AN EXPANDING UNIVERSE OF SPINNING SPHERES

Homer G. Ellis

Abstract. A novel but elementary geometric construction produces on the seven-dimensional manifold of rotated spheres in Euclidean three-space a finslerian geometry whose geodesics are interpreted as the paths of free, spinning, spherical particles moving through de Sitter's expanding universe. A particle of nonzero inertial rest mass typically follows a helical track and exhibits behavior remindful of the phenomenon of "Zitterbewegung" of spinning electrons first deduced by Schrödinger from Dirac's relativistic wave equation. Its velocity vector and its spin vector precess about the axial direction of the helix, with their projections onto that direction at all times parallel or at all times antiparallel. Particles of zero rest mass follow straight tracks at the speed of light with their spin vectors parallel or antiparallel to their velocity vectors, thereby replicating behavior of spinning photons predicted by the quantum theory of light.

The four-dimensional manifold whose points are the spheres of Euclidean three-space $\mathbb{E}^3$ can be coordinatized by $[\![\, R, \mathbf{s} \,]\!]$, this designating the sphere S of radius R with its center C at position $\mathbf{s}$. If $[\![\, R + dR, \mathbf{s} + \mathbf{ds} \,]\!]$ designates a neighboring sphere S', and $d\alpha$ is the radian measure of the angle in which S and S' intersect, then

$$\begin{aligned} d\alpha^2 &= (ds^2 - dR^2)/R^2 \\ &= e^{2t} ds^2 - dt^2, \end{aligned} \tag{1}$$

where $ds := |\mathbf{ds}|$ and $t := -\ln R$ (see Fig. 1). As this is precisely the metric of de Sitter's expanding universe, one can consider that universe to *be* this manifold of spheres, the event at $[\![\, t, \mathbf{s} \,]\!]$ in de Sitter's universe being then the two-sphere of radius e^{-t} centered at position $\mathbf{s}$ in $\mathbb{E}^3$. One gains thereby the advantage of reducing the ever mysterious notion of time (t) to a purely spatial concept $(-\ln R)$, along with the satisfaction of producing a space-time cosmological model out of the whole cloth of Euclidean space.(1) This satisfaction is tempered, however, by the apparent absence of a way to extend the construction to a metric for spheres that are "spinning" in a sense that makes sense. The difficulty lies in the fact that neighboring spheres will intersect in the same angle whether spinning or not.

A plan of escape from this cul-de-sac grows out of the realization that radian measure of an angle is simply a ratio of arc lengths, which suggests that some alternative characterization of $d\alpha$ as a ratio of distances might admit the needed extension. Of several such characterizations, the one that does the job is this: If each point P of the sphere S is moved radially, to produce a magnification of S by the factor $1 + dR/R$, and subsequently is translated by the vector $\mathbf{ds}$, then P arrives at a point P' on the neighboring sphere S'. Generically, there are only two such points P for which the displacement vector $\overrightarrow{PP'}$ is orthogonal to S, namely, the two points where the line through C and the center C' of S' intersects S. Of these two points P one has moved a distance $dR + ds$ in the direction of $\overrightarrow{CP}$, the other a distance $dR - ds$ in the direction of $\overrightarrow{CP}$. The product of the ratios of these distances to R is exactly the negative of the $d\alpha^2$ of Eq. (1), even when, as in Fig. 2, S and S' fail to intersect, so that there is no angle to measure.

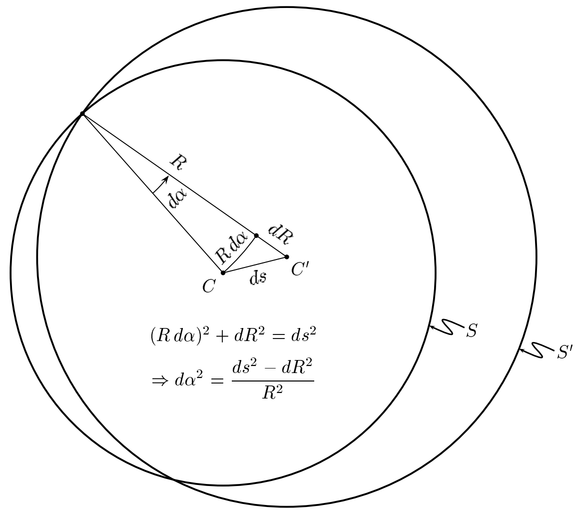


FIG. 1. INTERSECTING NEIGHBORING SPHERES S AND S' IN EUCLIDEAN THREE-SPACE, WITH ANGULAR SEPARATION $d\alpha$ — SHOWN IN CROSS SECTION THROUGH THEIR CENTERS.

Now the means of escape is at hand. It is to include a rotation with the magnification and the translation, find the points P of S for which $\overrightarrow{PP'}$ is orthogonal to S, compute the ratios to R of distances moved, as before, and use these to define a measure of the separation of S' from S. A hurdle or two remain, however. The first is the need to specify the manifold that will take the role played by the sphere manifold in the nonrotating case. Clearly, this will be the manifold of rotat*ed* spheres, a point of which is a sphere with a center, a radius, and a rotational position relative to some standard position for all spheres with the same center. This seven-dimensional manifold $\mathcal{M}$, diffeomorphic to $\mathbb{R}\times\mathbb{E}^3\times\mathrm{SO}(3)$, can be coordinatized by $[\![\, R, \mathbf{s}, \phi, \theta, \psi \,]\!]$, where ϕ, θ, and ψ are Euler angles that together specify the rotational position of the sphere with respect to the standard reference frame at $\mathbf{s}$. With $t := -\ln R$ as before, a path in this manifold can be taken to represent a spherical particle, moving through space and time, spinning as it goes.

Let the rotated sphere S designated by $[\![\, R, \mathbf{s}, \phi, \theta, \psi \,]\!]$ undergo the combined infinitesimal rotation, expansion, and translation represented by $[\![\, dR, \mathbf{ds}, d\phi, d\theta, d\psi \,]\!]$. Let P be a point of S, and let $\mathbf{u} = \overrightarrow{CP}$, the position vector of P relative to the center C of S. Then the rotation moves P to a point whose position vector relative to C is $\mathbf{u} + \boldsymbol{\delta} \times \mathbf{u}$, where $\boldsymbol{\delta} := [\![\, (\cos\phi)d\theta + (\sin\phi)(\sin\theta)d\psi, (\sin\phi)d\theta - (\cos\phi)(\sin\theta)d\psi, d\phi + (\cos\theta)d\psi \,]\!]$. The magnification multiplies this vector by $1 + dR/R$, and the translation adds $\mathbf{ds}$. Thus the requirement that the final position P' of P be collinear with C and P, equivalent to the requirement that $\overrightarrow{PP'}$ be orthogonal to S, reduces to the equation

$$(1 + dR/R)(\mathbf{u} + \boldsymbol{\delta} \times \mathbf{u}) + \mathbf{ds} = (1 + \rho)\mathbf{u}, \tag{2}$$

for some number ρ. When the term of second order in the infinitesimals is discarded, this equation simplifies to

$$(\rho - dR/R)\mathbf{u} = \boldsymbol{\delta} \times \mathbf{u} + \mathbf{ds}. \tag{3}$$

In the generic case that $\Delta := (\rho - dR/R)[(\rho - dR/R)^2 + \delta^2] \neq 0$, the solution of this equation is

$$\mathbf{u} = [(\rho - dR/R)^2\mathbf{ds} + (\rho - dR/R)(\boldsymbol{\delta} \times \mathbf{ds}) + (\boldsymbol{\delta} \cdot \mathbf{ds})\boldsymbol{\delta}]/\Delta. \tag{4}$$

Because P lies on S, $\mathbf{u}\cdot\mathbf{u} = R^2$, which is equivalent to

$$(\rho - dR/R)^4 + [\delta^2 - (ds/R)^2](\rho - dR/R)^2 - [\boldsymbol{\delta}\cdot(\mathbf{ds}/R)]^2 = 0. \tag{5}$$

The numbers ρ that satisfy this equation are the distance ratios with which to build a separation measure on $\mathcal{M}$ and make good our escape. Generically, there are four such numbers, two of them real, the others complex. The remaining hurdle is to decide how best to use them.

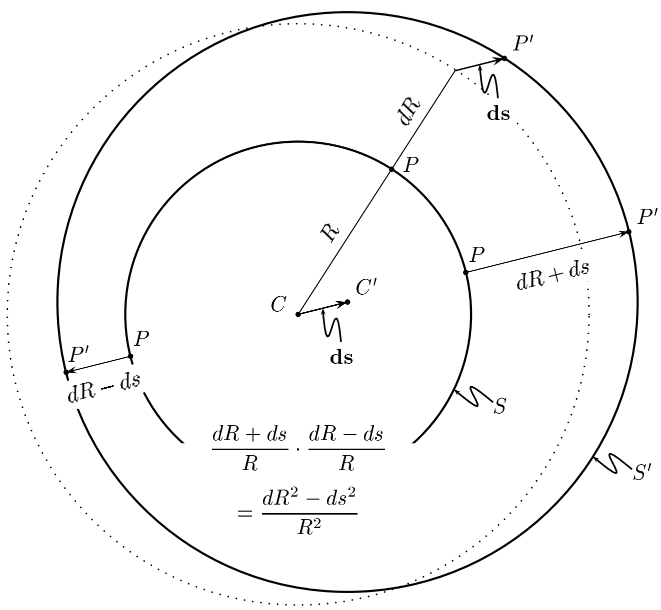


FIG. 2. NONINTERSECTING NEIGHBORING SPHERES S AND S' IN EUCLIDEAN THREE-SPACE, SEPARATED BY THE "TWO-POINT" DISTANCE $\sqrt{(dR^2 - ds^2)/R^2}$ — SHOWN IN CENTERED CROSS SECTION.

Inasmuch as only the two real roots of Eq. (5) correspond to real points of S, one might think it best to construct the separation measure from their product. Investigation shows, however, that this yields a measure with an incurable degeneracy. If on the other hand one chooses to build the separation measure from the product of all four of these ratios, then smooth sailing lies ahead, but no longer on the broad Sea of Riemann, rather on the vaster Ocean of finslerian Geometry.

A finslerian geometry on a manifold such as $\mathcal{M}$ assigns to each smooth path $p\colon [a,b] \to \mathcal{M}$ an integrated length $I(p) := \int_a^b L(p,\dot{p})$, subject for present purposes essentially only to the restrictions that L be positively homogeneous of degree one in the velocity $\dot{p}$ (so that $I(p)$ will be independent of path parametrization) and that the metric tensor G, loosely described as $dx^M \otimes g_{MN} dx^N$, where $g_{MN}(x,v) := \partial^2[(1/2)L^2(x,v)]/\partial v^M \partial v^N$ if $x = [\![\, x^K \,]\!]$ and $v = v^K(\partial/\partial x^K)$, be nondegenerate. The homogeneity allows the finslerian metric function L to be reconstructed from G via the equation $L^2(x,v) = v^M g_{MN}(x,v) v^N$. Riemannian geometries are those finslerian geometries for which $g_{MN}(x,v)$ is independent of v.[2,3]

The product of the four roots of Eq. (5) is expressible both as

$$dt^2(dt^2 - e^{2t}ds^2 + \delta^2) - e^{2t}(\mathbf{ds}\cdot\boldsymbol{\delta})^2 \tag{6}$$

and as

$$(dt^2 + \delta^2)(dt^2 - e^{2t}ds^2) + e^{2t}|\mathbf{ds}\times\boldsymbol{\delta}|^2, \tag{7}$$

where again $t := -\ln R$. With this product we can now impress upon $\mathcal{M}$ a finslerian geometry by defining L as follows: Let $[\![\, p^K \,]\!] := [\![\, x^K(p) \,]\!], = [\![\, t, \mathbf{s}, \phi, \theta, \psi \,]\!]$ for short, and $[\![\, \dot p^K \,]\!] := [\![\, dx^K(p)\dot p \,]\!] = [\![\, (p^K)^{\cdot} \,]\!], = [\![\, \dot t, \dot{\mathbf{s}}, \dot\phi, \dot\theta, \dot\psi \,]\!]$ for short. Let $\boldsymbol{\sigma} := [\![\, (\cos\phi)\dot\theta + (\sin\phi)(\sin\theta)\dot\psi,$ $(\sin\phi)\dot\theta - (\cos\phi)(\sin\theta)\dot\psi, \dot\phi + (\cos\theta)\dot\psi \,]\!]$. Then

$$\begin{aligned} L(p,\dot p) &:= \left|\dot t^2(\dot t^2 - e^{2t}\dot s^2 + \sigma^2) - e^{2t}(\dot{\mathbf{s}}\cdot\boldsymbol{\sigma})^2\right|^{1/4} \\ &= \left|(\dot t^2 + \sigma^2)(\dot t^2 - e^{2t}\dot s^2) + e^{2t}|\dot{\mathbf{s}}\times\boldsymbol{\sigma}|^2\right|^{1/4}. \end{aligned} \tag{8}$$

As is seen most clearly in (7) above, this finslerian geometry incorporates the geometrically derived de Sitter space-time metric of Eqs. (1) and envelops it in additonal structure involving the rotations of the spheres. The geodesic paths of the finslerian geometry will be taken to represent freely spinning spherical particles moving through de Sitter's universe under the influence only of the gravitational effects attributable to the cosmic expansion.

Computing the Euler–Lagrange equations for stationary paths of I, and applying to them the inverse of the metric tensor G to isolate the derivatives, one arrives at the following equations for affinely parametrized geodesics:

$$\dot E = C_0, \tag{9}$$

$$\dot{\mathbf{v}} = C_1\mathbf{v} + C_2\boldsymbol{\sigma} + C_3(\mathbf{v}\times\boldsymbol{\sigma}), \tag{10}$$

$$\dot{\boldsymbol{\sigma}} = C_4\mathbf{v} + C_5\boldsymbol{\sigma} + C_6(\mathbf{v}\times\boldsymbol{\sigma}), \tag{11}$$

in which $E := \dot t, \mathbf{v} := e^t\dot{\mathbf{s}}$, and

$$\begin{aligned} C_0 &= -(E^2v^2 - B^2)/D, \\ C_1 &= -E[(E^4 + B^2)(E^2 - v^2) + (E^4 - B^2)(E^2 - \sigma^2)]/D(E^4 + B^2), \\ C_2 &= -2BE(E^2v^2 - B^2)/D(E^4 + B^2), \\ C_3 &= -B^2/(E^4 + B^2), \\ C_4 &= -2BE[(E^4 + B^2) + E^2(E^2 - \sigma^2)]/D(E^4 + B^2), \\ C_5 &= 2E^3(E^2v^2 - B^2)/D(E^4 + B^2), \\ C_6 &= BE^2/(E^4 + B^2), \end{aligned} \tag{12}$$

with $B = \mathbf{v}\cdot\boldsymbol{\sigma}$ and $D = 2E^2 + v^2 - \sigma^2$.

It is straightforward to show that ϵ, d, and $\mathbf{c}$ defined as follows are constants of the motion:

$$\begin{aligned} \epsilon &:= E^2(E^2 - v^2 + \sigma^2) - B^2 \\ &= (E^2 + \sigma^2)(E^2 - v^2) + |\mathbf{v}\times\boldsymbol{\sigma}|^2, \end{aligned} \tag{13}$$

$$d := e^tB, \tag{14}$$

$$\mathbf{c} := e^t(E^2\mathbf{v} + B\boldsymbol{\sigma}). \tag{15}$$

(Choosing arc length for the path parameter when $\epsilon \neq 0$ restricts the values of ϵ to 1, 0, and -1.) If $d = 0$, then the particle's scaled velocity vector $\mathbf{v}$ and spin vector $\boldsymbol{\sigma}$ are at all times orthogonal to one another. Each, if not $\mathbf{0}$, maintains a fixed direction in space, $\mathbf{v}$'s direction being that of the vector $\mathbf{c}$. The particle's track through space is, therefore, a straight line in its own, unwavering equatorial plane. If $\mathbf{c} = \mathbf{0}$, then $\mathbf{v} = \mathbf{0}$, so the particle sits in one place spinning (or not, if $\boldsymbol{\sigma} = \mathbf{0}$) and shrinking as time moves on (indeed, moving time onward *by* shrinking). If $d \neq 0$, then $\mathbf{c}$ is a nonzero vector around which both $\mathbf{v}$ and $\boldsymbol{\sigma}$ precess, with, as in Fig. 3, $\mathbf{v}$ and either $\boldsymbol{\sigma}$ or $-\boldsymbol{\sigma}$ keeping $\mathbf{c}$ between them at all times (except, of course, when $\mathbf{v}$, $\boldsymbol{\sigma}$ or $-\boldsymbol{\sigma}$, and $\mathbf{c}$ are all parallel). We shall see that in fact the particle in question moves on a helical track whose axis is aligned with $\mathbf{c}$.

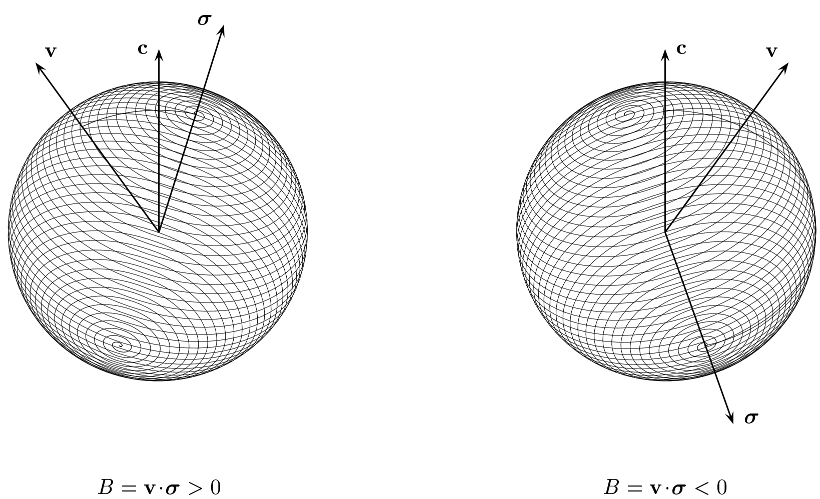


FIG. 3. GEODESICALLY SPINNING SPHERES WITH THEIR SCALED VELOCITY VECTORS $\mathbf{v}$ AND THEIR SPIN VECTORS $\boldsymbol{\sigma}$ PRECESSING AROUND THE FIXED VECTOR $\mathbf{c}$. BECAUSE $e^t B$ IS A CONSTANT OF THE MOTION, THE CASES $B > 0$ AND $B < 0$ DO NOT MIX ON A SINGLE GEODESIC.

Resolving $\mathbf{s}$, $\dot{\mathbf{s}}$, and $\mathbf{v}$ into their components $\mathbf{s}_\parallel$, $\dot{\mathbf{s}}_\parallel$, and $\mathbf{v}_\parallel$ parallel to $\mathbf{c}$, and $\mathbf{s}_\perp$, $\dot{\mathbf{s}}_\perp$, and $\mathbf{v}_\perp$ perpendicular to $\mathbf{c}$ allows us to express the curvature $\kappa_\perp$ of the projection of the particle's track onto the plane through the origin perpendicular to $\mathbf{c}$ as follows:

$$\kappa_\perp = \frac{|\dot{\mathbf{s}}_\perp \times (\dot{\mathbf{s}}_\perp)^\cdot|}{|\dot{\mathbf{s}}_\perp|^3} = \frac{e^t|\mathbf{v}_\perp \times \dot{\mathbf{v}}_\perp|}{|\mathbf{v}_\perp|^3}. \tag{16}$$

Some calculating then shows that

$$R_\perp := \frac{1}{\kappa_\perp} = \frac{(E^4 + B^2)|\mathbf{v} \times \mathbf{c}|}{c^2|B|}, \tag{17}$$

and further that $(R_\perp)^\cdot = 0$. Thus the projection, having constant radius of curvature $R_\perp$, is a circle, and the track lies, therefore, on a right circular cylinder whose axis is parallel to $\mathbf{c}$. The center of that circle, through which the axis of the cylinder must pass, is located by the vector

$$\mathbf{C} := \mathbf{s}_\perp + R_\perp \frac{\mathbf{c} \times \dot{\mathbf{s}}_\perp}{|\mathbf{c} \times \dot{\mathbf{s}}_\perp|} \operatorname{sgn}(B), \tag{18}$$

another constant of the motion.

When $E^2 - v^2 > 0$, $= 0$, < 0 the particle is conventionally said to be traveling "slower than light, at the same speed as light, faster than light." In de Sitter's as in every ordinary space-time no free particle can be in one of these states now and another later. Here that is not the case: a single geodesic with $\epsilon = 1$, for example, can have $E^2 - v^2 > 0$ now, $= 0$ later, and < 0 even later. The second of Eqs. (13) clearly implies, however, that if at any time the particle is traveling "slower than light," then ϵ must be positive. For this reason the geodesics on which $\epsilon > 0$ will be taken to represent particles of nonzero inertial rest mass. A picture of such a particle's helical track in $\mathbb{E}^3$, produced by numerical integration of the geodesic equations, is shown in Fig. 4.

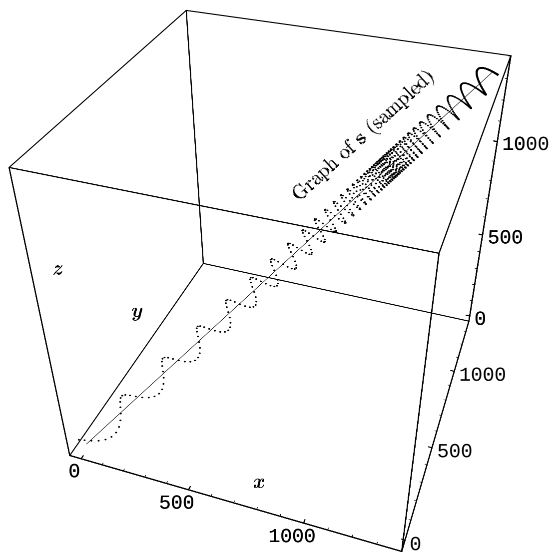
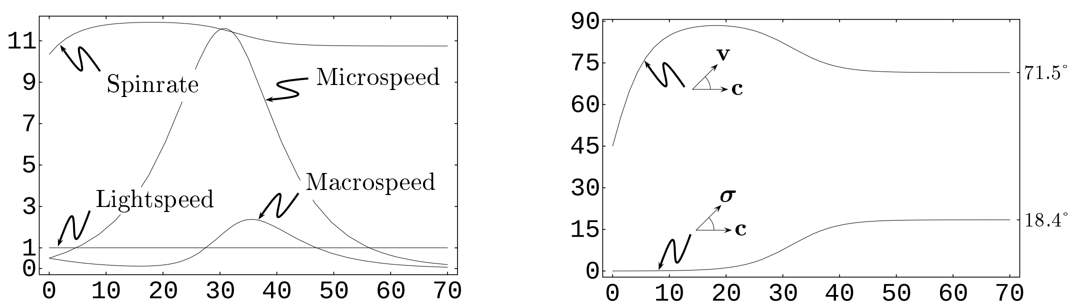


FIG. 4. A PORTION OF A TYPICAL HELICAL TRACK OF A FREE SPINNING PARTICLE OF NONZERO INERTIAL REST MASS. THE PARAMETER INTERVAL IS $[0, 70]$, SAMPLED AT INTERVALS OF $.05$. THE PARTICLE MOVES FROM LOWER LEFT TO UPPER RIGHT ON A CYLINDER OF RADIUS $R_\perp = 50$, WHOSE AXIS VECTOR $\mathbf{c} \approx .00045 \cdot [\![1, 1, 1]\!]$. SOME INITIAL CONDITIONS ARE $t \approx -8.78$, $E \approx .13$, $\mathbf{v} \approx [\![.086, -.009, .038]\!]$, $\boldsymbol{\sigma} \approx [\![5.000, 5.003, 5.001]\!]$, $v_\parallel/E = .5$, $v_\perp/E = .5$, AND $v/E = \sqrt{.5} \approx .707$ (SPEED OF LIGHT $= 1$).

The visible compression of the coils of the helix reflects the well-known phenomenon that in de Sitter's universe all freely moving test particles come asymptotically to rest at a point in space (though continuing to spread apart as space itself expands).[(4)] The mere existence of these coils, owed specifically to the inclusion of spin by way of the finslerian geometry, brings to mind the quantum mechanical phenomenon of "Zitterbewegung" of a spinning electron. This "jitter motion," whose existence Schrödinger deduced from Dirac's relativistic wave equation,[(5,6)] is a "microscopic" oscillatory perturbation of the "macroscopic" propagation motion of the electron. The microscopic "zitterspeed" equals the speed of light, but the "macrospeed" is less. In one of the manifestations of Zitterbewegung the electron appears to follow a helical path that winds around a line representing its macroscopic path of propagation through space.[(7)] In the present development the quantities $v_\parallel/E$ (macrospeed) and $v_\perp/E$ (microspeed), scaled so that lightspeed $= 1$, play roles somewhat analogous to the macroscopic speed and the microscopic (zitter)speed of the helical

Zitterbewegung manifestation. The de Sitter phenomenon affects both the macrospeed, causing the compression of the coils, and the microspeed, causing the circulatory motion (but not the spinning) to stop. Figure 5 displays these effects explicitly, along with the variations of the angles that the velocity $\mathbf{v}$ and the spin vector $\boldsymbol{\sigma}$ make with the axis vector $\mathbf{c}$ of the helix, and of the particle's spinrate $(2\pi)^{-1}(\sigma/E)$ (in revolutions per unit of time t).



Speeds and Spinrate

Angles

FIG. 5. GRAPHS OF SPEEDS, SPINRATE, AND ANGLES FOR THE SPINNING PARTICLE FOLLOWING THE HELICAL TRACK OF FIG. 4. AFTER FALLING TO A LOCAL MINIMUM JUST ABOVE .1, THE MACROSPEED RISES TO A MAXIMUM JUST UNDER 2.4 AS THE MICROSPEED TRAVERSES ITS PEAK NEAR 11.6 (LIGHTSPEED = 1). INITIALLY THE ANGLES THAT $\mathbf{v}$ AND $\boldsymbol{\sigma}$ MAKE WITH $\mathbf{c}$ ARE 45° AND APPROXIMATELY $.01^\circ$, RESPECTIVELY; AT THE END, APPROXIMATELY 71.5° AND 18.4°. THEIR SUM, THE ANGLE BETWEEN $\mathbf{v}$ AND $\boldsymbol{\sigma}$, TENDS ASYMPTOTICALLY TO 90°.

The falling of the macrospeed to a local minimum produces the compression of the helical coils seen in Fig. 4. Its subsequent rise to a maximum while the microspeed is traversing its peak and the spinrate is decreasing is responsible for the expansion of the coils after the compression. This interplay among kinematical variables can be interpreted as a subtle transfer of spin inertia and orbital (micro)inertia to linear (macro)inertia as the angle from $\mathbf{c}$ to $\boldsymbol{\sigma}$ increases and the angle from $\mathbf{c}$ to $\mathbf{v}$ decreases. For a clear understanding of these unusual behaviors it is essential to remember that we are not examining motion of a pointlike particle. Instead, we are looking at eccentric motion of a center of a spinning sphere whose radius, according to the relation $R = e^{-t}$, is about 6495 initially, when $t \approx -8.78$, and about 1.47 at the end, when $t \approx -.38$. Moreover, the "four-point" derivation of the finslerian metric function of Eqs. (8) makes evident that the differential interaction of this spinning sphere with itself, captured in the "stationarizing" of the finslerian arc length integral, is an interaction taking place on the sphere itself, far from its helically moving center.

If a helical track and precessing spin and velocity are typical for a free spinning particle of nonzero rest mass, what is typical for a free spinning particle of zero rest mass, defined as one traveling "at the same speed as light," thus on a geodesic on which $E^2 - v^2 = 0$ at all times? For such a particle ϵ must be 0 (one can show), and then the second of Eqs. (13) implies that $\mathbf{v} \times \boldsymbol{\sigma} = \mathbf{0}$, hence that $\boldsymbol{\sigma}$ must be parallel or antiparallel to $\mathbf{v}$. Equations (15) and (17) then entail that $R_\perp = 0$, thus that the track is straight. This behavior replicates some of the behavior of spinning photons predicted by the quantum theory of light, and does so without the aid of a Hilbert space, an operator, a bra, or a ket.

It is both remarkable and highly suggestive that, departing from the very elementary construction on Euclidean spheres presented here, we can a) arrive at a purely geometrical theory of the kinematics of free, spinning particles in an expanding universe, b) upon arrival, look about and find that we have somewhat unintentionally modeled certain exotic behaviors of such particles, behaviors first encountered in the quantum mechanical study of spinning electrons and photons, and c) looking back, come to suspect that we have peered a little deeper into the mystery of time. This short trip is perhaps in itself a good day's journey, but it only foreshadows the labor, the pleasure, and the satisfaction of many (maybe even *infinitely* many) days beyond to be spent sailing the high seas of the Ocean of Finslerian Geometry. For just as the construction of the Riemannian angle, or "two-point," metric of Eq. (1) can be extended from the manifold of spheres in Euclid's space to the manifold of hyperspheres in Minkowski's space-time to produce a theory of "space-time–time" (as I outlined in Ref. 1 and have elaborated in Ref. 8), the construction of the finslerian "four-point" metric function of Eqs. (8) can in direct analogy be extended from the manifold of rotated Euclidean spheres to the manifold of Lorentz rotated Minkowskian hyperspheres to make a theory of spinning particles in space-time-time, then further to the manifold of rotated hyperspheres of space-time-time, and extended yet again — time after time after time . . .

IN MEMORIAM. Throughout the writing of this paper came often to mind fond memories of Asim Orhan Barut (1926–1994), a kind and gentle spirit ever seeking the light.

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First version: March, 1995
Revised: September, 1995
Revised: October, 1995
Revised: October, 1996

Homer G. Ellis
Department of Mathematics
University of Colorado at Boulder
395 UCB
Boulder, Colorado 80309-0395

Telephone: (303) 492-7754 (office); (303) 499-4027 (home)
Email: ellis@euclid.colorado.edu
Fax: (303) 492-7707