SPACE-TIME–TIME: FIVE-DIMENSIONAL KALUZA–WEYL SPACE

Homer G. Ellis

Abstract. Space-time–time couples Kaluza's five-dimensional geometry with Weyl's conformal space-time geometry to produce an extension of space-time theory that goes beyond what either of the Weyl and the Kaluza theories can achieve by itself. Kaluza's "cylinder condition" is replaced by an "exponential expansion constraint" that causes translations along the secondary time dimension to induce both the electromagnetic gauge transformations found in the Kaluza and the Weyl theories and the metrical gauge transformations unique to the Weyl theory, related exactly as Weyl had postulated. A space-time–time geodesic describes a test particle whose rest mass $\mathring{m}$, space-time momentum $\mathring{m}u^\mu$, and electric charge q, all defined kinematically, evolve in accord with definite dynamical laws. Its motion, projected onto space-time, is governed by four apparent forces: the Einstein gravitational force, the Lorentz electromagnetic force, a force proportional to the electromagnetic four-potential, and a force proportional to a scalar field's gradient $d(\ln\phi)$. The particle appears suddenly at an event $\mathcal{E}_1$ with $q = -\phi(\mathcal{E}_1)$ and disappears at an event $\mathcal{E}_2$ with $q = \phi(\mathcal{E}_2)$. At $\mathcal{E}_1$ and $\mathcal{E}_2$ the gradient force infinitely dominates the others, causing $\mathcal{E}_1$ and $\mathcal{E}_2$ to occur preferentially in the valley depths of the potential $\ln\phi$ — this suggests the possibility of explaining some aspects of atomic structure without invoking quantum theory. Test particles with $\mathring{m} = 0$ and $q \neq 0$ can exist, but must follow paths p for which $\phi(p) = \text{const}$, and must have $q = \pm\phi(p)$. A ϕ-wave form with a null propagation vector can carry such particles from place to place at the speed of light, keeping them immune to the influence of an accompanying electromagnetic wave form. Test particles sharing a common event $\mathcal{E}$ of appearance, with $q = -\phi(\mathcal{E})$, or disappearance, with $q = \phi(\mathcal{E})$, can be made to "interact" by demanding that the sum of their space-time–time momenta at $\mathcal{E}$ vanish. This space-time–time conservation law would comprise for such interactions both conservation of space-time momentum and conservation of electric charge.

1. INTRODUCTION

The theory I am going to describe here employs Kaluza's five-dimensional geometry of 1919 [1], altered to a form that encompasses Weyl's conformal space-time geometry of 1918 [2]. These two early geometrical enlargements of Einstein's space-time theory of gravity to include Maxwell's theory of electromagnetism were somewhat successful, each in its own way, but they bore no apparent relation to one another. Properly joined, they make a theory, the theory of "space-time–time," that goes well beyond what either is able to achieve by itself, and that differs essentially from standard gauge theories of Kaluza–Klein type.

Weyl, to take into account the freedom to specify arbitrarily at each space-time event a scale against which to measure the lengths of tangent vectors at that event, enlarged the study of individual space-time metrics to the study of whole families of conformally related space-time metrics. He postulated that transport of a tangent vector keeping its covariant derivative equal to zero need not preserve its length with respect to any of these metrics. The consequent "nonintegrability" of lengths of vectors transported in this manner around closed circuits he ascribed to inexactness of an electromagnetic covector (1-form) potential A

whose exterior derivative $d_\wedge A$ manifests as an electromagnetic field. Conformal transitions $G \to e^{2\lambda} G$ between metrics coincided with transitions $A \to A + d\lambda$ between potentials. Weyl referred to invariance under these combined transitions as "gauge" invariance, so the transitions have come to be known as "gauge transformations."

Kaluza, taking a different tack, enlarged the study of space-time to the study of five-dimensional metric manifolds $\mathcal{M}$ whose cross sections transverse to the fifth dimension are space-time manifolds. To account for the unobservability of this extra dimension he postulated that translations of $\mathcal{M}$ in its direction should induce isometries of the metric $\hat{G}$ of $\mathcal{M}$. This condition, which he termed "cylinder condition," can be formulated as the requirement that there exist on $\mathcal{M}$ a vector field ξ, in the direction of the extra dimension, such that $\mathcal{L}_\xi \hat{G} = 0$, where $\mathcal{L}_\xi$ denotes Lie differentiation along ξ. The electromagnetic field grows out of nonintegrability of the distribution of hyperplanes orthogonal to ξ, which traces back to inexactness of a space-time electromagnetic covector potential $\mathring{A}$. Transformations $\mathring{A} \to \mathring{A} + d\lambda$ leaving $d_\wedge \mathring{A}$ and therefore the electromagnetic field unchanged coincide with refoliations of $\mathcal{M}$ by space-time cross sections.

Space-time–time theory brings together these seemingly disparate approaches to the task of producing a unified theory of gravity and electromagnetism. It accomplishes this simply by replacing the isometry equation $\mathcal{L}_\xi \hat{G} = 0$ in Kaluza's cylinder condition by the conformality equation $\mathcal{L}_\xi \hat{G} = 2G$, where G is the "space-time part" of $\hat{G}$. This modification causes translations of $\mathcal{M}$ along ξ to induce conformal transformations of the space-time metrics of the cross sections of $\mathcal{M}$ transverse to ξ. The result is a natural hybrid of the Kaluza and the Weyl geometries that retains and enhances the most useful characters of its parents while attenuating to benign and useful form those that have caused difficulty. Most notably, it retains both Kaluza's extra dimension and Weyl's association of metrical with electromagnetic gauge changes. Also, it converts the objectionable nonintegrability of length transference in the Weyl geometry to integrability without sacrificing the principle that length, because it is a comparative measure, depends on selection of a scale at each point, that is, on choice of a gauge. In the process it lends to the fifth dimension an essential significance that the Kaluza geometry fails to provide. This significance arises from a geometrical construction that compels interpretation of the fifth dimension as a secondary temporal dimension [3], in contrast to its more usual interpretation as a spatial dimension whose unobservability has to be excused.

The picture that emerges from application of this hybrid geometry to the modeling of physical systems has in it some surprising, unorthodox representations of elementary physical phenomena, quantum phenomena included. Taken on their own terms they offer the possibility of adding to our image of the world a certain coherency not present in existing representations. Whether they are accurate will be, of course, a matter for investigation.

The geometry of space-time–time is a special case of the geometry of what may be called Kaluza–Weyl spaces, which conform to the requirement that $\mathcal{L}_\xi \hat{G} = 2G$, but are unrestricted as to dimensionality of the carrying manifold and signature of the metric. Sections 2–6 below present the bare bones of this Kaluza–Weyl geometry, including a discussion of its gauge transformations and ending with equations for its geodesics. Section 7 develops the dynamics of test particles following space-time–time geodesics. Section 8 draws inferences about the behavior of these test particles and proposes a conservation law for their interactions. Section 9, the last, is devoted to remarks speculative and prospective in nature. A subsequent paper will present field equations appropriate to the Kaluza–Weyl geometry.

2. KALUZA–WEYL SPACES

Let $\mathcal{M}$ be a manifold and $\hat{G}$ a symmetric, nondegenerate metric on $\mathcal{M}$. The condition on $\mathcal{M}$ and $\hat{G}$ that will replace Kaluza's "cylinder condition" as formulated in Sec. 1 is the

> **Exponential Expansion Constraint (EEC)**: There exists on $\mathcal{M}$ a vector field ξ such that $\mathcal{L}_\xi \hat{G} = 2G$, where $G := \hat{G} - \big(\hat{G}\xi\xi\big)^{-1}\big(\hat{G}\xi \otimes \hat{G}\xi\big)$.

When this constraint is satisfied let us call $\hat{G}$ a **Kaluza–Weyl metric** and the pair $\{\mathcal{M}, \hat{G}\}$ a **Kaluza–Weyl space**.

For proper interpretation of the EEC the cotangent space T_P of $\mathcal{M}$ at a point P must be understood as the space of all linear mappings of the tangent space T^P into $\mathbb{R}$, and the tensor product $T_P \otimes T_P$ as the space of all linear mappings of T^P into T_P, one such being $\hat{G}(P)$. This makes $\hat{G}\xi$ a covector field on $\mathcal{M}$ (the "metric dual" of ξ), and $\hat{G}\xi\xi$ a scalar field on $\mathcal{M}$ (the "square length" of ξ), whereupon G is seen to be the orthogonal projection of $\hat{G}$ along ξ, in that $G\xi = 0$ and $\hat{G}v = Gv$ if $\hat{G}\xi v = 0$. Implicit in the EEC is that $\hat{G}\xi\xi$ vanishes nowhere, that, to put it differently, ξ is nowhere null with respect to $\hat{G}$; a consequence is that ξ itself vanishes nowhere.

A prototype for Kaluza–Weyl metrics is the de Sitter space-time metric, which in the Lemaître coordinate system takes the form

$$\hat{G} = e^{2t}(dx \otimes dx + dy \otimes dy + dz \otimes dz) - R^2(dt \otimes dt), \tag{1}$$

where R is the (uniform) space-time radius of curvature [3, 4]. Here $\xi = \partial/\partial t$, $\hat{G}\xi = -R^2 dt$, $\hat{G}\xi\xi = -R^2$, and $G = e^{2t}(dx \otimes dx + dy \otimes dy + dz \otimes dz)$.

Space-time–time metrics are those five-dimensional Kaluza–Weyl metrics $\hat{G}$ for which G has a space-time signature. Prototypes are the hyper-de Sitter metrics $\hat{G}_\pm$ given by

$$\hat{G}_\pm = e^{2\zeta}(dx \otimes dx + dy \otimes dy + dz \otimes dz - dt \otimes dt) \pm R^2(d\zeta \otimes d\zeta). \tag{2}$$

For both metrics $\xi = \partial/\partial\zeta$ and $G = e^{2\zeta}(dx \otimes dx + dy \otimes dy + dz \otimes dz - dt \otimes dt)$; but $\hat{G}_+\xi\xi = R^2$, whereas $\hat{G}_-\xi\xi = -R^2$, which of course reflects the fact that $\hat{G}_+$ has diagonal signature $+++-+$ and $\hat{G}_-$ has it $+++--$. Like $\hat{G}$ in Eq. (1), each of $\hat{G}_+$ and $\hat{G}_-$ gives to its carrying manifold $\mathcal{M}$ a uniform radius of curvature R.

If one removes the factor $e^{2\zeta}$ from Eq. (2), the resulting metrics will satisfy Kaluza's isometry equation $\mathcal{L}_\xi \hat{G} = 0$, but the ambiguity of signature will remain. More generally, if $\hat{G}$ satisfies either the cylinder condition or the EEC, and G has signature $+++-$ then $\hat{G}$'s signature will be $+++-+$ or $+++--$, according as $\hat{G}\xi\xi > 0$ or $\hat{G}\xi\xi < 0$. Thinking it necessary to choose between these signatures for $\hat{G}$, Kaluza apparently opted for $+++-+$.[1] As the first three $+$'s refer to spatial dimensions, one naturally is tempted to say that this causes Kaluza's extra dimension to be spatial also. But that is mere verbal analogy — it lacks any real justification in the form of a conceptual parallelism between the fifth dimension, its coordinate generated along ξ, and the three dimensions of physical space represented by the first three coordinates. Indeed, the geometric construction described in [3] makes it clear that the natural parallelism is with the fourth, temporal dimension. That parallelism is in fact on display here in the similarity between the exponential role that t plays in the de Sitter metric and the exponential role that ζ plays in the hyper-de Sitter metrics. Its existence is the reason why I attach the label **space-time–time** to every five-dimensional Kaluza–Weyl space $\{\mathcal{M}, \hat{G}\}$ for which G has a space-time signature, irrespective of whether $\hat{G}\xi\xi > 0$ or $\hat{G}\xi\xi < 0$. There is, however, no implication that the secondary time dimension is interchangeable with the primary. The secondary is a child of the primary, not a clone.

3. CANONICAL FORMS OF KALUZA–WEYL METRICS

Let $\{\mathcal{M}, \hat{G}\}$ be a Kaluza–Weyl space. One sees easily that

$$\hat{G} = G + \hat{\epsilon}\phi^2(A \otimes A), \tag{3}$$

where $\phi := \left|\hat{G}\xi\xi\right|^{1/2}$, $A := \left(\hat{G}\xi\xi\right)^{-1}\hat{G}\xi$, and $\hat{\epsilon} := \operatorname{sgn}\left(\hat{G}\xi\xi\right) = 1$ or -1. The projected metric G, the scalar field ϕ, and the covector field A behave in the following ways under Lie differentiation along ξ: $\mathcal{L}_\xi\phi = 0$, $\mathcal{L}_\xi A = 0$, and $\mathcal{L}_\xi G = 2G$. This is demonstrable by a few simple calculations. First, $G\xi = \hat{G}\xi - \left(\hat{G}\xi\xi\right)^{-1}\left(\hat{G}\xi \otimes \hat{G}\xi\right)\xi = \hat{G}\xi - \left(\hat{G}\xi\xi\right)^{-1}\left(\hat{G}\xi\xi\right)\hat{G}\xi$, so $G\xi = 0$ (as noted previously). Next, because $\mathcal{L}_\xi\xi = 0$, one has that $\mathcal{L}_\xi\left(\hat{G}\xi\right) = \left(\mathcal{L}_\xi\hat{G}\right)\xi = 2G\xi = 0$ and $\mathcal{L}_\xi\left(\hat{G}\xi\xi\right) = \left(\mathcal{L}_\xi\left(\hat{G}\xi\right)\right)\xi = 0$, so that clearly $\mathcal{L}_\xi\phi = 0$ and $\mathcal{L}_\xi A = 0$. From Eq. (3) it then follows that $\mathcal{L}_\xi\hat{G} = \mathcal{L}_\xi G$, whence $\mathcal{L}_\xi G = 2G$.

A decomposition of G comes from integrating the differential equation $\mathcal{L}_\xi G = 2G$, the result being that $G = e^{2C}\mathring{G}$, where C is a scalar field, $\mathcal{L}_\xi C = 1$, $\mathring{G}$ is a metric of the same signature as G, and $\mathcal{L}_\xi\mathring{G} = 0$. Application of $\mathcal{L}_\xi$ to both sides of

$$\hat{G} = e^{2C}\mathring{G} + \hat{\epsilon}\phi^2(A \otimes A) \tag{4}$$

then shows that this representation for $\hat{G}$, under the conditions that $\mathcal{L}_\xi C = 1$ and the Lie derivatives along ξ of $\mathring{G}$, ϕ, and A all vanish, is sufficient to make $\hat{G}$ satisfy the EEC

[1] Kaluza was not committed to this choice, indeed seemed willing to let it go the other way if by so doing he could overcome a "serious difficulty" pointed out to him by Einstein [1, p. 971].

(with respect to ξ). With these conditions the representation (4) therefore constitutes a characterization of Kaluza–Weyl metrics.

Now let us introduce (by a standard construction) a coordinate system $[\![x^\mu, \zeta]\!]$ adapted to ξ so that $\xi = \partial/\partial\zeta$.[2] As a covector field, A has in $[\![x^\mu, \zeta]\!]$ the expansion $A = A_\mu dx^\mu + A_\zeta d\zeta$. But $A_\zeta = A(\partial/\partial\zeta) = A\xi = (\hat{G}\xi\xi)^{-1}\hat{G}\xi\xi = 1$, so $A = A_\mu dx^\mu + d\zeta$. Further, $0 = \mathcal{L}_\xi A = \mathcal{L}_{\partial/\partial\zeta} A = (\partial A_\mu/\partial\zeta) dx^\mu$, so $\partial A_\mu/\partial\zeta = 0$; thus the A_μ depend on the coordinates x^κ alone, and not on ζ. Also, $\partial\phi/\partial\zeta = \mathcal{L}_\xi\phi = 0$, so ϕ is a function of the x^κ only. Arguing similarly about G, we arrive at the adapted coordinates version of Eq. (3), viz.,

$$\hat{G} = dx^\mu \otimes g_{\mu\nu} dx^\nu + \hat{\epsilon}\phi^2 (A_\mu dx^\mu + d\zeta) \otimes (A_\nu dx^\nu + d\zeta), \tag{3$'$}$$

with $\partial\phi/\partial\zeta = \partial A_\mu/\partial\zeta = 0$ and $\partial g_{\mu\nu}/\partial\zeta = 2g_{\mu\nu}$. To do the same for Eq. (4), let us specify the scalar field C, which is at our disposal. The possibilities are $C = \zeta + \theta$, thus $e^{2C} = e^{2\zeta}e^{2\theta}$, with $\partial\theta/\partial\zeta = 0$. Inasmuch as the factor $e^{2\theta}$ can be absorbed into $\mathring{G}$, we can, without loss of generality, take $\theta = 0$. Then $G = e^{2\zeta}\mathring{G}$ and $g_{\mu\nu} = e^{2\zeta}\mathring{g}_{\mu\nu}$, with $\partial\mathring{g}_{\mu\nu}/\partial\zeta = 0$. Let us also introduce the covector field $\mathring{A} := A - d\zeta$, for which $A = \mathring{A} + d\zeta$, $\mathring{A} = \mathring{A}_\mu dx^\mu$, $\mathring{A}_\mu = A_\mu$, $\mathcal{L}_\xi\mathring{A} = 0$, and $\partial\mathring{A}_\mu/\partial\zeta = 0$. Then Eq. (4) takes the forms

$$\begin{aligned}\hat{G} &= e^{2\zeta}\mathring{G} + \hat{\epsilon}\phi^2\big(\mathring{A} + d\zeta\big) \otimes \big(\mathring{A} + d\zeta\big)\\ &= e^{2\zeta}\big(dx^\mu \otimes \mathring{g}_{\mu\nu} dx^\nu\big) + \hat{\epsilon}\phi^2\big(\mathring{A}_\mu dx^\mu + d\zeta\big) \otimes \big(\mathring{A}_\nu dx^\nu + d\zeta\big),\end{aligned} \tag{4$'$}$$

with $\partial\phi/\partial\zeta = \partial\mathring{A}_\mu/\partial\zeta = \partial\mathring{g}_{\mu\nu}/\partial\zeta = 0$. These forms are canonical for Kaluza–Weyl metrics. They differ from the analogous canonical forms for metrics satisfying Kaluza's cylinder condition precisely by the presence of the factor $e^{2\zeta}$. This factor produces surprising effects, as we shall see.

4. GAUGE TRANSFORMATIONS

When $\{\mathcal{M}, \hat{G}\}$ is a space-time–time, the tensor field $F := -2d_\wedge A$ will come to be identified as the electromagnetic field tensor. We shall have then that $F = -2d_\wedge\big(\mathring{A} + d\zeta\big) = -2d_\wedge\mathring{A}$, thus that $\mathring{A}$ takes the role of electromagnetic four-covector potential. Klein, who independently formulated and refined the Kaluza geometry [5], and Einstein, who introduced refinements of his own [6], used the same identification of F for the Kaluza (–Klein) theory, deviating somewhat from Kaluza's choice. They further recognized that the electromagnetic gauge transformations $\mathring{A} \to \mathring{A}' := \mathring{A} + d\lambda$ such that $\mathcal{L}_\xi\lambda = 0$ are generated by transformations $\zeta \to \zeta' := \zeta - \lambda$ such that $\partial\lambda/\partial\zeta = 0$, which follows from $A = \mathring{A}' + d\zeta' = \mathring{A} + d\zeta$ and $d_\wedge A = d_\wedge\mathring{A}' = d_\wedge\mathring{A}$. This recognition was the first step on the road to the gauge theories that now abound in theoretical physics. Missing from Kaluza–Klein theory and from these later gauge theories, however, is any remembrance

[2] Here μ and other Greek letter indices will range, if $d > 1$, from 1 to $d-1$, where $d := \dim\mathcal{M}$; if $d = 1$, then the only coordinate is ζ, so μ does not enter the picture. M and other upper case roman indices will range from 1 to d.

of Weyl's earlier association of electromagnetic gauge transformations with (conformal) gauge transformations of the metric of space-time.[3] In space-time–time this association is preserved through intermediation of the coordinate transformation $\zeta' = \zeta - \lambda$, for the conformality relation $\mathring{G}' = e^{2\lambda}\mathring{G}$ is a clear implication of $G = e^{2\zeta}\mathring{G} = e^{2\zeta'}\mathring{G}'$.

The coordinate transformations that generate the electromagnetic and the metrical gauge transformations, *being* coordinate transformations, do not alter the metric of space-time–time. This is a principal advantage that the space-time–time geometry has over the Weyl geometry. Weyl, working without the aid of a fifth dimension, impressed his infinitude of conformally related space-time metrics onto one four-dimensional manifold. That is very much like drawing all the maps of the world on a single sheet of paper, a practice that would conserve paper but confound navigators. In effect, the space-time–time geometry economizes on paper but avoids the confusion of maps on maps, by drawing a selection of the maps on individual sheets, then stacking the sheets so that each of the remaining maps can be generated on command by slicing through the stack in a particular way. The Kaluza–Klein geometry does much the same, but the cylinder condition restricts its stack to multiple copies of a single map, with no new maps producible by slicing.

5. CONNECTION FORMS AND COVARIANT DIFFERENTIATIONS

A coframe system $\{\omega^\mu, \omega^d\}$ that will facilitate computation of connection forms for the Kaluza–Weyl space $\{\mathcal{M}, \hat{G}\}$ is defined as follows: relabeling the coordinates x^μ as $x^{\mu'}$, let $\omega^\mu := dx^{\mu'} J_{\mu'}{}^\mu$, with $[J_{\mu'}{}^\mu]$ and its inverse matrix $[J_\mu{}^{\mu'}]$ independent of ζ; let $\omega^d := \phi A$. In this system $\mathring{G}$ has the expansion $\mathring{G} = \omega^\mu \otimes \mathring{g}_{\mu\nu}\omega^\nu$, where $\mathring{g}_{\mu\nu} = J_\mu{}^{\mu'}\mathring{g}_{\mu'\nu'}J_\nu{}^{\nu'}$, and $\hat{G}$ takes the semi-orthogonal form

$$\hat{G} = e^{2\zeta}\big(\omega^\mu \otimes \mathring{g}_{\mu\nu}\omega^\nu\big) + \hat{\epsilon}\big(\omega^d \otimes \omega^d\big). \tag{5}$$

Upon identifying the frame system $\{e_\mu, e_d\}$ to which $\{\omega^\mu, \omega^d\}$ is dual, one has

$$\begin{aligned} e_\mu &= J_\mu{}^{\mu'}\big(\partial_{\mu'} - \mathring{A}_{\mu'}\partial_\zeta\big) = \partial_\mu - \mathring{A}_\mu\partial_\zeta, \\ e_d &= \phi^{-1}\xi = \phi^{-1}\partial_\zeta, \end{aligned} \tag{6}$$

to go with

$$\begin{aligned} \omega^\mu &= dx^{\mu'} J_{\mu'}{}^\mu, \\ \omega^d &= \phi A = \phi\big(\mathring{A}_{\mu'}dx^{\mu'} + d\zeta\big) = \phi\big(\mathring{A}_\mu\omega^\mu + d\zeta\big), \end{aligned} \tag{7}$$

where $\partial_{\mu'} := \partial/\partial x^{\mu'}$, $\partial_\zeta := \partial/\partial\zeta$, $\partial_\mu := J_\mu{}^{\mu'}\partial_{\mu'}$, and $\mathring{A}_\mu = J_\mu{}^{\mu'}\mathring{A}_{\mu'}$, so that $\mathring{A} = \mathring{A}_\mu\omega^\mu$. The components $\mathring{g}_{\mu\nu}$ and $\mathring{A}_\mu$ are independent of ζ. The vector field e_d is the unit

[3] Weyl himself contributed to this amnesia by transferring his allegiance over to an association of electromagnetic gauge transformations with electron wave field phase shifts [7], the association that London extracted from the theories of Weyl and of Kaluza and Klein [8].

normalization of ξ and is orthogonal to each of the vector fields e_μ. It is not difficult to see that $\mathfrak{L}_\xi e_\mu = \mathfrak{L}_\xi e_d = 0$ and $\mathfrak{L}_\xi \omega^\mu = \mathfrak{L}_\xi \omega^d = 0$. Thus we have a frame system and its dual coframe system that are Lie constant along ξ, but with the further property that e_d has unit length and is orthogonal to each e_μ. Their constancy along ξ makes them gauge invariant: coordinate gauge transformations $\zeta \to \zeta - \lambda$ leave them unchanged.[4]

For the exterior derivatives of ω^μ and ω^d we have

$$d_\wedge \omega^\mu = C_\kappa{}^\mu{}_\lambda \omega^\lambda \wedge \omega^\kappa, \tag{8}$$

and

$$\begin{aligned} d_\wedge \omega^d &= -(1/2)\phi F + d\phi \wedge A \\ &= -(1/2)\phi F_{\kappa\lambda}\omega^\lambda \wedge \omega^\kappa + \phi^{-1}\phi_{.\lambda}\omega^\lambda \wedge \omega^d, \end{aligned} \tag{9}$$

with $C_\kappa{}^\mu{}_\lambda$ skew-symmetric in κ and λ and independent of ζ, and with

$$F := -2 d_\wedge A = -2 d_\wedge \mathring{A} = F_{\kappa\lambda}\omega^\lambda \wedge \omega^\kappa, \tag{10}$$

where

$$F_{\kappa\lambda} = \mathring{A}_{\lambda.\kappa} - \mathring{A}_{\kappa.\lambda} - 2\mathring{A}_\mu C_\kappa{}^\mu{}_\lambda, \tag{11}$$

also skew-symmetric in κ and λ and independent of ζ. Here $f_{.\mu} := \partial_\mu f$, for scalar fields f.

The torsionless covariant differentiation $\hat{\mathbf{d}}$ on $\mathcal{M}$ that is compatible with $\hat{G}$ has connection forms $\hat\omega_K{}^M$ such that $\hat{\mathbf{d}} e_K = \hat\omega_K{}^M \otimes e_M$ and $\hat{\mathbf{d}}\omega^M = -\hat\omega_K{}^M \otimes \omega^K$. They can be expressed as follows:

$$\begin{aligned} \hat\omega_\kappa{}^\mu &= \omega_\kappa{}^\mu + \big[\phi^{-1}\mathring{g}_\kappa{}^\mu + \hat\epsilon(1/2)e^{-2\zeta}\phi\mathring{F}_\kappa{}^\mu\big]\omega^d, \\ \hat\omega_\kappa{}^d &= -\big[\hat\epsilon e^{2\zeta}\phi^{-1}\mathring{g}_{\kappa\lambda} - (1/2)\phi F_{\kappa\lambda}\big]\omega^\lambda + \phi^{-1}\phi_{.\kappa}\omega^d, \\ \hat\omega_d{}^\mu &= \big[\phi^{-1}\mathring{g}^\mu{}_\lambda - \hat\epsilon(1/2)e^{-2\zeta}\phi\mathring{F}^\mu{}_\lambda\big]\omega^\lambda - \hat\epsilon e^{-2\zeta}\phi^{-1}\mathring{\phi}^{.\mu}\omega^d, \end{aligned} \tag{12}$$

and

$$\hat\omega_d{}^d = 0.$$

In these and in subsequent equations raising of an index with $\mathring{g}^{\mu\nu}$ is indicated by insertion of a $^\circ$, unless one is already present, as in $\mathring{g}_\kappa{}^\mu := \mathring{g}_{\kappa\lambda}\mathring{g}^{\lambda\mu} = \delta_\kappa{}^\mu$ and $\mathring{A}^\mu := \mathring{A}_\lambda \mathring{g}^{\lambda\mu}$. Also,

$$\omega_\kappa{}^\mu := \big(\mathring\Gamma_\kappa{}^\mu{}_\lambda + \mathring\Delta_\kappa{}^\mu{}_\lambda\big)\omega^\lambda, \tag{13}$$

where

$$\mathring\Gamma_\kappa{}^\mu{}_\lambda := (1/2)\big(\mathring{g}_{\nu\lambda.\kappa} + \mathring{g}_{\kappa\nu.\lambda} - \mathring{g}_{\kappa\lambda.\nu}\big)\mathring{g}^{\nu\mu} - \big(C_\kappa{}^\mu{}_\lambda + \mathring{C}_{\kappa\lambda}{}^\mu + \mathring{C}_{\lambda\kappa}{}^\mu\big) \tag{14}$$

[4] In the terminology of fibre bundle theory the e_μ and the tangent subspace they span at a point are "horizontal," and e_d and the subspace it spans at a point are "vertical," as determined with reference to the covector field A, standing in for a bundle connection 1-form.

and

$$\mathring{\Delta}_\kappa{}^\mu{}_\lambda := -\big(\mathring{A}_\kappa \mathring{g}^\mu{}_\lambda + \mathring{g}_\kappa{}^\mu \mathring{A}_\lambda - \mathring{g}_{\kappa\lambda}\mathring{A}^\mu\big), \tag{15}$$

with $\mathring{C}_{\kappa\lambda}{}^\mu := \mathring{g}_{\lambda\nu} C_\kappa{}^\nu{}_\pi \mathring{g}^{\pi\mu}$.

A covariant differentiation $\mathbf{d}$ on $\mathcal{M}$, related to but distinct from $\hat{\mathbf{d}}$, is fixed by the stipulations that $\mathbf{d}e_\kappa = \omega_\kappa{}^\mu \otimes e_\mu$ and $\mathbf{d}e_d = 0$, or, equally well, by $\mathbf{d}\omega^\mu = -\omega_\kappa{}^\mu \otimes \omega^\kappa$ and $\mathbf{d}\omega^d = 0$. This is a direct analog of the covariant differentiation in Weyl's geometry, as it satisfies $\mathbf{d}G = 2A \otimes G$, the principal characterizing condition of Weyl's affine connection. Although $\mathbf{d}$ is not in general torsionless, Tor $\mathbf{d} = d_\wedge \omega^d \otimes e_d = \big[-(1/2)F + \phi^{-1} d\phi \wedge A\big] \otimes \xi$, so that the components of torsion in directions orthogonal to ξ vanish: $\omega^\mu(\text{Tor }\mathbf{d}) = (d_\wedge\omega^d)(\omega^\mu e_d) = 0$.

6. GEODESIC EQUATIONS

Let $p\colon I \to \mathcal{M}$ be a path in $\mathcal{M}$, with parameter interval I, and let the components of its velocity $\dot{p}$ be $\{\dot{p}^\mu, \dot{p}^d\}$, in the frame system $\{e_\mu, e_d\}$. For the acceleration of p generated by the covariant differentiation $\hat{\mathbf{d}}$ one has $\ddot{p} = \ddot{p}^\mu e_\mu(p) + \ddot{p}^d e_d(p)$, where

$$\ddot{p}^\mu = (\dot{p}^\mu)^{\cdot} + \dot{p}^\kappa \hat{\omega}_\kappa{}^\mu(p)\dot{p} + \dot{p}^d \hat{\omega}_d{}^\mu(p)\dot{p} \tag{16}$$

and

$$\ddot{p}^d = \big(\dot{p}^d\big)^{\cdot} + \dot{p}^\kappa \hat{\omega}_\kappa{}^d(p)\dot{p} + \dot{p}^d \hat{\omega}_d{}^d(p)\dot{p}. \tag{17}$$

The condition that p be an affinely parametrized geodesic path of $\hat{\mathbf{d}}$ is that $\ddot{p} = 0$, which is equivalent to $\ddot{p}^\mu = 0$ and $\ddot{p}^d = 0$. These are equivalent, respectively, to

$$\big(e^{2\zeta}\dot{p}^\mu\big)^{\cdot} + e^{2\zeta}\dot{p}^\kappa \mathring{\Gamma}_\kappa{}^\mu{}_\lambda \dot{p}^\lambda = \hat{\epsilon}\phi\dot{p}^d \mathring{F}^\mu{}_\lambda \dot{p}^\lambda - e^{2\zeta}\dot{p}^\kappa \mathring{g}_{\kappa\lambda}\dot{p}^\lambda \mathring{A}^\mu + \hat{\epsilon}\dot{p}^d\dot{p}^d\phi^{-1}\mathring{\phi}^{\cdot\mu} \tag{18}$$

and

$$\big(\hat{\epsilon}\phi\dot{p}^d\big)^{\cdot} = e^{2\zeta}\dot{p}^\kappa \mathring{g}_{\kappa\lambda}\dot{p}^\lambda, \tag{19}$$

in which for brevity the compositions with p of the various scalar fields are implicit rather than express.

As one knows, $\ddot{p} = 0$ implies that $\big[\hat{G}(p)\dot{p}\dot{p}\big]^{\cdot} = 0$, thus that $\hat{G}(p)\dot{p}\dot{p}$ is constant. This takes the form

$$e^{2\zeta}\dot{p}^\kappa \mathring{g}_{\kappa\lambda}\dot{p}^\lambda + \hat{\epsilon}\dot{p}^d\dot{p}^d = \epsilon, \tag{20}$$

where $\epsilon := \operatorname{sgn}\big(\hat{G}(p)\dot{p}\dot{p}\big) = 1$, 0, or -1, provided that the parametrization of p is by arc length when $\hat{G}(p)\dot{p}\dot{p} \neq 0$.

7. TEST PARTICLE DYNAMICS IN SPACE-TIME–TIME

When the Kaluza–Weyl space $\{\mathcal{M}, \hat{G}\}$ is a space-time–time, its geodesics can be interpreted as histories of test particles, just as is done with space-time geodesics. It then becomes of interest to learn the dynamics governing the motions of such test particles. These dynamics will be, of course, only the kinematics imposed on the test particles by the space-time–time geometry, but dressed up in labels such as momentum, mass, charge, and force. For space-time test particles the procedure is relatively straightforward, geodesics in space-time having no kinematical variables to be interpreted as mass or electric charge, and only the gravitational force to be assigned a kinematical identity. In space-time–time there is a great deal more to be interpreted than in space-time. To arrive at useful interpretations we are bound to rely on formal similarities with extant equations and concepts, but we must accept whatever dynamics the kinematics dictate, and *firmly repress* the natural tendency to insist upon complete agreement with preconceived notions of particle properties and behavior derived from theories based on other, more restrictive geometries, or on no geometry at all.

It will be convenient to have the signature of the space-time part of the metric be $---+$; this causes the signature of $\hat{G}$ to be $---++$ if $\hat{\epsilon}=1$, and $---+-$ if $\hat{\epsilon}=-1$. To begin, let us define the **space-time–time momentum covector** P of the test particle following the geodesic path p to be the metric dual of its velocity, that is, $P := \hat{G}(p)\dot{p}$. Because $\hat{G}$ is $\hat{\mathbf{d}}$-covariantly constant, $\dot{P} = \hat{G}(p)\ddot{p}$, and therefore the geodesic equation $\ddot{p}=0$ is equivalent to $\dot{P}=0$. Analysis of the latter equation will yield the desired interpretations.

In the adapted coframe system $\{\omega^\mu, \omega^d\}$ the momentum P has the expansion $P = P_\kappa \omega^\kappa(p) + P_d \omega^d(p)$, where

$$P_\kappa = e^{2\zeta}\dot{p}^\mu \mathring{g}_{\mu\kappa} \tag{21}$$

and

$$P_d = \hat{\epsilon}\dot{p}^d. \tag{22}$$

The covariant derivative of P has the expansion $\dot{P} = \dot{P}_\kappa \omega^\kappa(p) + \dot{P}_d \omega^d(p)$, where

$$\begin{aligned} \dot{P}_\kappa &= (P_\kappa)^{\cdot} - P_\mu \hat{\omega}_\kappa{}^\mu(p)\dot{p} - P_d \hat{\omega}_\kappa{}^d(p)\dot{p} \\ &= (P_\kappa)^{\cdot} - P_\mu \mathring{\Gamma}_\kappa{}^\mu{}_\lambda \dot{p}^\lambda - \phi P_d F_{\kappa\lambda}\dot{p}^\lambda + e^{2\zeta}\dot{p}^\mu \mathring{g}_{\mu\nu}\dot{p}^\nu \mathring{A}_\kappa - \hat{\epsilon} P_d P_d \phi^{-1}\phi_{.\kappa} \end{aligned} \tag{23}$$

and

$$\begin{aligned} \dot{P}_d &= (P_d)^{\cdot} - P_\mu \hat{\omega}_d{}^\mu(p)\dot{p} - P_d \hat{\omega}_d{}^d(p)\dot{p} \\ &= (P_d)^{\cdot} + P_d \phi^{-1}\phi_{.\mu}\dot{p}^\mu - e^{-2\zeta}\phi^{-1}P_\mu \mathring{g}^{\mu\nu}P_\nu. \end{aligned} \tag{24}$$

Let

$$\mathring{m} := (P_\mu \mathring{g}^{\mu\nu} P_\nu)^{1/2} = e^{2\zeta}(\dot{p}^\mu \mathring{g}_{\mu\nu}\dot{p}^\nu)^{1/2} \tag{25}$$

and

$$\begin{aligned} q &:= P\xi(p) = \phi P_d = \hat{\epsilon}\phi\dot{p}^d \\ &= \hat{\epsilon}\phi^2 A(p)\dot{p} = \hat{\epsilon}\phi^2\big(\mathring{A}_\mu \dot{p}^\mu + \dot{\zeta}\big). \end{aligned} \tag{26}$$

In terms of these Eq. (20) becomes

$$e^{-2\zeta}\mathring{m}^2 + \hat{\epsilon}(q/\phi)^2 = \epsilon, \tag{27}$$

and the equations $\dot{P}_\kappa = 0$ and $\dot{P}_d = 0$, equivalent to $\dot{P} = 0$, are seen to be further equivalent to

$$\begin{aligned}(P_\kappa)^{\cdot} &= P_\mu \mathring{\Gamma}_\kappa{}^\mu{}_\lambda \dot{p}^\lambda + qF_{\kappa\lambda}\dot{p}^\lambda - e^{-2\zeta}\mathring{m}^2 \mathring{A}_\kappa + \hat{\epsilon}(q/\phi)^2\phi^{-1}\phi_{.\kappa} \\ &= e^{-2\zeta}\big(P_\mu \mathring{\Gamma}_\kappa{}^{\mu\lambda} P_\lambda + q\mathring{F}_\kappa{}^\lambda P_\lambda - \mathring{m}^2 \mathring{A}_\kappa\big) + \hat{\epsilon}(q/\phi)^2\phi^{-1}\phi_{.\kappa}\end{aligned} \tag{28}$$

and

$$\dot{q} = e^{-2\zeta}\mathring{m}^2. \tag{29}$$

Equation (29) can be recast in light of Eq. (27) as

$$\dot{q} = \epsilon - \hat{\epsilon}(q/\phi)^2. \tag{30}$$

Together with Eqs. (26) and (27) it implies that

$$\big(\mathring{m}^2\big)^{\cdot} = 2\big[-\mathring{m}^2 \mathring{A}_\kappa + \hat{\epsilon}e^{2\zeta}(q/\phi)^2\phi^{-1}\phi_{.\kappa}\big]\dot{p}^\kappa. \tag{31}$$

The scalar $\mathring{G}(p)\dot{p}\dot{p}$, recognizable also as $\dot{p}^\mu \mathring{g}_{\mu\nu}\dot{p}^\nu$ and as $e^{-4\zeta}\mathring{m}^2$, may be positive, zero, or negative on different geodesics and, generally, on different portions of the same geodesic. It is the square length of the projection $\dot{p}^\mu e_\mu(p)$ along ξ of the velocity $\dot{p}$, as measured by the space-time metric $\mathring{G}$ of signature $---+$. Wherever on p this scalar is positive, that is, wherever the space-time projection of $\dot{p}$ is timelike, we can introduce a proper-(primary)time parameter $\mathring{\tau}$ such that

$$(\mathring{\tau})^{\cdot} = \big(\mathring{G}(p)\dot{p}\dot{p}\big)^{1/2} = (\dot{p}^\mu \mathring{g}_{\mu\nu}\dot{p}^\nu)^{1/2} = e^{-2\zeta}\mathring{m}, \tag{32}$$

and with it define space-time velocity components u^λ by $u^\lambda := \dot{p}^\lambda/(\mathring{\tau})^{\cdot}$. Then Eqs. (28), (29), and (31) can transmute to

$$\frac{dP_\kappa}{d\mathring{\tau}} = P_\mu \mathring{\Gamma}_\kappa{}^\mu{}_\lambda u^\lambda + qF_{\kappa\lambda}u^\lambda - \mathring{m}\mathring{A}_\kappa + \hat{\epsilon}e^{2\zeta}\frac{(q/\phi)^2}{\mathring{m}}\phi^{-1}\phi_{.\kappa}, \tag{33}$$

$$\frac{dq}{d\mathring{\tau}} = \mathring{m}, \tag{34}$$

and

$$\frac{d\mathring{m}}{d\mathring{\tau}} = \Big[-\mathring{m}\mathring{A}_\kappa + \hat{\epsilon}e^{2\zeta}\frac{(q/\phi)^2}{\mathring{m}}\phi^{-1}\phi_{.\kappa}\Big]u^\kappa. \tag{35}$$

Equations (33) and (34) are coupled equations of motion for the test particle; they have the subsidiary equation (35) as a consequence.

Finally, recall that by convention $P_\nu \mathring{g}^{\nu\mu} =: \mathring{P}^\mu$. From this follows that $\mathring{P}^\mu = e^{2\zeta}\dot{p}^\mu$, $\mathring{m} = \left(\mathring{P}^\mu \mathring{g}_{\mu\nu} \mathring{P}^\nu\right)^{1/2}$, and, wherever $\mathring{m}^2 > 0$, $\mathring{P}^\mu = \mathring{m}u^\mu$ and we may replace the covector equation (33) by the equivalent vector equation

$$\frac{d(\mathring{m}u^\mu)}{d\mathring{\tau}} + (\mathring{m}u^\kappa)\mathring{\Gamma}_\kappa{}^\mu{}_\lambda u^\lambda = q\mathring{F}^\mu{}_\lambda u^\lambda - \mathring{m}\mathring{A}^\mu + \hat{\epsilon}e^{2\zeta}\frac{(q/\phi)^2}{\mathring{m}}\phi^{-1}\mathring{\phi}^{\cdot\mu}, \tag{33$'$}$$

which is a reformulation in present terms of the geodesic equation (18).

Comparison of Eqs. (33) and (33′) with the classical relativistic equations of motion for an electrically charged particle makes credible the interpretation of $\mathring{m}$ as **rest mass** and q as **electric charge** of the test particle, of P_μ as components of its **space-time momentum covector**, and of $\mathring{m}u^\mu$ as components of its **space-time momentum vector**. These interpretations adopted, four "forces" appear in Eqs. (33) and (33′) as drivers of the momentum rates $dP_\kappa/d\mathring{\tau}$ and $d(\mathring{m}u^\mu)/d\mathring{\tau}$:

1. the Einstein force attributable to the gravitational field and other space-time geometry fields, as manifested in the connection coefficients $\mathring{\Gamma}_\kappa{}^\mu{}_\lambda$;
2. the Lorentz force attributable to the electromagnetic field manifested in the tensor F;
3. a force proportional to the electromagnetic potential field embodied in the covector $\mathring{A}$;
4. a force proportional to the scalar field gradient $d(\ln\phi)$ $(= \phi^{-1}d\phi = \phi^{-1}\phi_{.\kappa}\omega^\kappa)$.

Of these only the first is present in space-time theory as part of the geometry. The first and the second show up in the geometry of Kaluza–Klein theory, and the fourth would as well but for the restriction that $\hat{G}\xi\xi$, and therefore ϕ, is constant, which Klein [5] and Einstein [6] added to Kaluza's cylinder condition.[5] Versions of the fourth occur in Kaluza's paper [1, Eq. (11a)] and, implicitly, in Jordan's elaboration of Kaluza's theory set out by Bergmann [9]. The third cannot be found in any of those theories — its existence here is owed specifically to the inclusion of Weyl's geometry in space-time–time theory by way of the EEC. As a force that will bend the tracks of particles in regions where the electromagnetic field F vanishes but the electromagnetic potential $\mathring{A}$ does not, its presence offers the chance of a new perspective on electron optics, on the Aharonov–Bohm effect in particular [10, 11].

In space-time theory rest mass and electric charge are extraneous to the geometry and are therefore of necessity put into equations of motion by hand, usually as constants. In space-time–time $\mathring{m}$ and q are kinematical variables of geometrical origin which remain constant only in special cases — witness Eq. (29), which allows q to be constant only if $\mathring{m} = 0$, and Eq. (31), which requires a delicate balance if $\mathring{m}$ is not to vary. The inconstancy of q stands in marked contrast to q's behavior in Kaluza (–Klein) theory, where instead of Eq. (29) one encounters $\dot{q} = 0$, which makes q a constant of the motion. Likewise, in Kaluza–Klein theory in place of Eq. (31) one has $\left(\mathring{m}^2\right)^{\cdot} = 0$, which makes $\mathring{m}$ a constant. But in Kaluza (–Jordan) theory Eq. (31) is replaced by $\left(\mathring{m}^2\right)^{\cdot} = 2\hat{\epsilon}(q/\phi)^2\phi^{-1}\phi_{.\kappa}\dot{p}^\kappa$, which permits $\mathring{m}$ to vary. Unlike q, which has the gauge-invariant definition $q := P\xi(p)$, $\mathring{m}$ as

[5] Einstein called the result "sharpened cylinder condition."

here defined is not gauge-invariant (although vanishing of $\mathring{m}$ is). Its variability therefore poses fewer questions than does that of q, and these can safely be excluded from the present discussion. The issues raised by the variability of q, on the other hand, are of considerable import. They will be addressed in the next section.

8. TEST PARTICLE BEHAVIOR IN SPACE-TIME–TIME

As a vacuum metric for space-time–time we can reasonably choose either of the hyper-de Sitter metrics of Eq. (2), expecting test particle behavior to differ somewhat between them. It will be instructive to study $\hat{G}_-$, taken in the form $\hat{G} = -\hat{G}_-$ so that the signature of $\hat{G}$ will be $- - - + +$ and the space-time signature will be $- - - +$ as in the preceding section. Then $\hat{\epsilon} = 1$, $\phi = R$ (the uniform space-time–time radius of curvature), and $\mathring{\Gamma}_\kappa{}^\mu{}_\lambda$, $\mathring{A}_\kappa$, $F_{\kappa\lambda}$, and $\phi_{.\kappa}$ are all zero.

Equation (28) is simply $(P_\kappa)^\cdot = 0$, from which $[\![P_\kappa]\!] = [\![-a, -b, -c, E]\!]$, a constant. With Eq. (21) this implies that

$$[\![\dot{x}, \dot{y}, \dot{z}, \dot{t}]\!] = [\![a, b, c, E]\!] e^{-2\zeta}, \tag{36}$$

which in turn implies, when $E \neq 0$, that $\mathbf{v} := [\![dx/dt, dy/dt, dz/dt]\!] = [\![a/E, b/E, c/E]\!]$, thus that the motion is uniform in space-time. The rest mass evolution equation (31) is $(\mathring{m}^2)^\cdot = 0$, so $\mathring{m}$ is constant; in fact $\mathring{m} := (P_\mu \mathring{g}^{\mu\nu} P_\nu)^{1/2} = \sqrt{E^2 - a^2 - b^2 - c^2}$, and therefore $\mathring{m} = |E|\sqrt{1 - |\mathbf{v}|^2}$, a familiar relation. For a test particle traveling slower than light, $\mathring{m}^2 > 0$ and $\epsilon = 1$, so the charge evolution equation (30) is $\dot{q} = 1 - (q/\phi)^2$. A solution of this equation representative of those consistent with $\mathring{m}^2 > 0$ is

$$q = \phi \tanh(\hat{\tau}/\phi), \tag{37}$$

where $\hat{\tau}$ is an arc length parameter for the test particle's geodesic. Equations (26) and (37) imply that $\dot{\zeta} = \phi^{-1}\tanh(\hat{\tau}/\phi)$. Further integration yields

$$\zeta = \ln(\mathring{m}\cosh(\hat{\tau}/\phi)) \tag{38}$$

and

$$[\![x, y, z, t]\!] = [\![x_0, y_0, z_0, t_0]\!] + [\![a, b, c, E]\!]\mathring{m}^{-2}\phi\tanh(\hat{\tau}/\phi). \tag{39}$$

Equations (32) are satisfied if $\mathring{\tau} = \mathring{m}^{-1}\phi\tanh(\hat{\tau}/\phi) = q/\mathring{m}$.

The geodesic path p that these equations describe is complete, in that $p(\hat{\tau})$ exists for $-\infty < \hat{\tau} < \infty$. The test particle experiences, therefore, a full, historically complete existence in space-time–time. Contrarily, however, its sojourn in space-time is constricted to the times between $t(-\infty)$ and $t(\infty)$, where $t(\pm\infty) = t_0 \pm \big(E/\mathring{m}^2\big)\phi$. In the eyes of a space-time observer the particle, if its energy E is positive, springs into full-blown existence at the event $\mathcal{E}_1$ whose coordinate vector is $[\![x, y, z, t]\!](-\infty)$, traveling from the very instant of its birth with uniform velocity toward the event $\mathcal{E}_2$ with coordinate vector $[\![x,y,z,t]\!](\infty)$, at which it vanishes, having lived a lifetime of precisely the span given by $t(\infty)-t(-\infty)=2\big(\phi E/\mathring{m}^2\big)$ in coordinate time, and by $\mathring{\tau}(\infty)-\mathring{\tau}(-\infty)=2(\phi/\mathring{m})$ in its own proper time, a span that tends to ∞ as $\mathring{m}$ is decreased to 0. This sudden appearance and disappearance is an artifact of the projecting of the $\hat{\tau}$-complete geodesic from the five dimensions of space-time–time onto the four of space-time. It is entirely analogous to what happens when geodesics are projected from the four dimensions of de Sitter's space-time onto the three dimensions of space. In the de Sitter case the projections terminate at points of space, in the hyper-de Sitter case, at points of space-time, that is, at events.

The behavior of q is particularly interesting. Beginning with the asymptotic value $-\phi$ at $\mathcal{E}_1$, q increases monotonically (linearly with respect to $\mathring{\tau}$, at the rate $\mathring{m}$), and finishes with the asymptotic value ϕ at $\mathcal{E}_2$. In the vacuum, where $F=0$, this has no direct effect on the particle motion, as the Lorentz force vanishes. But in a nonvacuum, where the charge evolution equation is still $\dot{q}=1-(q/\phi)^2$, similar behavior, including sudden appearance of the particle at an event $\mathcal{E}_1$ and disappearance at an event $\mathcal{E}_2$, will persist generically, with the notable consequence that the particle will respond to an electromagnetic field initially as negatively charged with $q=-\phi(\mathcal{E}_1)$, but ultimately as positively charged with $q=\phi(\mathcal{E}_2)$, passing through a state of electrical neutrality at some intermediate event. To explain, if one can, how this behavior could be consistent with empirical observations will require a detailed investigation not to be undertaken here. Such an explanation clearly would center on the specifics of the transition from the negatively charged state to the positively charged state, particularly on how the ambient fields affect the time and place of that transition. That the undertaking can produce remarkable dividends is suggested by the following considerations.

The coupling of the fourth force to the momentum rates in Eqs. (33) and (33′) involves the factor $e^{2\zeta}$, not present in the couplings of the first three. At the terminal events $\mathcal{E}_1$ and $\mathcal{E}_2$ this factor goes to ∞, with the singular result that the fourth force effectively becomes infinitely strong, overwhelms the first three, and takes control of all aspects of the particle's space-time trajectory except the precise timing of its terminal events. If the potential $\ln\phi$ has valleys, then at the two ends of its space-time history the particle, no matter what it might do or where it might go in the interim, will be forced down into one of those valleys, to oscillate to and fro through its greatest depth with ever higher frequency and ever smaller amplitude. In more picturesque language, the particle, though it wander hither and yon in midlife, must with high probability be born shaking near a valley bottom and die trembling in a similar place. What is more, the electric charge of the particle at birth or at death will be determined environmentally by the value of ϕ at the event in question, thus will be more a characteristic of the space-time–time than of the individual particle. That all this suggests the possibility of explaining some aspects of atomic structure without invoking quantum theory is apparent. By way of illustration it is interesting to contemplate the vacuum space-time–time modified so that $\phi = Re^{f(\rho)}$, where $\rho = \sqrt{x^2+y^2+z^2}$. The valleys of $\ln\phi$ bottom out at the local minima of $f(\rho)$. If, for example, $f(\rho) = -\cos\bigl(2\pi\sqrt{\rho/\rho_1}\,\bigr)$, the bottoms are stationed where $\rho = n^2\rho_1$, for $n = 0, 1, 2, 3, \ldots$, which for $n = 1, 2, 3, \ldots$ mimics the spacing of circular electron orbits in the Bohr model of the hydrogen atom if ρ_1 is the ground state radius. A test particle in this space-time–time can live and die in one of these valleys while another, born at the same place and time with the same electric charge but with greater or lesser energy, migrates to some other valley to perform its disappearing act.[6]

The vacuum equations of motion admit solutions with $\mathring{m} = 0$, $\epsilon = 1$, $q = \pm\phi$, and $\dot{p}^\mu \neq 0$, thus admit massless charged test particles traveling at the speed of light. Such particles exist also in nonvacuum space-time–times of the same signature, but only in highly restricted circumstances. Each of them must follow a path confined to a level surface of ϕ, for if $\mathring{m} = 0$, then $\dot{q} = 0$, so q must be both constant and equal to $\pm\phi(p)$, and therefore $\phi(p) = |q| = \text{const}$. Moreover, satisfaction of the second of Eqs. (28) requires in general that if ϕ is time-independent, then $d\phi$ must vanish on p and the forces corresponding to the first two terms inside the parentheses must balance one another. In the modified-vacuum example above, these requirements cannot be met, as $d\phi = 0$ necessitates that the orbits be circular, and there is no Lorentz force to balance the resulting centrifugal force term. If, however, the vacuum is further modified to include a Coulomb potential in the form $\mathring{A} = (Q/\rho)dt$ with $Q > 0$, then the requisite balance can be attained with $q = -\phi(p)$. In that space-time–time will be found, therefore, negatively charged, massless particles circulating at lightspeed in the valley bottoms of ϕ, where $\rho = n^2\rho_1$, and (unstably) on the ridge crests of ϕ, where $\rho = \bigl(n+\frac{1}{2}\bigr)^2\rho_1$. Also to be found in those locations are charged, massless particles for which $\dot{p}^\mu = 0$, their space-time existences confined to single events $\mathcal{E}$, their charges restricted to $q = \pm\phi(\mathcal{E})$, and their ζ dependencies given by $\zeta = \zeta_0 + \hat{\tau}/q$. Such test particles can exist, in fact, in every space-time–time at every space-time event $\mathcal{E}$, if any, at which $d\phi = 0$, but at no other event.

A ϕ-wave form with a null propagation vector v^μ can carry charged, massless particles from place to place at the speed of light with $\dot{p}^\mu \propto v^\mu$, provided as above that the $\mathring{\Gamma}$- and the F-terms sum to zero in Eqs. (28), as they do when, for example, $\mathring{\Gamma}_\kappa{}^\mu{}_\lambda = 0$ and F describes an electromagnetic wave form with propagation vector v^μ, so that $F_{\kappa\lambda}\dot{p}^\lambda \propto F_{\kappa\lambda}v^\lambda = 0$. A useful example to study is the vacuum modified so that $\phi = U(t-x)$ and $\mathring{A} = V(t-x)y(dt - dx)$, representing planar ϕ-wave and $\mathring{A}$-wave forms propagating in the positive x direction at lightspeed, with $d\phi = U'(t-x)(dt-dx)$ and with $F = 2V(t-x)(dt\wedge dy - dx\wedge dy)$, a linearly polarized plane-wave solution of the vacuum Maxwell equations. With $\epsilon = \hat{\epsilon} = 1$, $\mathring{m} = 0$, and U nowhere constant, the equations of motion integrate to

$$[\![\dot{x}, \dot{y}, \dot{z}, \dot{t}]\!] = [\![1, 0, 0, 1]\!](a + b\hat{\tau})e^{-2\zeta} \tag{40}$$

[6] A proper interpretation, based on the geometrical construction described in [3], is not that the test particle is "born" at $\mathcal{E}_1$ and "dies" at $\mathcal{E}_2$, but that it only *appears* at $\mathcal{E}_1$ and *disappears* at $\mathcal{E}_2$.

and

$$\zeta = \zeta_0 + \hat{\tau}/q, \tag{41}$$

where $c = t - x$ (a constant of the motion), a is a constant, $b = U'(c)/U(c)$, and $q = \pm U(c)$. When $b = 0$, $d\phi(p) = 0$, so the particle rides along secure in a wave trough bottom of ϕ or balanced on a wave crest or wave shoulder. If $b > 0$, then $\dot{t}$ and $\dot{x}$ will switch from positive to negative when $\hat{\tau} = -a/b$, and $t(-a/b)$ and $x(-a/b)$ will be maximum values. Its five-dimensional proper time $\hat{\tau}$ increasing, the particle will cease to advance and begin to retreat in ordinary (space-time) time, maintaining, however, the velocity $[\![dx/dt, dy/dt, dz/dt]\!] = [\![1, 0, 0]\!]$. A space-time observer will perhaps interpret this as two particles on one track that at a certain instant jointly vanish without a trace. This variation on the disappearing act takes place where $U'(t - x) > 0$, which puts the particle(s) on the downward sloping front side of a ϕ wave. If $b < 0$, t and x will have minimum values, and the particle(s) will seem to appear out of nowhere on the back side of a ϕ wave.

It is worth emphasizing that the electromagnetic field in this example exerts no force on the charged, massless particles. Because these "charged photons" are moving with the same velocity with which the electromagnetic field is propagating, they are immune to its influence. In the previous example, on the other hand, the static Coulomb field supplies the only apparent force (other than the even more fictitious centrifugal force) exerted on such particles, but only because it holds them with precision in the valley bottoms and on the ridge crests of $\ln\phi$, where the gradient force vanishes.

Let us note in passing that space-time–time also provides geodesic paths for massless, electrically neutral test particles traveling at the speed of light, free of the bondage suffered by the particles discussed above. They are the paths on which $\epsilon = \mathring{m} = q = 0$. According to Eq. (28) these particles are acted upon by the Einstein force, but not by any other of the four "forces" identified in Sec. 7. The vector potential $\mathring{A}$ does, however, affect them indirectly by way of the equation $\dot{\zeta} = -\mathring{A}_\mu \dot{p}^\mu$, which follows from Eqs. (26). It is only this subtle effect that can cause such a "neutrino" particle's space-time history to have a beginning or an ending event, absent which the particle would be unable to participate in an "interaction" of the kind I shall now propose.

Several test particles of the types described above, including in particular the charged, massless particles that disappear the instant they appear, can have in common an event $\mathcal{E}$ at which each either appears, with $q = -\phi(\mathcal{E})$ or $q = 0$, or disappears, with $q = \phi(\mathcal{E})$ or $q = 0$. They can be made to "interact" by demanding that the asymptotic values of their kinematical variables obey a "conservation law" of some sort. A natural candidate is this

> **Space-Time–Time Conservation Law.** The sum of the asymptotic space-time–time momenta at the space-time event $\mathcal{E}$ of all the particles whose space-time trajectories begin or end at $\mathcal{E}$ is zero.

This law would comprise for such interactions both the conservation of space-time four-momentum and the conservation of electric charge.

9. REMARKS

It is possible to look upon space-time–time with its fields and test particles as a purely geometric, deterministic substructure underlying quantum theory, somewhat as the molecular structure of gases underlies their thermodynamical theory. The statistical indeterminacies and probabilistic predictions characteristic of quantum theory would, on this view, arise from the variability and preferential tendencies of asymptotic endings of individual test particle tracks in space. When two or more test particles in thrall to a ϕ field interact at an event $\mathcal{E}$, that event can occur anywhere in space, but is more likely to happen near some one of $\ln\phi$'s valley bottoms than off in the highlands. That $\mathcal{E}$ will occur exactly at the bottom is, however, unlikely. Even though an infinitely growing force causes the particles to oscillate about the bottom depth with increasing frequencies and diminishing amplitudes, only by the merest chance will they terminate their space-time histories (thus consummate their interaction) precisely there. Instead, they will disappear from space-time at some nearby point while the force is still trying to have its way with them — the magician's method of escape from bondage, so to speak. A random selection of such groups of interacting particles will produce a statistical cloud of interaction events whose density will peak at the valley bottoms of $\ln\phi$.

The proposed space-time–time conservation law speaks loosely of "all the particles whose space-time trajectories begin or end at $\mathcal{E}$." In truth no "particles" have been identified in space-time–time, only geodesics to be treated as possible paths of "test particles." Of these geodesics innumerably many have space-time projections that begin or end at $\mathcal{E}$, but only a few of those would be expected to have space-time projections that terminate at a prescribed second event.[7] Someone setting out to analyze the "interaction" between a particle appearing or disappearing at one event and another particle appearing or disappearing at another event, on the basis of particle "exchanges" (both direct and indirect via intermediate events), might soon begin assembling such projections of geodesics into diagrams of the Feynman type, using the space-time–time conservation law as the guide to vertex formation.

By making a test particle's electric charge q at an interaction event $\mathcal{E}$ be determined solely by the ambient geometry, through the asymptotic equation $q = \pm\phi(\mathcal{E})$, space-time–time theory explains at one stroke both the discreteness and the uniformity of electric charge, that is to say, of the *passive* electric charge of particles that can be treated as test particles. The price of this is that every such particle for which $\mathring{m} > 0$ must over the course of its lifetime watch its charge account balance go inexorably from negative to positive. On the face of it this would seem to present a grave difficulty for any attempt to identify space-time–time test particles with, for example, electrons. To deny that possibility at this stage would, however, be premature. The difficulty might be resolved if, for

[7]In the space-time–time vacuum only one of them does, as the terminal events $\mathcal{E}_1$ and $\mathcal{E}_2$ fully determine the integration constants in Eqs. (38) and (39).

instance, in regions of space-time where electron charges are actually measured ϕ is essentially constant (equal to e if $-e$ is the measured electron charge), and an electron ejected from an atom and passing through such a region maintains q close to $-\phi$ while there but quickly runs its charge balance up to ϕ as it is being captured by a target — provided, of course, that space-time–time geodesics reflecting this behavior can be found. Whether such a resolution is realizable is a question demanding further investigation.

The ambiguity involved in considering both $\hat{\epsilon} = 1$ and $\hat{\epsilon} = -1$ to yield a legitimate space-time–time metric $\hat{G}$ can be resolved by absorbing both cases into an enlarged geometry. It suffices to expand ζ into a complex coordinate and ϕ and $\mathring{A}$ into complex-valued fields. To keep the space-time metric $\mathring{G}$ real under coordinate gauge transformations $\zeta \to \zeta - \lambda$, with $\lambda = \mu + i\nu$, requires introduction of the additional transformation $\phi \to \phi e^{-i\nu}$. Thus, if $\zeta' = \zeta - \lambda$, then $\hat{G} = e^{i2\nu}\hat{G}'$, where $\hat{G}' = e^{2\zeta'}\mathring{G}' + \phi'^2\big(\mathring{A}' + d\zeta'\big) \otimes \big(\mathring{A}' + d\zeta'\big)$, with $\mathring{G}' = e^{2\mu}\mathring{G}$, $\mathring{A}' = \mathring{A} + d\lambda$, and $\phi' = \phi e^{-i\nu}$. The phase shift in $\phi e^{-i\nu}$ is reminiscent of London's electron wave field phase shift that Weyl embraced.[4]

This enlargement of the geometry *via* partial complexification can be accomplished for Kaluza–Weyl metrics in general by modifying the EEC to specify that ξ be a complex vector field (which entails that the corresponding dimension of $\mathcal{M}$ become complexified). The EEC also lends itself to modification in two less drastic ways: alteration of the condition $\mathcal{L}_\xi \hat{G} = 2G$ to the less restrictive (a) $\mathcal{L}_\xi G = 2G$ or the more restrictive (b) $\mathcal{L}_\xi \hat{G} = 2\hat{G}$.

The effect of changing to (a) is to admit a ζ-dependence of ϕ and of $\mathring{A}$. This version of space-time–time geometry is the one described in [3] (an extended description appears in [12]). What is apparently the same or an equivalent geometry was studied from a projective viewpoint by R. L. Ingraham, with results presented in a sequence of papers that appeared mainly in *Il Nuovo Cimento*, beginning in 1952 (see [13] and references therein). The focus was primarily on the theory of fields satisfying equations invariant under the conformal group of Minkowskian space-time, these fields being defined on the five-dimensional projective spaces whose points are the (hyper)spheres of Minkowskian space-time. To the limited extent that they can be compared, the physical interpretations I have adopted and those of Ingraham differ appreciably. Whereas Ingraham's address primarily the problem of deriving from the geometry a concept of "mass," the interpretations I have imposed on the space-time–time geometry speak to both the mass concept and the concept of "electric charge," and call for a much more fundamental revision of the latter concept than of the former.

Changing to (b) produces a metric $\hat{G}$ of the form $\hat{G} = e^{2\zeta}\bar{G}$, where $\bar{G} = \mathring{G} + \hat{\epsilon}\phi^2\big(\mathring{A} + d\zeta\big) \otimes \big(\mathring{A} + d\zeta\big)$ with $\mathring{G}$, ϕ, and $\mathring{A}$ independent of ζ. In this context the coordinate change $\zeta \to \zeta - \lambda$ produces the usual metric gauge transformation $\mathring{G} \to e^{2\lambda}\mathring{G}$, but produces instead of the usual gradient gauge transformation the mutilated version $\mathring{A} \to e^{\lambda}\big(\mathring{A} + d\lambda\big)$, of no obvious utility. A five-dimensional metric like $\bar{G}$, but with $\phi^2\mathring{G}$ in place of $\mathring{G}$, and only $\mathring{G}$ and $\mathring{A}$ independent of ζ, was arrived at in a formalistic manner by Vladimirov [14], who suggested for ϕ the forms $\phi = \tilde{\phi}(x^\mu)\exp(iK\zeta)$ and $\phi = \exp(K\zeta)$, with K a real constant; these would correspond respectively to $\mathcal{L}_\xi \hat{G} = 2iK\hat{G}$ and $\mathcal{L}_\xi \hat{G} = 2K\hat{G}$. Unlike the EEC modified by (a), which when ϕ and $\mathring{A}$ are independent of ζ yields the same geometry as the unmodified EEC, thus can support the same physical interpretations, the EEC modified by (b) pushes the Kaluza and the Weyl geometries into an awkward and unnatural union, therefore seems unlikely to become a source of deep insight into the foundations of physics.

REFERENCES

1. Th. Kaluza, Zum Unitätstheorie der Physik, *S.–B. Preuss. Akad. Wiss.* **1921**, 966–972 (1921); communicated to Einstein in 1919.
2. H. Weyl, Gravitation und Elektrizität, *S.–B. Preuss. Akad. Wiss.* **1918**, 465–480 (1918); Eine neue Erweiterung der Relativitätstheorie, *Ann. d. Physik* **59**, 101–133 (1919).
3. H. G. Ellis, Time, the grand illusion, *Found. Phys.* **4**, 311–319 (1974); Erratum: **5**, p. 193 (1975); selected for Honorable Mention in the 1973 Essay Competition of the Gravity Research Foundation, New Boston, New Hampshire.
4. E. Schrödinger, *Expanding Universes* (Cambridge Univ. Press, Cambridge, U. K., 1956), pp. 28–33.
5. O. Klein, Quantentheorie und fünfdimensionale Relativitätstheorie, *Z. Physik* **37**, 895–906 (1926); Zur fünfdimensionalen Darstellungen der Relativitätstheorie, **46**, 188–208 (1927).
6. A. Einstein, Zu Kaluzas Theorie des Zusammenhanges von Gravitation und Elektrizität, *S.–B. Preuss. Akad. Wiss., Phys.–math. Kl.* **1927**, 23–30 (1927).
7. H. Weyl, *Gruppentheorie und Quantenmechanik* (S. Hirzel, Leipzig, 1928), p. 88; *The Theory of Groups and Quantum Mechanics*, translated from the German 2nd ed. (revised) (Dover, New York, 1950), p. 100.
8. F. London, Quantenmechanische Deutung der Theorie von Weyl, *Z. Physik* **42**, 375–389 (1927).
9. P. G. Bergmann, Unified theory with fifteen field variables, *Ann. of Math.* **49**, 255–264 (1948).
10. W. Ehrenberg and R. Siday, The refractive index in electron optics and the principles of dynamics, *Proc. Phys. Soc.* (London) **B62**, 8–21 (1949).
11. Y. Aharonov and D. Bohm, Significance of electromagnetic potentials in the quantum theory, *Phys. Rev.* **115**, 485–491 (1959).
12. H. G. Ellis, Five-dimensional conformal relativity theory, in *Abstracts of Contributed Papers*, 8th International Conference on General Relativity and Gravitation, University of Waterloo, Waterloo, Ontario, Canada, 1977, p. 138.
13. R. L. Ingraham, Conformal relativity. VI. – The general theory, *Nuovo Cimento* **50 B**, 233–279 (1979).
14. Y. S. Vladimirov, The unified field theory, combining Kaluza's five-dimensional and Weyl's conformal theories, *Gen. Rel. Grav.* **14**, 1167–1181 (1982).

First version: Space-Time-Time (1989)
Current version: April, 2001
Revised: July, 2001

Homer G. Ellis
Department of Mathematics
University of Colorado at Boulder
395 UCB
Boulder, Colorado 80309-0395

Telephone: (303) 492-7754 (office); (303) 499-4027 (home)
Email: ellis@euclid.colorado.edu
Fax: (303) 492-7707