CALCULUS 3

October 13, 2010

2nd TEST

YOUR NAME:

○ **002** J. HARPER . . . . (9AM)
○ **003** C. MESA . . . . (10AM)
○ **004** A. SPINA . . . . (11AM)
○ **005** A. SPINA . . . . (12PM)
○ **006** M. STACKPOLE . . . . (1PM)
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SHOW ALL YOUR WORK

final answers without any supporting work
will receive no credit *even if they are right!*

No calculators allowed.
No cheat-sheets allowed.

Partial credit will be given for any **reasonable amount of work pointing in the right direction** towards the solution of your problem. You will not get any partial credit for memorizing formulas and not knowing how to use them, or for anything you write that is not directly related to the solution of your problem.

If your tests contains **more than one solution or answer** to a problem or part of a problem, and one of them is wrong, then it will be **the wrong one** the one that **counts** for your grading!

DO NOT WRITE INSIDE THIS BOX!

problem	points	score
1	15 pts	
2	15 pts	
3	15 pts	
4	15 pts	
5	15 pts	
6	10 pts	
7	15 pts	
TOTAL	100 pts	

1. **[15 pts]** Let $\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$ be the position vector of the point (x, y, z). Show that, if a is a constant,
$$\operatorname{grad}\left(\|\vec{r}\|^a\right) = a\,\|\vec{r}\|^{a-2}\,\vec{r}\,, \qquad \vec{r} \neq \vec{0}\,.$$

SOLUTION:

$$\begin{aligned}
\operatorname{grad}\left(\|\vec{r}\|^a\right) &= \operatorname{grad}\left(x^2+y^2+z^2\right)^{a/2} \\
&= \frac{a}{2}\left(x^2+y^2+z^2\right)^{a/2-1}\left(2x\vec{\imath}+2y\vec{\jmath}+2z\vec{k}\right) \\
&= \frac{a}{2}\left(x^2+y^2+z^2\right)^{(a-2)/2} 2\left(x\vec{\imath}+y\vec{\jmath}+z\vec{k}\right) \\
&= a\left(\sqrt{x^2+y^2+z^2}\right)^{a-2}\vec{r} \\
&= a\|\vec{r}\|^{a-2}\vec{r}
\end{aligned}$$

2. **[15 pts]** Approximate π using
$$\pi = f(C, r) = \frac{C}{2r}\,,$$
where the circumference of the circle C and the radius r are given by
$$C = 60 \pm 6\,\text{cm}\,, \qquad \text{and} \qquad r = 10 \pm 1\,\text{cm}\,.$$
Estimate the maximum error in your approximation.

SOLUTION:

$$\boxed{\pi \approx f(60, 10) = \frac{60}{2\times 10} = 3.0}$$

The change in f can be approximated by the differential

$$\begin{aligned}
\Delta f \;\approx\; \mathrm{d}f &= \left.\frac{\partial f}{\partial C}\right|_{(C_0,r_0)}\Delta C + \left.\frac{\partial f}{\partial r}\right|_{(C_0,r_0)}\Delta r \\
&= \frac{1}{2r_0}\,\Delta C - \frac{C_0}{2r_0^2}\,\Delta r\,.
\end{aligned}$$

A bound on the error ε can be found by writing

$$\boxed{\varepsilon = \left|\frac{1}{2r_0}\,\Delta C - \frac{C_0}{2r_0^2}\,\Delta r\right| \le \left|\frac{1}{2r_0}\right|\,|\Delta C| + \left|\frac{C_0}{2r_0^2}\right|\,|\Delta r| = \varepsilon_{\max}}$$

Using
$$\begin{aligned}
C_0 &= 60\,, & |\Delta C| &= 6\,, \\
r_0 &= 10\,, & |\Delta r| &= 1\,,
\end{aligned}$$
we get
$$\varepsilon_{\max} = \frac{1}{2\times 10}\times 6 + \frac{60}{2\times 10^2}\times 1 = 0.3 + 0.3 = 0.6\,,$$
that is
$$\boxed{\pi = 3.0 \pm 0.6}$$

3. **[15 pts]** Given $z = f(x, y)$, $x = x(u, v)$, $y = y(u, v)$ and $x(1, 2) = 5$, $y(1, 2) = 3$, calculate z_u at $(u, v) = (1, 2)$ in terms of some of the numbers p, m, c, t, a, k, h, q, where
$$\begin{aligned}
&f_x(1,2) = p\,, \quad f_y(1,2) = c\,, \quad x_u(1,2) = a\,, \quad y_u(1,2) = h\,, \\
&f_x(5,3) = m\,, \quad f_y(5,3) = t\,, \quad x_v(1,2) = k\,, \quad y_v(1,2) = q\,.
\end{aligned}$$

SOLUTION:

$$z = f(x,y) \qquad x = x(u,v) \qquad y = y(u,v)$$

$$\Rightarrow \quad \frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\boxed{z_u(1,2) = f_x(5,3)\,x_u(1,2) + f_y(5,3)\,y_u(1,2) = m \times a + t \times h}$$

4. **[15 pts]** The equation $F(x,y,z) = k$ defines z implicitly as a function of x and y. Find z_x and z_y in terms of F_x, F_y and F_z.

SOLUTION:

1st way.

$$F(x,y,z) \equiv 0 \quad \Rightarrow \quad \Delta F = \left(\frac{\partial F}{\partial x}\right)_{y,z}\Delta x + \left(\frac{\partial F}{\partial y}\right)_{x,z}\Delta y + \left(\frac{\partial F}{\partial z}\right)_{x,y}\Delta z = 0$$

$$\Rightarrow \quad \Delta z = -\frac{(\partial F/\partial x)_{y,z}}{(\partial F/\partial z)_{x,y}}\Delta x - \frac{(\partial F/\partial y)_{x,z}}{(\partial F/\partial z)_{x,y}}\Delta y$$

$$\Rightarrow \quad \boxed{\begin{array}{ccccccc} z_x & \equiv & \left(\dfrac{\partial z}{\partial x}\right)_y & = & -\dfrac{(\partial F/\partial x)_{y,z}}{(\partial F/\partial z)_{x,y}} & = & -\dfrac{F_x}{F_z} \\ z_y & \equiv & \left(\dfrac{\partial z}{\partial y}\right)_x & = & -\dfrac{(\partial F/\partial y)_{x,z}}{(\partial F/\partial z)_{x,y}} & = & -\dfrac{F_y}{F_z} \end{array}}$$

2nd way.

We can think the loss of z as an independent variable as

$$F(X(x,y), Y(x,y), Z(x,y)) = k\,, \quad \text{with} \quad X(x,y) = x\,, \quad Y(x,y) = y\,.$$

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial F}{\partial Y}\frac{\partial Y}{\partial x} + \frac{\partial F}{\partial Z}\frac{\partial Z}{\partial x} = \frac{\partial F}{\partial X} + \frac{\partial F}{\partial Z}\frac{\partial Z}{\partial x} = 0 \qquad \Rightarrow \qquad \frac{\partial Z}{\partial x} = -\frac{\partial F/\partial X}{\partial F/\partial Z}$$

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial F}{\partial Y}\frac{\partial Y}{\partial y} + \frac{\partial F}{\partial Z}\frac{\partial Z}{\partial y} = \frac{\partial F}{\partial Y} + \frac{\partial F}{\partial Z}\frac{\partial Z}{\partial y} = 0 \qquad \Rightarrow \qquad \frac{\partial Z}{\partial y} = -\frac{\partial F/\partial Y}{\partial F/\partial Z}$$

Going back to the original variables we have

$$\boxed{z_x = -\frac{F_x}{F_z} \qquad \text{and} \qquad z_y = -\frac{F_y}{F_z}}$$

5. **[15 pts]** The surface S is represented by the equation $w = 0$, where $w = F(x,y,z) = x^2 - \dfrac{y}{z^2}\cdot$ Find an equation for the tangent plane to S at the point $(1,1,1)$.

SOLUTION:

An equation for a plane with normal $\vec{n}$ and going through $\vec{r}_0$ is

$$\vec{n}\cdot(\vec{r} - \vec{r}_0) = 0\,,$$

where

$$\vec{n} = n_1\vec{i} + n_2\vec{j} + n_3\vec{k}\,, \qquad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}\,, \qquad \text{and} \qquad \vec{r}_0 = x_0\vec{i} + y_0\vec{j} + z_0\vec{k}\,.$$

For our problem,

$$\vec{r}_0 = \vec{i} + \vec{j} + \vec{k}\,,$$

and for the normal vector $\vec{n}$ we can take

$$\mathbf{n} = \operatorname{grad} F(1,1,1)\,,$$

where

$$\begin{aligned} \operatorname{grad} F(1,1,1) &= F_x(1,1,1)\vec{i} + F_y(1,1,1)\vec{j} + F_z(1,1,1)\vec{k} \\ &= \left[2x\vec{i} - \frac{1}{z^2}\vec{j} + \frac{2y}{z^3}\vec{k}\right]_{(x,y,z)=(1,1,1)} \\ &= 2\vec{i} - \vec{j} + 2\vec{k}\,. \end{aligned}$$

Hence, an equation for the plane is

$$\boxed{2(x-1) - (y-1) + 2(z-1) = 0} \qquad \text{or} \qquad \boxed{2x - y + 2z = 3}$$

6. **[10 pts]** Compute the critical points of $f(x,y) = 2x^2 - 3xy + 8y^2 + x - y$ and classify them.

SOLUTION:

$$\begin{aligned} \frac{\partial f}{\partial x} &= 4x - 3y + 1 = 0 \\ \frac{\partial f}{\partial y} &= -3x + 16y - 1 = 0 \end{aligned} \qquad \Rightarrow \qquad \boxed{(x_c, y_c) = \left(-\frac{13}{55}, \frac{1}{55}\right)}$$

$$\boxed{f_{xx}\, f_{yy} - f_{xy}^2 = 4 \times 16 - 9 = 55 > 0 \qquad \Rightarrow \qquad \left(-\frac{13}{55}, \frac{1}{55}\right) \text{ is a } \textbf{local minimum}}$$

7. **[15 pts]** Use the method of *Lagrange multipliers* to find the maximum and minimum values of $f(x,y) = x^3 - y^2$ subject to the constraint $g(x,y) = x^2 + y^2 = 1$.

SOLUTION:

$$\begin{cases} \operatorname{grad} f = \lambda \operatorname{grad} g \\ g(x,y) = 1 \end{cases} \qquad \Rightarrow \qquad \begin{cases} 3x^2 = 2\lambda x \\ -2y = 2\lambda y \\ x^2 + y^2 = 1\,. \end{cases}$$

From the second equation $y = 0$ or $\lambda = -1$.

- $y = 0 \quad \Rightarrow \quad x = \pm 1\,, \quad \Rightarrow \quad \boxed{(1,0)\,, \quad (-1,0)}$
- $y \neq 0 \ \ \& \ \ \lambda = -1 \quad \Rightarrow \quad 3x^2 = -2x \quad \Rightarrow \quad x = 0 \ \text{ or } \ x = -\frac{2}{3}\,.$
  - $x = 0 \quad \Rightarrow \quad y^2 = 1 \quad \Rightarrow \quad \boxed{(0,1)\,, \quad (0,-1)}$
  - $x = -\frac{2}{3} \quad \Rightarrow \quad y^2 = \frac{5}{9} \quad \Rightarrow \quad \boxed{\left(-\frac{2}{3}, \frac{\sqrt{5}}{3}\right), \quad \left(-\frac{2}{3}, -\frac{\sqrt{5}}{3}\right)}$

Evaluating f at these points we get

$$f(1,0) = 1\,, \qquad f(-1,0) = f(0,\pm 1) = -1\,, \qquad f\left(-\frac{2}{3}, \pm\frac{\sqrt{5}}{3}\right) = -\frac{23}{27}\,.$$

Therefore,

$$\boxed{\min_{x^2+y^2=1} f(x,y) = -1\,, \qquad \text{at} \qquad (-1,0) \quad \text{and} \quad (0,\pm 1)}$$