modules over an algebra $\mathcal{A}$ which are connected by the smooth operator homotopy F_t and p is a number such that $[F_t] \in \mathcal{L}^p$ and $(F_t^2 - 1) \in \mathcal{L}^{\frac{p}{2}}$ for $0 \leq t \leq 1$. Choose m such that $2m \geq p+1$. Then $\mathrm{Ch}_{2m}(F_0) = \mathrm{Ch}_{2m}(F_1)$ in $HC^{2m}(\mathcal{A})$. If moreover $\frac{dF_t}{dt} \in \mathcal{L}^p$ one can choose m such that $2m \geq p$.*

Proof. Let Tch_{2m}^k denote the k-th component of the character of the constructed above chain, providing the cobordism between the cycles associated with $(\mathcal{H}, F_0, \gamma)$ and $(\mathcal{H}, F_1, \gamma)$, $k = 1, 3, \ldots, 2m+1$. It can be defined under the conditions on m specified in the theorem. According to the Theorem 3

$$\mathrm{Ch}_{2m}(F_1) - \mathrm{Ch}_{2m}(F_0) = (b + B)\, Tch_{2m}$$

Now,

$$Tch_{2m}^{2m+1}(a_0, a_1, \ldots a_{2m+1}) = \mathrm{const} \fint_c \rho_c(a_0) \nabla_c(\rho_c(a_1)) \ldots \nabla_c(\rho_c(a_{2m+1})) = 0$$

(since the term under the $\fint_c$ does not contain dt). Hence Tch_{2m} can be considered as the $2m-1$ chain (is in the image of S), and the result follows. $\square$

Remark 44. Suppose we have two Fredholm modules $(\mathcal{H}, F_0, \gamma)$ and $(\mathcal{H}, F_1, \gamma)$ such that $F_0 - F_1 \in \mathcal{L}^p$ and $F_i^2 - 1 \in \mathcal{L}^{\frac{p}{2}}$, $i = 0, 1$. Then $\mathrm{Ch}_{2m}(F_0) = \mathrm{Ch}_{2m}(F_1)$, $2m \geq p$. Indeed, we can apply the Theorem 43 to the linear homotopy $F_t = F_0 + t(F_1 - F_0)$, and need only to verify that $F_t^2 - 1 \in \mathcal{L}^{\frac{p}{2}}$. But

$$F_t^2 - 1 = (F_0^2 - 1) + t\big(F_0(F_1 - F_0) + (F_1 - F_0)F_0\big) + t^2(F_1 - F_0)^2$$

The first and the last terms in the right hand side are always in $\mathcal{L}^{\frac{p}{2}}$, and since the left hand side is in $\mathcal{L}^{\frac{p}{2}}$ for $t = 1$, $\big(F_0(F_1 - F_0) + (F_1 - F_0)F_0\big) \in \mathcal{L}^{\frac{p}{2}}$.

Corollary 45. *Let e be an idempotent in $M_N(\mathcal{A})$, and $(\mathcal{H}, F, \gamma)$ be an even Fredholm module over $\mathcal{A}$. Construct the Fredholm operator $F_e = e(F \otimes 1)e : \mathcal{H}^+ \otimes \mathbb{C}^N \to \mathcal{H}^- \otimes \mathbb{C}^N$ (where $\mathcal{H}^+$ and $\mathcal{H}^-$ are determined by the grading). Then*

$$\mathrm{index}(F_e) =< \mathrm{Ch}^*(F), \mathrm{Ch}_*(e) >$$

Here $\mathrm{Ch}_*(e)$ is the usual Chern character in the cyclic homology.

Proof. By replacing $\mathcal{A}$ by $M_N(\mathcal{A})$ we reduce the situation to the case when $e \in \mathcal{A}$. Now we apply Connes' construction, which uses the homotopy $F_t = F + t(1-2e)[F, e]$ which connects F (obtained when $t = 0$) with the operator $F_1 = eFe + (1-e)F(1-e)$, obtained when $t = 1$. Note that $1 - F_t^2 \in \mathcal{L}^{\frac{p}{2}}$. Indeed,

$$F_t^2 - 1 = \big(F^2 - 1\big) + \big(t(1-2e)[F, e]\big)^2 + t\big([F, (1-2e)[F, e]]\big)$$

The first two terms are clearly in $\mathcal{L}^{\frac{p}{2}}$. As for the third one, it can be rewritten as $-2[F, e][F, e] + (1-2e)[F, [F, e]] = -2[F, e]^2 + (1-2e)[(F^2-1), e] \in \mathcal{L}^{\frac{p}{2}}$.

The operator F_1 commutes with e, and homotopy does not change the pairing. Hence it is enough to prove the result in the case when F and e commute. In this case in the formula for the pairing all of the terms involving commutators are 0 , hence the only term with nonzero contribution is $\mathrm{Ch}^0(F)(e) = Tr\,\gamma e(1-F^2)^m = Tr\,\gamma(e - (eFe)^2)^m = \mathrm{index}(F_e)$ by the well known formula. $\square$

In [Con85] Connes provides canonical construction, allowing one to associate with every p-summable Fredholm module such that $F^2 - 1 \neq 0$ another one for which $F^2 - 1 = 0$, and which defines the same K-homology class. This allows to reduce the definition of the character of a general Fredholm module to the case when $F^2 = 1$.

The construction is the following. Given the Fredholm module $(\mathcal{H}, F, \gamma)$ one first constructs the Hilbert space $\widetilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$ with the grading given by $\widetilde{\gamma} = \gamma \oplus (-\gamma)$. An element $a \in \mathcal{A}$ acts by $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Then one constructs an operator $\widetilde{F}$, such that $\widetilde{F} - F' \in \mathcal{L}^p$ and $\widetilde{F}^2 = 1$; here by F' we denote $\begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix}$. The character of the Fredholm module $(\mathcal{H}, F, \gamma)$ is then *defined* to be the character of the $(\widetilde{\mathcal{H}}, \widetilde{F}, \widetilde{\gamma})$.

Theorem 46. *Let* $(\mathcal{H}, F, \gamma)$ *be an even finitely summable Fredholm module over the algebra* $\mathcal{A}$, *and let* p *be a real number such that* $[F, a] \in \mathcal{L}^p$ *and* $(F^2 - 1) \in \mathcal{L}^{\frac{p}{2}}$. *Then class of* $\mathrm{Ch}^*(F)$ *defined in* (5.5) *in the periodic cyclic cohomology coincides with the Chern character, as defined by Connes [Con85].*

Proof. First, let us consider the Fredholm module $(\widetilde{\mathcal{H}}, F', \widetilde{\gamma})$ over the algebra $\widetilde{\mathcal{A}}$ - the algebra $\mathcal{A}$ with adjoined unit (acting by the identity operator). Then $\mathrm{Ch}_{2m}(F')$ defines a class in the cyclic cohomology of $\widetilde{\mathcal{A}}$, where we choose $2m \geq p$. Since $Tr\, \widetilde{\gamma}(1 - (F')^2)^m = 0$, it defines a class in the reduced cyclic cohomology of $\widetilde{\mathcal{A}}$, and hence in the cyclic cohomology of $\mathcal{A}$. It coincides with the class defined by the Fredholm module $(\mathcal{H}, F, \gamma)$.

The Theorem 43 and the Remark 44 show that the classes defined by the Fredholm modules $\mathrm{Ch}(\widetilde{F})$ and $\mathrm{Ch}(F')$ coincide. To finish the proof we note that $\mathrm{Ch}(\widetilde{F})$ coincides with the Chern character as defined in [Con85].

$\square$

The proof of the Theorem 43 also provides an explicit transgression formula. We just need to compute explicitly formula for

$$Tch^k_{2m}(a_0, a_1, \dots a_k) =$$

$$\frac{(-1)^{m-\frac{k-1}{2}}(m)!}{(m+\frac{k+1}{2})!} \sum_{i_0+i_1+\dots+i_k=m-\frac{k-1}{2}} \fint_c \rho_c(a_0)\theta_c^{i_0}\nabla_c(\rho_c(a_1))\theta_c^{i_1}\dots\nabla_c(\rho_c(a_k))\theta_c^{i_k} \quad (5.12)$$

Since $\theta_c^{i_l} = \sum_{r+q=i_l-1} dt\widehat{\otimes}(F_t^2-1)^r\frac{dF_t}{dt}(F_t^2-1)^q$ one can rewrite this formula as

$$\frac{(-1)^{m-\frac{k-1}{2}}(m)!}{(m+\frac{k+1}{2})!} \int_0^1 \Big(\sum_{i_0+i_1+\dots+i_k=m-\frac{k-1}{2}} \sum_{l=0}^{k} \sum_{r+q=i_l-1} (-1)^l Tr\,\gamma a_0(F_t^2-1)^{i_0}$$

$$[F_t, a_1](F_t^2-1)^{i_1}\dots[F_t,a_l](F_t^2-1)^r\frac{dF_t}{dt}(F_t^2-1)^q\dots[F_t,a_k](F_t^2-1)^{i_k}\Big)dt \quad (5.13)$$

Finally we can write the answer as

$$Tch^k_{2m}(a_0, a_1, \dots a_k) =$$

$$-\frac{(m)!}{(m+\frac{k+1}{2})!} \int_0^1 \Big(\sum_{i_0+\dots+i_k+i_{k+1}=m-\frac{k+1}{2}} \sum_{l=0}^{k} (-1)^l Tr\,\gamma a_0(F_t^2-1)^{i_0}$$

$$[F_t, a_1](1-F_t^2)^{i_1}\dots[F_t,a_l](1-F_t^2)^{i_l}\frac{dF_t}{dt}(1-F_t^2)^{i_{l+1}}\dots[F_t,a_k](1-F_t^2)^{i_{k+1}}\Big)dt$$

$$(5.14)$$

where k is an odd number between 1 and $2m-1$.

All the considerations above can be repeated in the case of an odd finitely summable Fredholm module $(\mathcal{H}, F)$ over an algebra $\mathcal{A}$. Here as before we suppose that $[F,a] \in \mathcal{L}^p$, $(F^2-1) \in \mathcal{L}^{\frac{p}{2}}$. We choose number m such that $n = 2m+1 \geq p$. The trace now is given by $\fint \xi = \sqrt{2i}\Gamma(n/2+1)Tr\,\xi$.

The corresponding Chern character $\mathrm{Ch}_{2m+1}(F)$ has components Ch^k_{2m+1} for $k = 1$, 3 , ... , $2m+1$, given by the formula

$$\mathrm{Ch}^k_{2m+1}(a_0, a_1, \ldots, a_k) = \frac{\Gamma(m+\frac{3}{2})\sqrt{2i}}{(m+\frac{k+1}{2})!}$$
$$\sum_{i_0+i_1+\cdots+i_k=m-\frac{k-1}{2}} Tr\, a_0(1-F^2)^{i_0}[F, a_1](1-F^2)^{i_1}\ldots[F, a_k](1-F^2)^{i_k} \quad (5.15)$$

If the two Fredholm modules are connected via the operator homotopy F_t one has the transgression formula

$$\mathrm{Ch}_{2m+1}(F_1) - \mathrm{Ch}_{2m+1}(F_0) = (b+B)\, Tch_{2m+1} \quad (5.16)$$

where Tch_{2m+1} is a $2m$ cyclic cochain having components Tch^k_{2m} for k even between 0 and $2m$, given by the formula:

$$Tch^k_{2m+1}(a_0, a_1, \ldots a_k) =$$
$$-\frac{\Gamma(m+\frac{3}{2})\sqrt{2i}}{(m+\frac{k}{2}+1)!}\int_0^1 \Big(\sum_{i_0+\cdots+i_k+i_{k+1}=m-\frac{k}{2}} \sum_{l=0}^{k} (-1)^l Tr\, a_0(F_t^2-1)^{i_0}$$
$$[F_t, a_1](1-F_t^2)^{i_1}\ldots[F_t, a_l](1-F_t^2)^{i_l}\frac{dF_t}{dt}(1-F_t^2)^{i_{l+1}}\ldots[F_t, a_k](1-F_t^2)^{i_{k+1}}\Big)dt$$
$$(5.17)$$

The proof of the Theorem 46 works in the odd situation as well and shows that $\mathrm{Ch}^*(F)$ coincides with the Chern character as defined by Connes. In particular, this allows to recover the spectral flow via the pairing with K-theory. More precisely, let $u \in M_N(\mathcal{A})$ be a unitary. Let $\mathrm{sf}(F\otimes 1, (F\otimes 1)^u)$ be the spectral flow of the operators

$F \otimes 1$ and $(F \otimes 1)^u = u((F \otimes 1)u^*$ acting on the space $\mathcal{H} \otimes \mathbb{C}^N$. The Chern character of the class of u in $K_1(\mathcal{A})$ is the periodic cyclic cycle defined by

$$Ch_*(u) = \frac{1}{2\sqrt{2\pi i}} \sum_{l=1}^{\infty} (-1)^l (l-1)!\, tr\left((u \otimes u^{-1})^l - (u^{-1} \otimes u)^l\right) \tag{5.18}$$

Then we have the following

Corollary 47. *Let $u \in \mathcal{A}$ be a unitary, and $(\mathcal{H}, F)$ be an odd Fredholm module over the algebra $\mathcal{A}$. Then* $< \mathrm{Ch}^*(F), \mathrm{Ch}_*(u) >= \mathrm{sf}(F \otimes 1, (F \otimes 1)^u)$

Remark 48. This is a finitely summable analogue of the result of Getzler [Get93] In the finitely summable case analytic formula for the spectral flow was derived in [CP]; use of the Theorem 46 allows to give a proof of the corollary 47 without using this formula.

5.2 Duality map

This section is devoted to the construction of the analogue of the Paschke-Voiculescu duality (cf. [Pas81]) in the cyclic (co)homology context. This is achieved by representing Paschke-Voiculescu duality as Connes' Poinare duality (cf. [Con94]) in KK-theory. The formulae in cyclic cohomology thus obtained are related to the Cuntz and Quillen explicit formulae (cf. [CQ95]) for the Godwillie's isomorphism ([Goo85]). There should also be a bivariant version of our construction, similarly with Valette's bivariant version of Paschke-Voiculescu duality [Val83].

Let $\mathcal{A}$ be an algebra, acting on a Hilbert space $\mathcal{H}$. Suppose that $\widetilde{\mathcal{B}}$ is another algebra, acting on $\mathcal{H}$ and the following condition is satisfied:

$$[a, b] \in \mathcal{L}^p, \forall a \in \mathcal{A}, b \in \widetilde{\mathcal{B}} \tag{5.19}$$

The natural norm on $\widetilde{\mathcal{B}}$ is given by

$$\|b\|_{\widetilde{\mathcal{B}}} = \|b\| + \sup_{\|a\|\leq 1} \|[a, b]\|_p, b \in \widetilde{\mathcal{B}}, a \in \mathcal{A} \tag{5.20}$$

Here $\|\cdot\|_p$ is an $\mathcal{L}^p$ norm and $\|\cdot\|$ is an operator norm. Let $\mathcal{B} = \widetilde{\mathcal{B}}/(\mathcal{L}^p \cap \mathcal{B})$. Let $\rho : \mathcal{B} \to \widetilde{\mathcal{B}}$ be a ($\mathbb{C}$-linear and unital) section of a natural projection $\widetilde{\mathcal{B}} \to \mathcal{B}$ (not necessarily a homomorphism), and let

$$\omega(b_0, b_1) = \rho(b_0 b_1) - \rho(b_0)\rho(b_1) \tag{5.21}$$

be its curvature. Clearly, $\omega(b_0, b_1) \in \mathcal{L}^p$, b_0, $b_1 \in \mathcal{B}$. Consider then the following map $s : \mathcal{A} \otimes \mathcal{B} \to End(\mathcal{H})$:

$$s(a \otimes b) = a\rho(b) \tag{5.22}$$

This clearly is not, in general, a homomorphism, but it is a homomorphism modulo $\mathcal{L}^p$. Indeed,

$$\begin{aligned}
&\Xi(a_0 \otimes b_0, a_1 \otimes b_1) := \\
&\quad s(a_0 a_1 \otimes b_0 b_1) - s(a_0 \otimes b_0)s(a_1 \otimes b_1) = a_0 a_1 \rho(b_0 b_1) - a_0 \rho(b_0) a_1 \rho(b_1) = \\
&\quad (a_0 a_1 \rho(b_0, b_1) - a_0 a_1 \rho(b_0)\rho(b_1)) + (a_0 a_1 \rho(b_0)\rho(b_1) - a_0 \rho(b_0) a_1 \rho(b_1)) \\
&\quad = a_0 a_1 \omega(b_0, b_1) + a_0 [a_1, \rho(b_0)]\rho(b_1)
\end{aligned} \tag{5.23}$$

is clearly in $\mathcal{L}^p$.

This means that we have an extension

$$0 \to \mathcal{L}^p \to \widetilde{\mathcal{A} \otimes \mathcal{B}} \to \mathcal{A} \otimes \mathcal{B} \to 0 \tag{5.24}$$

To such an extension with a section Connes associated (cf. [Con85]) for every odd $n \geq 2p-1$ a class in the cyclic cohomology $HC^n(\mathcal{A} \otimes \mathcal{B})$, represented by the canonical cyclic cocycle Ch_s^n.

$$\begin{aligned} Ch_s^n(x_0, x_1, \ldots, x_n) \\ = \frac{(-1)^{\frac{n+1}{2}}\sqrt{2\pi i}}{\left(\frac{n-1}{2}\right)!}\Big(&\operatorname{Tr} \Xi(x_0, x_1)\Xi(x_2, x_3)\ldots\Xi(x_{n-1}, x_n) \\ &- \operatorname{Tr} \Xi(x_n, x_0)\Xi(x_1, x_2)\ldots\Xi(x_{n-2}, x_{n-1})\Big) \end{aligned} \tag{5.25}$$

Here $x_j \in \mathcal{A} \otimes \mathcal{B}$ and Ξ is defined in (5.23). Connes shows that the class of the cocycle Ch_s^n is independent of the choice of the section s, and in particular, of the choice of $\rho : \mathcal{B} \to \widetilde{\mathcal{B}}$, and also that $SCh_s^n = Ch_s^{n+2}$, hence a class $Ch_s^* \in HP^1(\mathcal{A} \otimes \mathcal{B})$ is well defined.

Now we have the natural products consistent via the Chern character with products in K-theory and K-homology.

$$\begin{aligned} HC_i^-(\mathcal{A}) \otimes_{\mathcal{A}} HC^n(\mathcal{A} \otimes \mathcal{B}) \to HC^{n-i}(\mathcal{B}) \\ HC_i^-(\mathcal{B}) \otimes_{\mathcal{B}} HC^n(\mathcal{A} \otimes \mathcal{B}) \to HC^{n-i}(\mathcal{A}) \end{aligned} \tag{5.26}$$

$$\begin{aligned} HP_i(\mathcal{A}) \otimes_{\mathcal{A}} HP^j(\mathcal{A} \otimes \mathcal{B}) \to HP^{j-i}(\mathcal{B}) \\ HP_i(\mathcal{B}) \otimes_{\mathcal{B}} HP^j(\mathcal{A} \otimes \mathcal{B}) \to HP^{j-i}(\mathcal{A}) \end{aligned} \tag{5.27}$$

The products (5.26) can be constructed by composing the natural map

$$HC_i^-(\mathcal{A}) = HC^{-i}(\mathbb{C},\mathcal{A}) \overset{\otimes\mathcal{B}}{\to} HC^{-i}(\mathcal{B},\mathcal{A}\otimes\mathcal{B}) \tag{5.28}$$

(here we use Jones-Cassel bivariant cyclic cohomology), and the intersection product

$$HC^{-i}(\mathcal{B},\mathcal{A}\otimes\mathcal{B}) \times HC^n(\mathcal{A}\otimes\mathcal{B},\mathbb{C}) \to HC^{n-i}(\mathcal{B},\mathbb{C}) = HC^{n-i}(\mathcal{B}) \tag{5.29}$$

The map (5.28) can be written explicitly as follows. We are given a chain $\{\alpha_{i+2k}\}$, $k = 0, 1, \dots$ in $CC_*^-(\mathcal{A})$. Using it we can construct an element in the bivariant complex $\mathrm{Hom}^S\left(Tot_*(\mathcal{B}), Tot_*(\mathcal{A}\otimes\mathcal{B})\right)$ which maps a chain $\{\beta_j\} \in Tot_*(\mathcal{B})$ to the chain $\{\sum_{k+l=n} Sh(\alpha_k \otimes \beta_l)\}$, n runs over even or odd numbers, depending on the parity of α and β. Here $Sh = sh + sh'$ (cf. [Lod92]).

Now we can define natural maps

$$\begin{aligned}
&\Psi^{i,n}_{\mathcal{A},\mathcal{B}} : HC_i^-(\mathcal{A}) \to HC^{n-i}(\mathcal{B})\\
&\Psi^{i,n}_{\mathcal{B},\mathcal{A}} : HC_i^-(\mathcal{B}) \to HC^{n-i}(\mathcal{A}), n \geq 2p-1
\end{aligned} \tag{5.30}$$

and

$$\begin{aligned}
&P\Psi^{i}_{\mathcal{A},\mathcal{B}} : HP_i(\mathcal{A}) \to HP^{1-i}(\mathcal{B})\\
&P\Psi^{i}_{\mathcal{B},\mathcal{A}} : HP_i(\mathcal{B}) \to HP^{1-i}(\mathcal{A}), i = 0,1
\end{aligned} \tag{5.31}$$

by taking products (5.26, 5.27) with Ch_s^* from 5.25. So, for example

$$\begin{aligned}
\Psi^{i,n}_{\mathcal{A},\mathcal{B}}(\{\alpha_{i+2k}\})(\{\beta_j\}) &= Ch_s^n\left(\sum_{k+l=n} Sh(\alpha_k\otimes\beta_l)\right)\\
&= Ch_s^n\left(\sum_{k+l=n} sh(\alpha_k\otimes\beta_l)\right)
\end{aligned} \tag{5.32}$$

where the last equality is due to the fact that $sh'(\alpha\otimes\beta)$ always starts with $1\otimes 1$, but Ch_s^n vanishes on such elements. The similar formula holds for the periodic case.

5.3 Explicit formula for the duality map.

To write an explicit formulae for the maps (5.30), (5.31) we first write down values of the "curvature" Ξ on the tensors which appear in shuffles. All the formulas below are the special cases of (5.23).

$$\Xi(a \otimes b, a' \otimes 1) = a[a', \rho(b)] \tag{5.33}$$

$$\Xi(a \otimes b, 1 \otimes b') = a\omega(b, b') \tag{5.34}$$

$$\Xi(a \otimes 1, a' \otimes b') = 0 \tag{5.35}$$

$$\Xi(1 \otimes b, a' \otimes b') = a'\omega(b, b') + [a', \rho(b)]\rho(b') \tag{5.36}$$

We will now compute $Ch_s^n(sh(\alpha \otimes \beta))$, with $\alpha = a_0 \otimes a_1 \otimes \dots a_k$, $\beta = b_0 \otimes b_1 \dots b_l$ with $k + l = n$. Recall that n is an odd number greater than $2p - 1$. We will compute contributions of the two terms in Ch_s^n separately.

We start with the computation of the contribution of the first term. Suppose first that $k = 0$ (and $l = n$). Then there is just one term in the $sh(\alpha \otimes \beta)$ – namely $(a_0 \otimes b_0, b_1, b_2, \dots, b_n)$. In this case the contribution of the first term is

$$\begin{aligned} &\operatorname{Tr} \Xi(a_0 \otimes b_0, 1 \otimes b_1)\Xi(1 \otimes b_2, 1 \otimes b_3) \dots \Xi(1 \otimes b_{n-1}, 1 \otimes b_n) \\ &\qquad = \operatorname{Tr} a_0 \omega(b_0, b_1)\omega(b_2, b_3) \dots \omega(b_{n-1}, b_n) \end{aligned} \tag{5.37}$$

where we have used identities (5.33)-(5.36)

Let now $k \geq 1$. We need to evaluate the expression

$$\operatorname{Tr} \Xi(x_0, x_1)\Xi(x_2, x_3) \dots \Xi(x_{n-1}, x_n) \tag{5.38}$$

where $x_0 = a_0 \otimes b_0$ and x_j-s are a shuffle of terms like $a \otimes 1$ and $1 \otimes b$. Observe that if x_j for j even has a form $a \otimes 1$, then (5.38) equals to 0, due to (5.35). Hence we need to consider only the shuffles in which this does not happen. In particular we should have $k \leq \frac{n+1}{2}$ (otherwise the corresponding term is 0). Also,the following observation will be useful.

Suppose in some shuffle for some m x_{2m}, x_{2m+1}, x_{2m+2} are of the form $1 \otimes b$, and x_{2m+3} is of the form $a \otimes 1$. Consider a new shuffle which coincides with the old one in all places except

$$\begin{aligned} x'_{2m} &= x_{2m} \\ x'_{2m+1} &= x_{2m+3} \\ x'_{2m+2} &= x_{2m+1} \\ x'_{2m+3} &= x_{2m+2} \end{aligned} \tag{5.39}$$

Then the sign of permutation in the new shuffle is the same as in the old one. Hence if one shuffle can be obtained from the other by a sequence of such operations (or their inverses) they signs coincide.

We will consider now two cases:

a) $x_1 = a_1 \otimes 1$. Let $a_j \otimes 1$ be in the place number $2i_j + 1$ ($i_1 = 0$, $i_j < i_{j'}$ if $j < j'$, and $i_k \leq \frac{n-1}{2}$). Note that for all the shuffles of such a form the sign of the permutation is the same, since all of them can be obtained one from the other by the operations described in (5.39). Hence it is enough to compute the sign of permutation for just one of them, say with $i_j = j - 1$, in which case it is seen to be $(-1)^{\frac{k(k-1)}{2}}$.

Hence the contribution of such shuffles is

$$c_{n,k} \sum_{0=i_1<i_2<\cdots<i_k\leq\frac{n-1}{2}} \operatorname{Tr} a_0[a_1,\rho(b_0)]\omega(b_1,b_2)\dots$$

$$[a_2,\rho(b_{2i_2-1})]\omega(b_{2i_2},b_{2i_2+1})\dots[a_j,\rho(b_{2i_j-j+1})]\dots \quad (5.40)$$

where

$$c_{n,k}=\frac{(-1)^{\frac{n+1}{2}+\frac{k(k-1)}{2}}\sqrt{2\pi i}}{\left(\frac{n-1}{2}\right)!} \quad (5.41)$$

The expression above is sum of the terms each of which is a trace of product of expressions of the form $\omega(b,b')$ and $[a,\rho(b)]$, and is obtained using (5.33)-(5.36).

b)$x_1=1\otimes b_1$. Let again $a_j\otimes 1$ be in the place number $2i_j+1$ (but now $i_1>0$). For all the shuffles of such a form the sign of the permutation is the same and equals $(-1)^{\frac{k(k-1)}{2}}$. Thus for these shuffles the contribution is

$$c_{n,k} \sum_{0<i_1<i_2<\cdots<i_k\leq\frac{n-1}{2}} \operatorname{Tr} a_0\omega(b_0,b_1)\dots[a_1,\rho(b_{2i_1})]\omega(b_{2i_1+1},b_{2i_1+2})\dots$$

$$\dots[a_j,\rho(b_{2i_j-j+1})]\dots \quad (5.42)$$

Adding the expressions (5.40) and (5.42) we obtain the formula for the first term:

$$c_{n,k} \sum_{0\leq i_1<i_2<\cdots<i_k\leq\frac{n-1}{2}} \operatorname{Tr} a_0\omega(b_1,b_2)\dots[a_1,\rho(b_{2i_1})]\omega(b_{2i_1+1},b_{2i_1+2})\dots$$

$$[a_2,\rho(b_{2i_2-1})]\omega(b_{2i_2},b_{2i_2+1})\dots[a_j,\rho(b_{2i_j-j+1})]\dots \quad (5.43)$$

For example, let $k = \frac{n+1}{2}$. In this case there is no contribution from (5.42), and only one term exists in (5.40), for which $i_j = j - 1$, and it is equal to

$$c_{n,k} \operatorname{Tr} a_0[a_1, \rho(b_0)][a_2, \rho(b_1)] \dots [a_k, \rho(b_{k-1})]$$

We will now compute the contribution of the second term in Ch_s^n. The computation is similar to the previous one.

First let $k = 0$. In this case we get

$$\frac{(-1)^{\frac{n+1}{2}}\sqrt{2\pi i}}{\left(\frac{n-1}{2}\right)!} \operatorname{Tr} \Xi(1 \otimes b_n, a_0 \otimes b_0)\Xi(1 \otimes b_1, 1 \otimes b_2) \dots \Xi(1 \otimes b_{n-2}, 1 \otimes b_{n-1})$$
$$= \operatorname{Tr} \left(a_0\omega(b_n, b_0) + [a_0, \rho(b_n)]\rho(b_0)\right) \omega(b_1, b_2) \dots \omega(b_{n-2}, b_{n-1}) \quad (5.44)$$

Suppose now that $k \geq 1$. Again $x_0 = a_0 \otimes b_0$. Notice that if x_j has a form $a \otimes 1$ for an odd j, $\Xi(x_j, x_{j+1}) = 0$ (where $x_{n+1} := x_0$) according to (5.35). Hence we need to consider only the shuffles in which terms of the form $a \otimes 1$ appear only in the even places. Let $a_j \otimes 1$ appear in the place number $2i_j$ (i.e. $x_{2i_j} = a_j \otimes 1$) . Here $j \geq 1$, $i_1 \geq 1$, the sequence $\{i_j\}$ is increasing and $i_k \leq \frac{n-1}{2}$. In particular, in this case the result will be nonzero only for $k \leq \frac{n-1}{2}$. Also, notice that x_n should always be equal to $1 \otimes b_l$.

The same argument as before shows that all the shuffles of such form have the same sign of permutation, which equals $(-1)^{\frac{k(k+1)}{2}} = (-1)^{\frac{k(k-1)}{2}}(-1)^k$

Using the formulas (5.33)-(5.36) we compute the expression for the second term, which equals

$$(-1)^k c_{n,k} \sum_{1\le i_1<i_2<\cdots<i_k\le \frac{n-1}{2}} \mathrm{Tr}\left(a_0\omega(b_l,b_0)+[a_0,\rho(b_l)]\rho(b_0)\right)\omega(b_1,b_2)\ldots$$

$$[a_1,\rho(b_{2i_1-1})]\omega(b_{2i_1},b_{2i_1+1})\ldots[a_j,b_{2i_j-j}]\ldots \quad (5.45)$$

For example for $k=\frac{n-1}{2}$ (then $l=k+1$) this expression equals to

$$(-1)^k c_{n,k}(\mathrm{Tr}\left(a_0\omega(b_{k+1},b_0)+[a_0,\rho(b_{k+1})]\rho(b_0)\right)[a_1,\rho(b_1)]\ldots[a_k,\rho(b_k)]))$$
$$=(-1)^k c_{n,k}(\mathrm{Tr}\, a_0\omega(b_{k+1},b_0)[a_1,\rho(b_1)][a_2,\rho(b_j)]\ldots[a_k,\rho(b_k)]$$
$$+\mathrm{Tr}\,\rho(b_0)[a_1,\rho(b_1)][a_2,\rho(b_j)]\ldots[a_k,\rho(b_k)][a_0,\rho(b_{k+1})]) \quad (5.46)$$

Finally subtracting the results just obtained we get the following expression for $Ch_s^n\left(sh(\alpha\otimes\beta)\right)$:

for $k=0$

$$\frac{(-1)^{\frac{n+1}{2}}\sqrt{2\pi i}}{\left(\frac{n-1}{2}\right)!}(\mathrm{Tr}\, a_0\omega(b_0,b_1)\omega(b_2,b_3)\ldots\omega(b_{n-1},b_n)-$$
$$\mathrm{Tr}(a_0\omega(b_n,b_0)+[a_0,\rho(b_n)]\rho(b_0))\omega(b_1,b_2)\ldots\omega(b_{n-2},b_{n-1})$$
$$=\frac{(-1)^{\frac{n+1}{2}}}{\left(\frac{n-1}{2}\right)!}(\mathrm{Tr}\, a_0\left(\omega(b_0,b_1)\ldots\omega(b_{n-1},b_n)-\omega(b_n,b_0)\ldots\omega(b_{n-2},b_{n-1})\right)$$
$$-\mathrm{Tr}\,\rho(b_0)\omega(b_1,b_2)\ldots\omega(b_{n-2},b_{n-1})[a_0,\rho(b_n)]) \quad (5.47)$$

for $k\ge 1$

$$c_{n,k}\Big(\sum_{0\le i_1<i_2<\dots<i_k\le\frac{n-1}{2}} \operatorname{Tr} a_0\omega(b_0,b_1)\dots[a_1,\rho(b_{2i_1})]\omega(b_{2i_1+1},b_{2i_1+2})\dots$$

$$[a_2,\rho(b_{2i_2-1})]\omega(b_{2i_2},b_{2i_2+1})\dots[a_j,\rho(b_{2i_j-j+1})]\dots$$

$$-(-1)^k\sum_{1\le i_1<i_2<\dots<i_k\le\frac{n-1}{2}} \operatorname{Tr}\left(a_0\omega(b_l,b_0)+[a_0,\rho(b_l)]\rho(b_0)\right)\omega(b_1,b_2)\dots$$

$$[a_1,\rho(b_{2i_1-1})]\omega(b_{2i_1},b_{2i_1+1})\dots[a_j,b_{2i_j-j}]\dots\Big) \quad (5.48)$$

where $c_{n,k}=\frac{(-1)^{\frac{n+1}{2}+\frac{k(k-1)}{2}}\sqrt{2\pi i}}{\left(\frac{n-1}{2}\right)!}$

Now we can write explicit formulas for the maps (5.30) and (5.31). Consider, for example, $\{\alpha_{l+2r}\}_{r=0}^{\infty}$, a l-cycle in the negative $b+B$ bicomplex. Then

$$\Psi^{l,n}_{\mathcal{A},\mathcal{B}}(\{\alpha_{l+2r}\}_{r=0}^{\infty})=(\{\phi_{n-l-2r}\}_{\frac{n-1-2l}{4}\le r\le\frac{n-l}{2}}) \quad (5.49)$$

where $\phi_{n-l-2r}\in C^{n-l-2r}(\mathcal{B})$ is given by the formula

$$\phi_{n-l-2r}(\beta)=Ch^n_s\left(sh(\alpha_{l+2r}\otimes\beta)\right) \quad (5.50)$$

and the right hand side comes from (5.47), (5.48)

5.4 Another formula for the duality map.

We will now derive another formula for $Ch^n_s\left(Sh(\alpha\otimes\beta)\right)$. It is based on the identity $Ch^n_s=BC^{n+1}_s$, where

$$C^{n+1}_s(x_0,x_1,\dots,x_n,x_{n+1})$$
$$:=\frac{(-1)^{\frac{n+1}{2}}\sqrt{2\pi i}}{\left(\frac{n+1}{2}\right)!}\operatorname{Tr} s(x_0)\Xi(x_1,x_2)\Xi(x_3,x_4)\dots\Xi(x_n,x_{n+1}) \quad (5.51)$$

These chains satisfy the equation

$$bC_s^l + BC_s^{l+2} = 0 \tag{5.52}$$

We will also need identities for shuffles (in the reduced complex):

$$\begin{aligned} &B\,sh(x\otimes y) - sh(B\,x\otimes y) - (-1)^{\deg x} sh(x\otimes B\,y) = \\ &-\big(b\,sh'(x\otimes y) - sh'(b\,x\otimes y) - (-1)^{\deg x} sh'(x\otimes b\,y)\big) \end{aligned} \tag{5.53}$$

$$B\,sh'(x\otimes y) = sh'(x\otimes B\,y) = sh'(B\,x\otimes y) = 0 \tag{5.54}$$

Hence we have

$$\begin{aligned} &Ch_s^n\,(sh(\alpha\otimes\beta)) = B\,C_s^{n+1}\,(Sh(\alpha\otimes\beta)) = \\ &C_s^{n+1}\,(sh(B\,\alpha\otimes\beta)) + (-1)^{\deg\alpha} C_s^{n+1}\,(sh(\alpha\otimes B\,\beta)) - b\,C_s^{n+1}\,(sh'(\alpha\otimes\beta)) \\ &-C_s^{n+1}\,(sh'(b\,\alpha\otimes\beta)) - (-1)^{\deg\alpha} C_s^{n+1}\,(sh'(\alpha\otimes b\,\beta)) = \\ &C_s^{n+1}\,(sh(B\,\alpha\otimes\beta)) + (-1)^{\deg\alpha} C_s^{n+1}\,(sh(\alpha\otimes B\,\beta)) - B\,C_s^{n+3}\,(sh'(\alpha\otimes\beta)) \\ &-C_s^{n+1}\,(sh'\,((b+B)\,\alpha\otimes\beta)) - (-1)^{\deg\alpha} C_s^{n+1}\,(sh'\,(\alpha\otimes(b+B)\,\beta)) \end{aligned}$$

But $B\,C_s^{n+3}\,(sh'(\alpha\otimes\beta)) = C_s^{n+3}\,(B\,sh'(\alpha\otimes\beta)) = 0$, and hence the result of the computation above is

$$\begin{aligned} &C_s^{n+1}\,(sh(B\,\alpha\otimes\beta)) + (-1)^{\deg\alpha} C_s^{n+1}\,(sh(\alpha\otimes B\,\beta)) - \\ &\qquad - C_s^{n+1}\,(sh'\,((b+B)\,\alpha\otimes\beta)) - (-1)^{\deg\alpha} C_s^{n+1}\,(sh'\,(\alpha\otimes(b+B)\,\beta)) \end{aligned} \tag{5.55}$$

The last two terms are cohomologically trivial, so we need to compute the contribution of the first two terms:

$$C_s^{n+1}\,(sh(B\,\alpha\otimes\beta)) + (-1)^{\deg\alpha} C_s^{n+1}\,(sh(\alpha\otimes B\,\beta)) \tag{5.56}$$

Suppose α and β are homogeneous chains, $\alpha = a_0 \otimes a_1 \otimes \cdots \otimes a_k$, $\beta = b_0 \otimes b_1 \otimes \cdots \otimes b_l$.

Compute first $C_s^{n+1}\left(sh\left((1 \otimes \alpha) \otimes \beta\right)\right)$. In this case $x_0 = 1 \otimes b_0$. Again, use of the formula (5.35) shows that the terms of the form $a \otimes 1$ appear only in the even places. Consider the shuffle in which $x_{2i_j} = a_j \otimes 1$ (here we must have $i_0 \geq 1$). As before, the sign of permutation for this shuffle is independent of the values of i_j, and equals $(-1)^{\frac{(k+2)(k+1)}{2}} = -(-1)^{\frac{k(k-1)}{2}}$. Using the formula (5.51), we obtain the following expression:

$$b_{n,k} \sum_{1 \leq i_0 < i_1 \cdots < i_k \leq \frac{n+1}{2}} \operatorname{Tr} \rho(b_0)\omega(b_1, b_2) \dots [a_0, \rho(b_{2i_0-1})]\omega(b_{2i_0}, b_{2i_0+1}) \dots$$

$$[a_j, \rho(b_{2i_j-j-1})]\omega(b_{2i_j-j}, b_{2i_j-j+1}) \dots \tag{5.57}$$

where

$$b_{n,k} = \frac{(-1)^{\frac{n+1}{2} + \frac{(k+1)(k+2)}{2}} \sqrt{2\pi i}}{(\frac{n+1}{2})!} \tag{5.58}$$

Hence

$$C_s^{n+1}\left(sh(B\,\alpha \otimes \beta)\right) =$$

$$b_{n,k} \sum_{\substack{1 \leq i_0 < i_1 \cdots < i_k \leq \frac{n+1}{2} \\ \lambda-\text{cyclic permutation}}} \operatorname{sgn}(\lambda) \operatorname{Tr} \rho(b_0)\omega(b_1, b_2) \dots [a_{\lambda(0)}, \rho(b_{2i_0-1})]\omega(b_{2i_0}, b_{2i_0+1})$$

$$\dots [a_{\lambda(j)}, \rho(b_{2i_j-j-1})]\omega(b_{2i_j-j}, b_{2i_j-j+1}) \dots \tag{5.59}$$

The second term can be computed similarly, and we get

$$C_s^{n+1}\left(sh(\alpha \otimes B\,\beta)\right)$$
$$= -b_{n,k} \sum_{\substack{1 \le i_1 \cdots < i_k \le \frac{n+1}{2} \\ \sigma - \text{cyclic permutation}}} \operatorname{sgn}(\sigma) \operatorname{Tr} a_0 \omega(b_{\sigma(0)}, b_{\sigma(1)}) \ldots$$
$$[a_1, b_{\sigma(2i_1-2)}] \ldots [a_j, b_{\sigma(2i_j-j-1)}] \ldots \quad (5.60)$$

Combining this two terms we get the resulting formula:

$$b_{n,k}\Big(\sum_{\substack{1 \le i_0 < i_1 \cdots < i_k \le \frac{n+1}{2} \\ \lambda - \text{cyclic permutation}}} \operatorname{sgn}(\lambda) \operatorname{Tr} \rho(b_0)\omega(b_1, b_2) \ldots [a_{\lambda(0)}, \rho(b_{2i_0-1})]$$
$$\omega(b_{2i_0}, b_{2i_0+1}) \ldots [a_{\lambda(j)}, \rho(b_{2i_j-j-1})]\omega(b_{2i_j-j}, b_{2i_j-j+1}) \cdots +$$
$$(-1)^{k+1} \sum_{\substack{1 \le i_1 \cdots < i_k \le \frac{n+1}{2} \\ \sigma - \text{cyclic permutation}}} \operatorname{sgn}(\sigma) \operatorname{Tr} a_0 \omega(b_{\sigma(0)}, b_{\sigma(1)}) \ldots [a_1, b_{\sigma(2i_1-2)}]$$
$$\ldots [a_j, b_{\sigma(2i_j-j-1)}] \ldots \Big) \quad (5.61)$$

where

$$b_{n,k} = \frac{(-1)^{\frac{n+1}{2} + \frac{(k+1)(k+2)}{2}} \sqrt{2\pi i}}{(\frac{n+1}{2})!}$$

5.5 Application to Fredholm modules.

We now apply the duality map to write the formulae for the character of a finitely summable Fredholm module.

Consider first the case of an odd Fredholm module. We suppose that the Fredholm is over a unital algebra $\mathcal{A}$, which is represented on a Hilbert space $\mathcal{H}$, and the Fredholm module is given by the operator F such that

$$[F, a] \in \mathcal{L}^p \tag{5.62}$$

$$1 - F^2 \in \mathcal{L}^p \tag{5.63}$$

Consider a unital algebra E generated by an idempotent e. Consider a map $\rho : E \to \mathcal{L}(\mathcal{H})$ given by

$$\rho(1) = 1 \tag{5.64}$$

$$\rho(e) = \frac{F+1}{2} \tag{5.65}$$

Then it is a homomorphism modulo $\mathcal{L}^p$, and according to the results of the previous sections we have a map

$$P\Psi^0_{E,\mathcal{A}} : HP_0(E) \to HP^1(\mathcal{A}) \tag{5.66}$$

Now, in the periodic cyclic homology of E we consider an element

$$\mathrm{Ch}(e) = e + \sum_{k=1}^{\infty} (-1)^k \frac{(2k)!}{k!} (e - \frac{1}{2}) \otimes e^{\otimes 2k} \tag{5.67}$$

which is the Chern character of the K-theory class of e. Using the duality map we construct an element $P\Psi^0_{E,\mathcal{A}}(\mathrm{Ch}(e)) \in HP^1(\mathcal{A})$. We can write an explicit formula for it , using the results of the previous section.

Proposition 49. *The following $2m+1$ cocycle with components $\phi_k(F)$ represents the class of $P\Psi^0_{E,\mathcal{A}}(\mathrm{Ch}(e))$, where $2m \geq p-1$*

$$\phi_k(F)(a_0, a_1 \ldots, a_k) =$$

$$\sqrt{2\pi i}\Big(\frac{(4m+1-k)!}{(2m+1)!(2m-\frac{k-1}{2})!2^{4m-k+2}} \sum_{\substack{j_0+j_1+\ldots j_{k+1}=2m-k \\ \lambda-\textit{cyclic permutation}}} \mathrm{Tr}\, F(1-F^2)^{j_0}[a_{\lambda(0)}, F]$$

$$(1-F^2)^{j_1} \ldots [a_{\lambda(k)}, F](1-F^2)^{j_{k+1}} +$$

$$\frac{(4m+2-k)!}{(2m+1)!(2m-\frac{k-1}{2})!2^{4m-k+2}} \sum_{j_1+j_2+\cdots+j_{k+1}=2m-k+1} \mathrm{Tr}\, a_0(1-F^2)^{j_1}[a_1, F]$$

$$(1-F^2)^{j_2} \ldots [a_k, F](1-F^2)^{j_{k+1}}\Big) \quad (5.68)$$

The same class can also be represented by the $2m-1$ cocycle with component $\phi'_k(F)$, where $2m \geq p$

$$\phi'_k(F)(a_0, a_1, \ldots a_k) =$$

$$-\sqrt{2\pi i}\Big(\frac{(4m-1-k)!}{(2m)!(2m-\frac{k+1}{2})!2^{4m-k}} \sum_{\substack{j_0+j_1+\ldots j_{k+1}=2m-k-1 \\ \lambda-\textit{cyclic permutation}}} \mathrm{Tr}\, F(1-F^2)^{j_0}$$

$$[a_{\lambda(0)}, F](1-F^2)^{j_1} \ldots [a_{\lambda(k)}, F](1-F^2)^{j_{k+1}} +$$

$$\frac{(4m-k)!}{(2m)!(2m-\frac{k+1}{2})!2^{4m-k}} \sum_{j_1+j_2+\cdots+j_{k+1}=2m-k} \mathrm{Tr}\, a_0(1-F^2)^{j_1}[a_1, F](1-F^2)^{j_2} \ldots$$

$$[a_k, F](1-F^2)^{j_{k+1}}\Big) \quad (5.69)$$

Proof. We are going to apply the formula (5.61). Choose an odd number $n \geq 2p-1$. We do the computation in the case $n = 4m+1$, which gives the first formula. The second formula appears in the case $n = 4m-1$. We notice first that we can rewrite the formula for $\mathrm{Ch}(e)$ as

$$\mathrm{Ch}(e) = e + \sum_{k=1}^{\infty} (-1)^k \frac{(2k)!}{k!} (e - \frac{1}{2})^{\otimes(2k+1)} \tag{5.70}$$

since we are working in the normalized complex. We also have

$$\rho\left(e - \frac{1}{2}\right) = \frac{F}{2} \tag{5.71}$$

$$\omega\left(\left(e - \frac{1}{2}\right), \left(e - \frac{1}{2}\right)\right) = \frac{1}{4}\left(1 - F^2\right) \tag{5.72}$$

Then application of the formula (5.61) gives

$$\begin{aligned}
&\phi_k(a_0, a_1, \ldots, a_k) = \\
&2^{-(4m-k+2)} b_{4m+1,k} \Big(\sum_{\substack{1 \le i_0 < i_1 \cdots < i_k \le 2m+1 \\ \lambda-\text{cyclic permutation}}} \mathrm{sgn}(\lambda) \operatorname{Tr} F(1-F^2)^{i_0-1}[a_{\lambda(0)}, F] \\
&(1-F^2)^{i_1-i_0-1} \ldots (1-F^2)^{i_j-i_{j-1}-1}[a_{\lambda(j)}, F](1-F^2)^{i_{j+1}-i_j-1} \ldots (1-F^2)^{2m+1-i_k} + \\
&(-1)^{k+1}(4m+2-k) \sum_{1 \le i_1 \cdots < i_k \le 2m+1} \operatorname{Tr} a_0(1-F^2)^{i_1-1}[a_1, F] \\
&\ldots (1-F^2)^{i_j-i_{j-1}-1}[a_j, F](1-F^2)^{i_{j+1}-i_j-1} \ldots \Big) = \\
&2^{-(4m-k+2)} b_{4m+1,k} \Big((\sum_{\substack{l_j \ge 0; l_0+l_1+\ldots l_{k+1} = 2m-k \\ \lambda-\text{cyclic permutation}}} \mathrm{sgn}(\lambda) \operatorname{Tr} F(1-F^2)^{l_0}[a_{\lambda(0)}, F] \\
&(1-F^2)^{l_1} \ldots (1-F^2)^{l_j}[a_{\lambda(j)}, F](1-F^2)^{l_{j+1}} \ldots (1-F^2)^{l_{k+1}} + \\
&(-1)^{k+1}(4m+2-k) \sum_{l_j \ge 0; l_1+l_2+\ldots l_{k+1} = 2m+1-k} \operatorname{Tr} a_0(1-F^2)^{l_1}[a_1, F] \\
&\ldots (1-F^2)^{l_j}[a_j, F](1-F^2)^{l_{j+1}} \ldots (1-F^2)^{l_{k+1}} \Big)
\end{aligned} \tag{5.73}$$

□

The map $P\Psi^0_{E,\mathcal{A}}$ depends only on the homomorphism $\widetilde{\rho} : E \to \mathcal{L}(\mathcal{H})/\mathcal{L}^p$, not on the map ρ itself. This implies the following

Lemma 50. *Let operator F define a p-summable Fredholm module over $\mathcal{A}$ and F' be an operator such that $F' - F \in \mathcal{L}^p$. Then F' also satisfies* (5.62), (5.63) *and*

$$[\phi(F)] = [\phi(F')] = [\phi'(F)] = [\phi'(F')] \tag{5.74}$$

where $[\cdot]$ denotes the class in the periodic cyclic cohomology.

Proof. It is immediate that F' satisfies (5.62), (5.63). Define the map $\rho' : E \to \mathcal{L}(\mathcal{H})$ by

$$\rho'(1) = 1 \tag{5.75}$$

$$\rho'(e) = \frac{F' + 1}{2} \tag{5.76}$$

Then ρ and ρ' clearly define the same homomorphisms $E \to \mathcal{L}(\mathcal{H})/\mathcal{L}^p$, and hence the maps $P\Psi^0_{E,\mathcal{A}}$, and $\left(P\Psi^0_{E,\mathcal{A}}\right)'$ constructed using ρ and ρ' coincide. □

We will now prove that the class of the cyclic cocycles ϕ and ϕ' defined above is actually the Chern character of the Fredholm module.

Theorem 51. *Let the operator F define a Fredholm module over an algebra $\mathcal{A}$. Then*

$$[\phi(F)] = [\phi'(F)] = \mathrm{Ch}(F) \tag{5.77}$$

where $\mathrm{Ch}(F)$ is the character of the Fredholm module as defined by Connes.

Proof. First, we notice that when $F^2 = 1$, our formulae are identical with the Connes formula, and hence define the Chern character. The Theorem now follows from the Lemma 5.5 exactly as the Theorem 46 follows from the Theorem 43. $\square$

Formulas for the even case can be obtained by suspension. The final result is given by

Proposition 52. *Let $(\mathcal{H}, F, \gamma)$ be a p-summable Fredholm module. Then the following $2m$ cocycle with components $\phi_k(F)$ represents the class of its Chern character, where $2m \geq p$*

$$\phi_k(F)(a_0, a_1 \dots, a_k) =$$
$$\Big(-\frac{(2m-k/2-1)!}{2(2m)!} \sum_{\substack{j_0+j_1+\dots j_{k+1}=2m-k-1 \\ \lambda-\textit{cyclic permutation}}} \operatorname{Tr} \gamma F (1-F^2)^{j_0} [a_{\lambda(0)}, F]$$
$$(1-F^2)^{j_1} \dots [a_{\lambda(k)}, F](1-F^2)^{j_{k+1}} +$$
$$\frac{(2m-k/2)!}{(2m)!} \sum_{j_1+j_2+\dots+j_{k+1}=2m-k} \operatorname{Tr} \gamma a_0 (1-F^2)^{j_1} [a_1, F](1-F^2)^{j_2} \dots$$
$$[a_k, F](1-F^2)^{j_{k+1}}\Big) \quad (5.78)$$

The same class can also be represented by the cocycle with component $\phi'_k(F)$

$$\phi'_k(F)(a_0, a_1, \dots a_k) = \Big(-\frac{(2m-k/2)!}{2(2m+1)!} \sum_{\substack{j_0+j_1+\dots j_{k+1}=2m-k \\ \lambda-\textit{cyclic permutation}}} \operatorname{Tr} \gamma F (1-F^2)^{j_0} [a_{\lambda(0)}, F]$$
$$(1-F^2)^{j_1} \dots [a_{\lambda(k)}, F](1-F^2)^{j_{k+1}} +$$
$$\frac{(2m-k/2+1)!}{(2m+1)!} \sum_{j_1+j_2+\dots+j_{k+1}=2m-k+1} \operatorname{Tr} \gamma a_0 (1-F^2)^{j_1} [a_1, F](1-F^2)^{j_2} \dots$$
$$[a_k, F](1-F^2)^{j_{k+1}}\Big) \quad (5.79)$$

Appendix A

CHARACTERISTIC MAP FOR WEAK ACTIONS OF LIE ALGEBRAS.

In [Con80], [Con86] Connes constructed characteristic map for the action of the Lie algebra on an associative algebra, equipped with a trace. These results where further extended in one direction by Nest and Tsygan in [NT95a, NT95b, NT99] where they construct operations on the cyclic complex, and in another direction by Connes and Moscovici [CM98, CM], to the case of Hopf algebra action. Here we extend Connes original construction to the case of the *weak* actions of the Lie algebra. This construction should also admit generalizations along the lines of Nest, Tsygan and Connes, Moscovici.

Let $\mathfrak{g}$ be a Lie algebra, and $\mathcal{A}$ be an associative algebra. We suppose for simplicity that $\mathcal{A}$ is unital, but everything in this section extends to the nonunital case along the lines of Section 3.3.

Suppose we are given a linear map

$$\lambda : \mathfrak{g} \to \mathrm{Der}(\mathcal{A}) \tag{A.1}$$

where $\mathrm{Der}(\mathcal{A})$ is a space of derivations of $\mathcal{A}$ and a bilinear sqewsymmetric map

$$\theta : \Lambda^2\mathfrak{g} \to \mathcal{A} \tag{A.2}$$

satisfying

$$[\lambda(l_1), \lambda(l_2)] - \lambda([l_1, l_2]) = \operatorname{ad}\theta(l_1, l_2) \tag{A.3}$$

and

$$\begin{aligned}\lambda(l_1)\theta(l_2, l_3) + \lambda(l_2)\theta(l_3, l_1) + \lambda(l_3)\theta(l_1, l_2) - \\ \theta([l_1, l_2], l_3) - \theta([l_3, l_1], l_2) - \theta([l_2, l_3], l_1) = 0\end{aligned} \tag{A.4}$$

If the conditions above are satisfied, we say that $\mathfrak{g}$ *weakly acts* on $\mathcal{A}$.

Suppose also we have a trace τ on $\mathcal{A}$ satisfying

$$\tau(\lambda(l)a) = \delta(l)\tau(a) \tag{A.5}$$

where $\delta : \mathfrak{g} \to \mathbb{C}$ is some character of the Lie algebra $\mathfrak{g}$.

Then to this data one associates a generalized chain $\mathcal{C}(\mathcal{X})$ of degree n to every polyvector $\mathcal{X} = X_1 \wedge X_2 \dots X_n \in \Lambda^n \mathfrak{g}$.

Indeed, let Ω^k be the space of polylinear sqewsymmetric maps $\Lambda^k \mathfrak{g} \to \mathcal{A}$ with the product ($\omega_1 \in \Omega^p$, $\omega_2 \in \Omega^q$)

$$\omega_1\omega_2(l_1, \dots, l_{p+q}) = \sum \operatorname{sgn}\sigma\omega_1(l_{\sigma(1)}, l_{\sigma(2)}, \dots, l_{\sigma(p)})\omega_2(l_{\sigma(p+1)}, l_{\sigma(p+2)} \dots, l_{\sigma(p+q)}) \tag{A.6}$$

whre summation is over all the permutations σ satisfying $\sigma(1) < \sigma(2) < \dots < \sigma(p)$, $\sigma(p+1) < \sigma(p+2) < \dots < \sigma(p+q)$

We define $\partial\Omega = \Omega$, and the restriction map r is the identity map.

The graded derivation ∇ is defined by $(\omega \in \Omega^p)$

$$\nabla\omega(l_1, \dots l_{p+1}) = \sum(-1)^{j-1}\lambda(l_j)\omega(l_1, \dots, \hat{l}_j, \dots l_{p+1}) + \sum_{i<j}(-1)^{i+j-1}\omega([l_i, l_j], l_1, \dots, \hat{l}_i, \dots \hat{l}_j, \dots, l_{p+1}) \quad \text{(A.7)}$$

The fact that ∇ is a derivation is verified by the standard computation.

We can take θ defined above as the curvature of our chain, as the following two propositions show.

Proposition 53.

$$\nabla\theta = 0 \quad \text{(A.8)}$$

Proof. We have

$$\begin{aligned}
&\nabla\theta(l_1, l_2, l_3) = \\
&\lambda(l_1)\theta(l_2, l_3) - \lambda(l_2)\theta(l_1, l_3) + \lambda(l_3)\theta(l_1, l_2) - \theta([l_1, l_2], l_3) + \theta([l_1, l_3], l_2) - \theta([l_2, l_3], l_1) = \\
&\lambda(l_1)\theta(l_2, l_3) + \lambda(l_2)\theta(l_3, l_1) + \lambda(l_3)\theta(l_1, l_2) - \theta([l_1, l_2], l_3) - \theta([l_3, l_1], l_2) - \theta([l_2, l_3], l_1) = 0
\end{aligned} \quad \text{(A.9)}$$

□

Proposition 54. *Consider θ from above as an element of Ω^2. Then $\nabla^2 = \operatorname{ad}\theta$.*

Proof. We again verify this equality by a direct computation.

$$
\begin{aligned}
&\nabla^2\omega(l_1,\dots l_{p+2}) = \\
&\sum(-1)^{j-1}\lambda(l_j)\nabla\omega(l_1,\dots,\hat{l}_j,\dots l_{p+2}) + \\
&\sum_{i<j}(-1)^{i+j}\nabla\omega([l_i,l_j],l_1,\dots,\hat{l}_i,\dots,\hat{l}_j,\dots,l_{p+2}) = \\
&\sum_{i<j}(-1)^{i+j}[\lambda(l_j),\lambda(l_i)]\omega(l_1,\dots,\hat{l}_i,\dots,\hat{l}_j,\dots l_{p+2}) + \\
&\sum(-1)^{k-1}\lambda(l_k)\sum_{i<j,\ i,j\neq k}(-1)^{i_k+j_k}\omega([l_i,l_j]l_1,\dots,\hat{l}_i,\dots,\hat{l}_j,\dots,\hat{l}_k\dots l_{p+2}) + \\
&\sum_{i<j}(-1)^{i+j}\lambda([l_i,l_j])\omega(l_1,\dots,\hat{l}_i,\dots,\hat{l}_j,\dots,l_{p+2}) + \\
&\sum_{i<j}(-1)^{i+j}\sum_{k\neq i,j}(-1)^{k_{ij}}\lambda(l_k)\omega([l_i,l_j],l_1,\dots,\hat{l}_i,\dots,\hat{l}_j,\dots,\hat{l}_k\dots,l_{p+2}) + \\
&\sum_{i<j}(-1)^{i+j+k_{ij}}\omega([[l_i,l_j],l_k],l_1,\dots,\hat{l}_i,\dots,\hat{l}_j,\dots,\hat{l}_k\dots,l_{p+2}) + \\
&\sum_{i<j}(-1)^{i+j}\sum_{k<r}(-1)^{k_{ij}+r_{ij}}\omega([l_i,l_j],[l_k,l_r],l_1,\dots,\hat{l}_i,\dots,\hat{l}_j,\dots,\hat{l}_k\dots,\hat{l}_r\dots,l_{p+2})
\end{aligned}
\tag{A.10}
$$

Here we used the following notations

$$
i_k = \begin{cases} i & \text{if } i<k \\ i-1 & \text{if } i>k \end{cases},\ j_k = \begin{cases} j & \text{if } j<k \\ j-1 & \text{if } j>k \end{cases} \text{ and}
$$

$$
k_{ij} = \begin{cases} k & \text{if } k<i \\ k-1 & \text{if } i<k<j \\ k-2 & \text{if } j<k \end{cases},\ r_{ij} = \begin{cases} r & \text{if } r<i \\ r-1 & \text{if } i<r<j \\ r-2 & \text{if } j<r \end{cases}
$$

It is easy to see that

$i_k + j_k + k = i + j + k_{ij}$ and this implies that

$$\sum (-1)^{k-1}\lambda(l_k) \sum_{i<j,\ i,j\neq k} (-1)^{i_k+j_k}\omega([l_i,l_j]l_1,\ldots,\hat{l}_i,\ldots,\hat{l}_j,\ldots,\hat{l}_k\ldots l_{p+2}) +$$

$$\sum_{i<j}(-1)^{i+j}\sum_{k\neq i,j}(-1)^{k_{ij}}\lambda(l_k)\omega([l_i,l_j],l_1,\ldots,\hat{l}_i,\ldots,\hat{l}_j,\ldots,\hat{l}_k\ldots,l_{p+2}) = 0 \quad \text{(A.11)}$$

Next, Jacobi identity implies that

$$\sum_{i<j}(-1)^{i+j+k_{ij}}\omega([[l_i,l_j],l_k],l_1,\ldots,\hat{l}_i,\ldots,\hat{l}_j,\ldots,\hat{l}_k\ldots,l_{p+2}) = 0 \quad \text{(A.12)}$$

Finally, the last term is 0 due to the antisymmetry property of ω. So we are left with

$$\sum_{i<j}(-1)^{i+j}[\lambda(l_j),\lambda(l_i)]\omega(l_1,\ldots,\hat{l}_i,\ldots,\hat{l}_j,\ldots l_{p+2}) +$$

$$\sum_{i<j}(-1)^{i+j}\lambda([l_i,l_j])\omega(l_1,\ldots,\hat{l}_i,\ldots,\hat{l}_j,\ldots,l_{p+2}) =$$

$$-\sum_{i<j}(-1)^{i+j}\,\mathrm{ad}\,\theta(l_i,l_j)\omega(l_1,\ldots,\hat{l}_i,\ldots,\hat{l}_j,\ldots,l_{p+2}) = [\theta,\omega](l_1,\ldots,l_{p+2}) \quad \text{(A.13)}$$

□

The graded derivation ∇' coincides with ∇.

Finally, the graded trace $\displaystyle⨍_{\mathcal{X}}$ is given by

$$⨍_{\mathcal{X}}\omega = \tau(\omega(X_1,X_2,\ldots,X_k)) \quad \text{(A.14)}$$

Proposition 55. *Let $\partial\mathcal{X}$ denote the boundary of $\mathcal{X}$ considered as an element of the* *complex of the Lie algebra homology of $\mathfrak{g}$ with coefficients in the module $\mathbb{C}$ with the action given by δ:*

$$\partial(X_1 \wedge X_2 \cdots \wedge X_n) = \sum_{i=1}^{n} (-1)^{i-1} \delta(X_i) X_1 \wedge X_2 \ldots \widehat{X}_i \cdots \wedge X_n + \sum_{i<j} (-1)^{i+j} [X_i, X_j] \wedge X_1 \wedge X_2 \ldots \widehat{X}_i \ldots \widehat{X}_j \cdots \wedge X_n \quad \text{(A.15)}$$

Then ($\deg \omega = n - 1$)

$$⨍_{\mathcal{X}} \nabla\omega = ⨍_{\partial\mathcal{X}} \omega \quad \text{(A.16)}$$

Proof. We have

$$⨍_{\mathcal{X}} \nabla\omega = \tau(\nabla\omega(X_1 \wedge X_2 \cdots \wedge X_n)) = \tau(\sum (-1)^{j-1} \lambda(X_j) \omega(X_1, \ldots, \hat{X}_j, \ldots X_n) + \sum_{i<j} (-1)^{i+j-1} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots \hat{X}_j, \ldots, X_n)) = \sum (-1)^{j-1} \delta(X_j) \tau(\omega(X_1, \ldots, \hat{X}_j, \ldots X_n)) + \sum_{i<j} (-1)^{i+j-1} \tau(\omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots \hat{X}_j, \ldots, X_n)) = ⨍_{\partial\mathcal{X}} \omega \quad \text{(A.17)}$$

□

We immediately have the following Corollaries:

Corollary 56.

$$\partial\mathcal{C}(\mathcal{X}) = \mathcal{C}(\partial\mathcal{X}) \quad \text{(A.18)}$$

Corollary 57. *Let $\mathcal{X}$ be a cycle in the Lie algebra complex. Then $\mathcal{C}(\mathcal{X})$ is a generalized cycle.*

Corollary 58. *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be homologous. Then the cycles $\mathcal{C}(\mathcal{X}_1)$ and $\mathcal{C}(\mathcal{X}_2)$ are cobordant.*

Consideration of the character of the cycle $\mathcal{C}(\mathcal{X})$ gives the following:

Theorem 59. *The map $\mathcal{X} \to \mathrm{Ch}(\mathcal{C}(\mathcal{X}))$ is a map of Lie algebra homology complex to the periodic bicomplex of the algebra $\mathcal{A}$.*

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