Darkholes: Nicer than blackholes — with a bright side, too*
(Does energy produce gravity?)

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The geometry of three-dimensional space guides the search for a better model than the blackhole with its unwelcome singularity. An elementary construction produces on the 4-manifold of 2-spheres in a Riemannian 3-space a space-time metric invariant under uniform conformal transformations of the 3-space. When the 3-space is Euclidean, the metric reduces to de Sitter's expanding universe metric. Generalization yields a space-time metric that retains the 'exponential expansion property' of the de Sitter metric. A strictly geometric action principle gives field equations which, because they do not adhere to Einstein's early confounding of energy and inertial mass with gravitating mass, admit solutions that escape the Penrose–Hawking singularity theorems. A spherically symmetric solution that is asymptotic to the Schwarzschild blackhole metric has, in place of a horizon and a singularity, an Einstein–Rosen 'bridge', or 'tunnel', connecting two asymptotically Euclidean regions. On one side the gravitational center attracts, and is dark but not black; on the other side it repels, and is bright. Travel and signaling from either side to the other *via* the tunnel are possible. Analysis of the Einstein tensor of this 'darkhole' (or 'darkhole–brighthole') suggests that not all energy produces gravity, and that calling energy 'negative', or its relationship to geometry 'exotic', is unjustified.

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Blackholes with singularities are not satisfactory models of real things, for at the singularities they lose their predictive powers, causing one to throw up the hands and mutter some such incantation as "Quantum effects take over." It is easy to construct a blackhole without a singularity: if, for $-\infty < \rho < \infty$,

$$\hat{G} = dt^2 - [d\rho - u(\rho)\,dt]^2 - r^2(\rho)\,d\Omega^2, \tag{1}$$

where $d\Omega^2 := d\vartheta^2 + (\sin\vartheta)^2 d\varphi^2$, then $\hat{G}$ has no singularity, provided that neither $u(\rho)$ nor $r(\rho)$ has, that $u(\rho)$ and $r(\rho)/\rho$ are bounded at $-\infty$ and at ∞, and that $r(\rho)$ is bounded away from 0. Blackness occurs if there is a region in which $|u(\rho)| > 1$, bounded by a sphere (or two spheres) on which $|u(\rho)| = 1$ — for the following reasons: Photon orbits are characterized by the equations

$$\frac{d\rho}{dt} = u(\rho) \pm \sqrt{1 - r^2(\rho)\left(\frac{d\Omega}{dt}\right)^2}. \tag{2}$$

In a region where $|u(\rho)| > 1$, either $u(\rho) > 1$ throughout or $u(\rho) < -1$ throughout. In the first case $d\rho/dt > 0$, in the second, $d\rho/dt < 0$. In either case all photons in that region are going in the same direction radially. None can have entered the region through a bounding sphere that all are approaching, and none can leave it through a bounding sphere from which all are retreating. In each case the sphere is a horizon for light, thus also for test particles moving slower than light.

The coordinate transformation $T = t + \int u(\rho)\,[1 - u^2(\rho)]^{-1}\,d\rho$ changes the expression of $\hat{G}$ to

$$\begin{aligned}\hat{G} = {} & \left[1 - u^2(\rho)\right] dT^2 \\ & - \left[1 - u^2(\rho)\right]^{-1} d\rho^2 - r^2(\rho)\,d\Omega^2.\end{aligned} \tag{3}$$

If $r(\rho) \sim \rho$ and $u^2(\rho) \sim 2m/\rho$ as $\rho \to \infty$, the metric behaves like the Schwarzschild metric of mass parameter m, so it can model the far field of a spherically symmetric gravitating object. Because $r^2(\rho)$ stays away from 0, any region interior to a horizon is spacious: it does not squeeze down to a point at which a singularity could develop, as the Schwarzschild inner region does. A construction analogous to the Kruskal–Fronsdal extension of the Schwarzschild metric would show that each horizon serves as a neck of a wormhole connecting two or more regions in which $|u(\rho)| < 1$.

Such a singularity-free blackhole cannot, of course, be a solution of the Einstein field equations. It must in fact escape in some way the Penrose–Hawking singularity theorems [1], and this it can do only by violating one of the hypotheses of those theorems. Among those hypotheses suspicion attaches most readily to the requirement that the Ricci tensor be everywhere nonnegative definite with respect to null or timelike vectors. This so-called 'energy condition' is conventionally taken to mean that the density of energy, in whatever form, that 'produces' a gravitational field must be nowhere on balance negative. As 'negative energy' is believed to be an attribute only of never observed 'exotic' matter, the energy condition is almost universally accepted as realistic. That acceptance, however, rests ultimately on a questionable identification that traces all the way back to Einstein's 1916 paper Die Grundlage der allgemeinen Relativitätstheorie [2].

*Revision and amplification of Darkholes: Blackholes' Better Behaved Cousins, selected for honorable mention in the Gravity Research Foundation 1999 Awards for Essays on Gravitation.

In that paper's §16, titled in translation The General Form of the Field Equations of Gravitation, Einstein seeks a tensorial equation to correspond to the Poisson equation $\nabla^2\phi = 4\pi\kappa\rho$, where ρ denotes the "density of matter". Drawing on the special theory of relativity's identification of "inert mass" with "energy, which finds its complete mathematical expression in . . . the energy-tensor", he concludes that "we must introduce a corresponding energy-tensor of matter T_σ^α". Further describing this energy-tensor as "corresponding to the density ρ in Poisson's equation", he goes on to invent the field equations $E_{\mu\nu} = \kappa T_{\mu\nu}$ that bear his name and have the built-in consequence that, wherever energy density is nonnegative for all observers, the Ricci tensor is nonnegative definite with respect to null and timelike vectors (here E is the Einstein tensor $\Phi - \frac{1}{2}\Psi G$, where G is the metric tensor, Φ is the Ricci tensor, and Ψ is the curvature scalar; see Appendix for definitional conventions).

The questionable identification referred to is the confounding of 'gravitating mass', which is the sole contributor to the "density of matter" in Poisson's equation, with "inert mass", thus with energy by way of $E = mc^2$. That all bodies respond alike to a gravitational field establishes the equivalence of '*passive*' (gravitat*ed*) mass with 'inertial' (inert) mass, but an equivalence between '*active*' (gravitat*ing*) mass and passive(–inertial) mass is in no way implied. The distinction between active mass and passive mass, well explicated by Bondi [3], is present already in Newton's gravitational equation $m_{\text{inertial}}\, a = -G\, M_{\text{active}}\, m_{\text{passive}}/r^2$, where M_{active} and m_{passive} are properties of entirely different bodies, one doing the acting, the other receiving the action.[1]

If active mass is not equivalent to passive mass, it is not equivalent to inertial mass, thus is not equivalent to energy. Unresolved, therefore, is whether all constituents of matter and energy gravitate, and of those that do, whether they attract or repel gravitationally. In an experiment by Kreuzer [4], two congruent, homogeneous bodies, differently constituted but weighing the same, were seen to exert the same gravitational attraction on test particles (within experimental precision). This indicates equality of the ratio of active to passive mass for the two macroscopic bodies, but it says nothing about the gravitational effects of energy, or of any particular species of the particles that make up matter. It is consistent with this observation to suppose, for example, that only nucleons produce gravitational effects, that energy and other particles such as electrons and neutrinos do not gravitate at all. To see this, consider an idealized Kreuzer experiment in which body A is is made of a single isotope of one element, each of whose atoms has p_{A} protons, the same number of electrons, and n_{A} neutrons, and body B is made of a single isotope of another element, each atom of which has p_{B} protons and electrons, and n_{B} neutrons, with $p_{\mathrm{A}} + n_{\mathrm{A}} = p_{\mathrm{B}} + n_{\mathrm{B}}$, and $p_{\mathrm{A}} > p_{\mathrm{B}}$. Next, perform the thought experiment of reversing beta decay in each atom of body A by stuffing $p_{\mathrm{A}} - p_{\mathrm{B}}$ of its atomic electrons, along with as many antineutrinos, into its nuclear protons, thus turning the protons into neutrons and the A atoms into B atoms, maintaining congruence all the while. Now the bodies are identical, and their weights are still the same — but so are their active masses, despite that antineutrinos have been added and binding energies have changed. It is conceivable that the binding energies and the antineutrinos have increased A's active gravitational mass, and that this increase is exactly compensated by a decrease owed to a loss of molecular kinetic energy necessary to maintain A's size and weight. It is also conceivable that they have *de*creased A's active mass, and that this is compensated by an increase of kinetic energy. It is, however, equally conceivable (and from a probabilistic standpoint even more likely) that the binding energies, the antineutrinos, and the kinetic energy produce *no* gravity — that only the nucleons and perhaps (but perhaps not) the electrons have nonzero active gravitational mass. Any contradiction of this in the form of a measurement of the gravity of an isolated electron, antineutrino, or quantum of energy would seem a distant prospect at best. Absent such a measurement, the 'energy condition' is an unproven hypothesis, nothing more.

In what rational way might one replace the Einstein field equations with others that allow violations of the 'energy condition'? Geometry should be the guide, according to Einstein, who likened his equations to a building with two wings, one made of fine marble (the geometrical tensor), the other of low-grade wood (the matter tensor) [5]. All the better, a purist says, if it is the geometry of real three-dimensional space, not the pseudogeometry of space-time in which 'time' is a fourth dimension, independent of and unrelated to the three spatial dimensions. Precisely that geometry is the guide for the construction that follows.

Let $\mathring{G}$ be a positive definite Riemannian metric on a three-dimensional manifold $\mathcal{M}$ that is geodesically complete with respect to $\mathring{G}$. The 2-sphere in $\mathcal{M}$ of radius R centered at the point C is the set of all points whose distance from C along a geodesic is R. The set of all such spheres is itself a four-dimensional manifold $\hat{\mathcal{M}}$. Let S and S' be neighboring spheres in $\mathcal{M}$, centered at C and C', of radii R and $R + dR$. Starting at C and following a geodesic through C' one arrives at a point P on S and a point P' on S' separated by a (directed) geodesic distance $dR + ds$, where ds is the geodesic distance from C to C' (see Fig. 1). Going in the other direction one arrives at points Q on S and Q' on S' separated by $dR - ds$. The product of these separations, each normalized by division by $R/\hat{R}$, where $\hat{R}$ is a positive constant, provides a normal hyperbolic metric $\hat{G}$ on $\hat{\mathcal{M}}$ that is invariant under all

[1]That Einstein confounded active mass with passive–inertial mass, knowingly or unknowingly, is borne out further by the statement in his §16 that for a "complete system (e.g. the solar system), the total mass of the system, and therefore its total gravitat*ing* action as well, will depend on the total energy of the system, and therefore on the *ponderable* energy together with the gravitational energy." (Emphasis added.)

uniform conformal transformations of $\mathring{G}$ ($\mathring{G} \to k\mathring{G}$ with k a positive constant), viz.,

$$\hat{G} = \hat{R}^2 \left(\frac{dR^2 - ds^2}{R^2} \right). \tag{4}$$

Assigning to each 2-sphere in $\mathcal{M}$ a time t related to its radius R by $t := -\ln(R/\hat{R})$ gives $\hat{G}$ the form

$$\hat{G} = \hat{R}^2\, dt^2 - e^{2t}\, ds^2. \tag{5}$$

Upon particularization of $\mathring{G}$ to be the metric of Euclidean 3-space, $\hat{G}$ reduces to the metric of de Sitter's expanding universe model [6], a solution of the Einstein vacuum equation

$$\hat{E} := \hat{\Phi} - \tfrac{1}{2}\hat{\Psi}\hat{G} = \Lambda\hat{G} \tag{6}$$

with cosmological constant $\Lambda = 3/\hat{R}^2$; $\hat{R}$ is the uniform space-time radius of curvature of this empty universe.

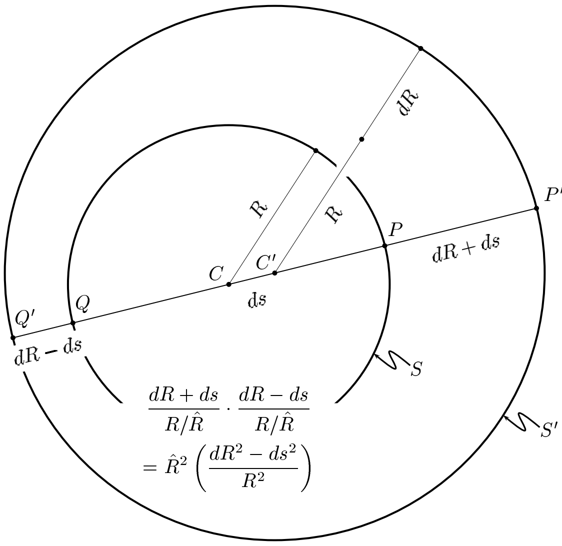

FIG. 1. Neighboring 2-spheres S and S' in the Riemannian 3-space $\{\mathcal{M}, \mathring{G}\}$, shown in cross section through the geodesic $Q'QCC'PP'$, separated by the 'two-point' distance $\hat{R}\sqrt{(dR^2 - ds^2)/R^2}$.

In tensor product form

$$\hat{G} = \hat{R}^2(dt \otimes dt) - e^{2t}\,\mathring{G} \tag{7a}$$
$$= \hat{R}^2(dt \otimes dt) - e^{2t}(dx^m \otimes \mathring{g}_{mn}\, dx^n). \tag{7b}$$

There is on $\hat{\mathcal{M}}$ a vector field ξ, namely, $\xi := \partial/\partial t$, with respect to which $\hat{G}$ has the following 'exponential expansion property': $\mathcal{L}_\xi \hat{G} = 2\,G$, where $G := \hat{G} - (\hat{G}\xi\xi)^{-1}(\hat{G}\xi \otimes \hat{G}\xi)$, $\mathcal{L}_\xi$ denoting Lie differentiation along ξ. (Note that $\hat{G}\xi = \hat{R}^2 dt$, $\hat{G}\xi\xi = \hat{R}^2$, and $G = -e^{2t}\,\mathring{G}$.) Generalizing, let $\hat{G}$ now be *any* space-time metric of signature $+---$ defined on a manifold $\hat{\mathcal{M}}$ on which there is a time-like vector field ξ with respect to which $\hat{G}$ has the exponential expansion property. One can show that (locally, at least) there exist on $\hat{\mathcal{M}}$ coordinate systems $[\![t, x^m]\!]$ for which $\xi = \partial/\partial t$ and $\hat{G}$ takes the form

$$\hat{G} = \phi^2\big(dt + \mathring{A}\big) \otimes \big(dt + \mathring{A}\big) - e^{2t}\,\mathring{G} \tag{8a}$$
$$= \phi^2\big(dt + \mathring{A}_m\, dx^m\big) \otimes \big(dt + \mathring{A}_n\, dx^n\big) - e^{2t}\big(dx^m \otimes \mathring{g}_{mn}\, dx^n\big), \tag{8b}$$

with ϕ, $\mathring{A}_m$, and $\mathring{g}_{mn}$ independent of t.

The Ricci tensor $\hat{\Phi}$ and curvature scalar $\hat{\Psi}$ of $\hat{G}$ are expressible in terms of those of $\mathring{G}$ and covariant derivatives of ϕ and $\mathring{A}$ with respect to $\mathring{G}$. One can define 'residuals' $\hat{\Phi}_\infty$ and $\hat{\Psi}_\infty$ of $\hat{\Phi}$ and $\hat{\Psi}$ (roughly, $\hat{\Phi}_\infty := \lim_{t\to\infty}(e^{-2t}\hat{\Phi})$, and, exactly, $\hat{\Psi}_\infty := \lim_{t\to\infty}\hat{\Psi}$). One then finds that $\hat{\Phi}_\infty = -3\,\phi^{-2}\hat{G}$ and $\hat{\Psi}_\infty = -12\,\phi^{-2}$, thus that

$$\hat{\Phi}_\infty - \tfrac{1}{2}\hat{\Psi}_\infty\hat{G} = \Lambda\hat{G}, \tag{9}$$

where $\Lambda := 3/\phi^2$. Comparison with the de Sitter model shows that the scalar field Λ could be termed the 'residual cosmological (non)constant', and the scalar field ϕ the 'residual (nonuniform) radius of curvature', of the generalized model. In the de Sitter model $\hat{\Phi}vv = -\Lambda\hat{G}vv$, which vanishes if v is a null vector, and is *negative* if v is timelike. Here the same is true of $\hat{\Phi}_\infty$.

Field equations are obtained from the strictly geometric action principle $\delta\mathcal{A} = 0$, where

$$\mathcal{A}(\phi, \mathring{A}_m) := \int_{\hat{\mathcal{D}}} \big(\hat{\Psi} - \hat{\Psi}_\infty\big)\, d\hat{V} \tag{10a}$$
$$= \int_{\mathcal{D}} \int_a^b \big(\hat{\Psi} - \hat{\Psi}_\infty\big)\, dt\, dV, \tag{10b}$$

the region $\hat{\mathcal{D}}$ having the cylindrical form $\hat{\mathcal{D}} = [a, b] \times \mathcal{D}$, where $\mathcal{D}$ is a bounded region of a cross section of $\hat{\mathcal{M}}$ transverse to ξ. The variations of ϕ and $\mathring{A}_m$ are to vanish on $[a, b] \times \partial\mathcal{D}$. The spatial metric $\mathring{G}$ is treated as given *a priori* on $\mathcal{D}$, and extended to $\hat{\mathcal{D}}$ by translations along ξ. Variation of ϕ yields the equation

$$\mathring{A}^k{}_{:k} - \mathring{A}^k\mathring{A}_k = \tfrac{3}{8}\phi^2 \mathring{F}_k{}^l \mathring{F}_l{}^k - \tfrac{1}{2}\mathring{\Psi}; \tag{11}$$

variation of $\mathring{A}_m$ yields

$$\mathring{F}^{mk}{}_{:k} + 3\,\phi^{-1}\mathring{F}^{mk}\phi_{.k} = 2\,\phi^{-2}\big(\phi^{-1}\mathring{\phi}^{.m} + 2\mathring{A}^m\big). \tag{12}$$

Here $(\ )_{.m} := \partial(\ )/\partial x^m$, $F_{mn} := \mathring{A}_{n.m} - \mathring{A}_{m.n}$, and insertion of a $\,^\circ\,$ indicates raising of an index by $\mathring{g}^{mn}$. The covariant differentiations indicated by a $_:$ are with respect to $\mathring{G}$. A constant factor $e^{-(a+b)/2}$ arising from the t integration has been absorbed into ϕ; this leaves in the equations *no* arbitrary coupling constant with which to finesse the 'energy condition' question.

Examining these field equations for a metric of the spherically symmetric form

$$\hat{G} = e^{2U(\rho)}\big[dt + V(\rho)\, d\rho\big]^2 - e^{2t}e^{-3U(\rho)}\left[d\rho^2 + r^2(\rho)\, d\Omega^2\right], \tag{13}$$

one finds them to be satisfied if

$$U' = -2V = \frac{m\hat{R}}{r^2}, \quad r'' = \frac{1-r'^2}{2r} - \frac{7}{8}\frac{m^2\hat{R}^2}{r^3}, \tag{14a}$$

$$U(\infty) = \ln\hat{R}, \quad r(0) = r_0, \quad \text{and} \quad r'(0) = 0, \tag{14b}$$

where each of m, $\hat{R}$, and r_0 is a constant, $\hat{R} > 0$, and $0 \leq m < m_{\text{crit}} := (2/\sqrt{7})(r_0/\hat{R})$.

The coordinate changes $T := \hat{R}\left[t + \int V(\rho)\,d\rho\right] = \hat{R}\left[t - \frac{1}{2}U(\rho)\right]$ and $\bar{\rho} := \rho/\hat{R}$ make

$$\hat{G} = e^{2\bar{U}(\bar{\rho})}dT^2 - e^{2T/\hat{R}}\left[e^{-2\bar{U}(\bar{\rho})}d\bar{\rho}^2 + \bar{r}^2(\bar{\rho})\,d\Omega^2\right], \tag{15}$$

where $\bar{U}(\bar{\rho}) := U(\rho) - \ln\hat{R}$ and $\bar{r}(\bar{\rho}) := e^{-U(\rho)}r(\rho)$.

On a human time scale the cosmological expansion factor $e^{2T/\hat{R}}$ can be treated as a constant, say $e^{2T_0/\hat{R}}$, and absorbed into the spatial metric by the transformations $\tilde{\rho} := e^{T_0/\hat{R}}\bar{\rho}$ and $\tilde{r}(\tilde{\rho}) := e^{T_0/\hat{R}}\,\bar{r}(\bar{\rho})$, to produce

$$\hat{G} \approx \hat{G}_{T_0} := e^{2\tilde{U}(\tilde{\rho})}dT^2 - e^{-2\tilde{U}(\tilde{\rho})}d\tilde{\rho}^2 - \tilde{r}^2(\tilde{\rho})\,d\Omega^2 \tag{16a}$$

$$= \left[1 - u^2(\tilde{\rho})\right]dT^2 - \left[1 - u^2(\tilde{\rho})\right]^{-1}d\tilde{\rho}^2 - \tilde{r}^2(\tilde{\rho})\,d\Omega^2 \tag{16b}$$

$$= d\tilde{t}^2 - \left[d\tilde{\rho} - u(\tilde{\rho})\,d\tilde{t}\right]^2 - \tilde{r}^2(\tilde{\rho})\,d\Omega^2, \tag{16c}$$

where $\tilde{U}(\tilde{\rho}) := \bar{U}(\bar{\rho})$, $u(\tilde{\rho}) := -\sqrt{1 - e^{2\tilde{U}(\tilde{\rho})}}$, and $\tilde{t} := T - \int u(\tilde{\rho})\left[1 - u^2(\tilde{\rho})\right]^{-1}d\tilde{\rho}$. Because Eqs. (16b) and (16c) replicate Eqs. (3) and (1), the previous discussion of horizons, blackness, and singularities applies directly to the metric $\hat{G}_{T_0}$.

Numerical integration produces the plots shown in Figs. (2–5), for which $r_0 = 1$, $\hat{R} = 10^6$, $m_{\text{crit}} \approx 7.6\times 10^{-7}$, $m = 0.5\,m_{\text{crit}}$, and $T_0 = 0$. The minimum of $r = r(0) = r_0 = 1$, whereas $\tilde{r}_{\min} \approx \tilde{r}(9.3\times 10^{-7}) \approx 1.93\times 10^{-6}$.

Equations (14) can also be integrated by hand (see Appendix). The result is that

$$\operatorname{sgn}(r'(\rho))\,\rho = \sqrt{(r - r_0)(r - \gamma^2 r_0)} + (1+\gamma^2)r_0 \ln\left(\frac{\sqrt{r - r_0} + \sqrt{r - \gamma^2 r_0}}{\sqrt{(1-\gamma^2)r_0}}\right), \tag{17}$$

where $\gamma := m/m_{\text{crit}} < 1$, and

$$U(\rho) = \ln\hat{R} + \frac{4}{\sqrt{7}}\ln\left(\frac{\sqrt{r - \gamma^2 r_0} + \operatorname{sgn}(\rho)\gamma\sqrt{r - r_0}}{(1+\gamma)\sqrt{r}}\right). \tag{18}$$

Equation (17) implicitly defines r as a function of ρ on the interval $-\infty < \rho < \infty$, with minimum value $r(0) = r_0$, and with $\operatorname{sgn}(r'(\rho)) = \operatorname{sgn}(\rho)$. It is clear from this equation that, as $\rho \to \pm\infty$, $r(\rho) \sim \infty$, $\rho/r(\rho) \sim \pm 1$, and, consequently, $r(\rho) \sim \pm\rho$. From this and $U' = m\hat{R}/r^2$ it follows that, as $\rho \to \infty$,

$$U(\rho) = U(\infty) + \int_\infty^\rho \frac{m\hat{R}}{r^2(\lambda)}\,d\lambda \tag{19a}$$

$$\sim U(\infty) + \int_\infty^\rho \frac{m\hat{R}}{\lambda^2}\,d\lambda = \ln\hat{R} - \frac{m\hat{R}}{\rho}, \tag{19b}$$

and, as $\rho \to -\infty$,

$$U(\rho) = U(-\infty) + \int_{-\infty}^\rho \frac{m\hat{R}}{r^2(\lambda)}\,d\lambda \tag{20a}$$

$$\sim U(-\infty) + \int_{-\infty}^\rho \frac{m\hat{R}}{\lambda^2}\,d\lambda = \ln\hat{R} + \frac{4}{\sqrt{7}}\ln\left(\frac{1-\gamma}{1+\gamma}\right) - \frac{m\hat{R}}{\rho}. \tag{20b}$$

Further,

$$\frac{\tilde{r}(\tilde{\rho})}{\tilde{\rho}} = \frac{\bar{r}(\bar{\rho})}{\bar{\rho}} = \hat{R}e^{-U(\rho)}\frac{r(\rho)}{\rho} \sim \begin{cases} 1 & \text{as } \rho \to \infty, \\ -\left(\dfrac{1+\gamma}{1-\gamma}\right)^{4/\sqrt{7}} & \text{as } \rho \to -\infty. \end{cases} \tag{21}$$

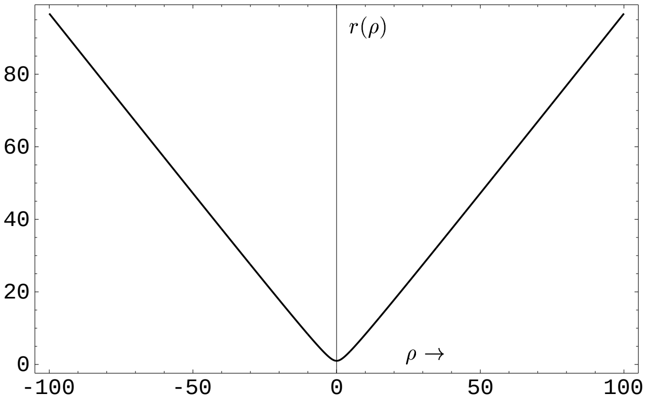

FIG. 2. Plot of the spatial geometric descriptor $r(\rho)$ on the interval $-100 < \rho < 100$.

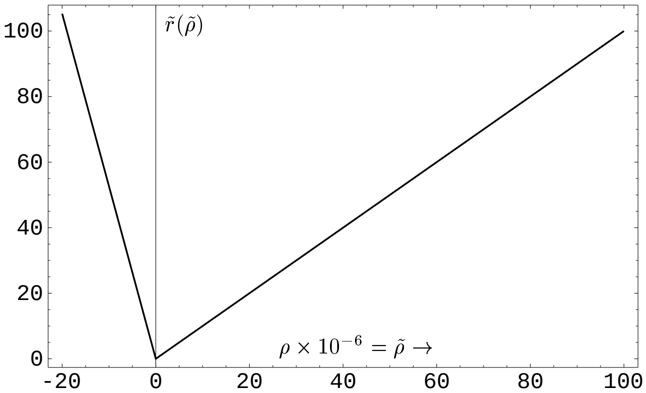

FIG. 3. Plot of the spatial geometric descriptor $\tilde{r}(\tilde{\rho})$, for $T_0 = 0$, on the interval $-2\times 10^7 < \rho < 10^8$.

Also,

$$u^2(\tilde\rho) = 1 - e^{2\tilde U(\tilde\rho)} = 1 - e^{2\bar U(\bar\rho)} = 1 - e^{2[U(\rho) - \ln \hat R]}$$

$$\sim \begin{cases} \dfrac{2m_{T_0}}{\tilde\rho} & \text{as } \rho \to \infty, \\ 1 - \left(\dfrac{1-\gamma}{1+\gamma}\right)^{8/\sqrt7}\left(1 - \dfrac{2m_{T_0}}{\tilde\rho}\right) & \text{as } \rho \to -\infty, \end{cases} \tag{22}$$

where $m_{T_0} := m e^{T_0/\hat R}$, and

$$\Lambda = 3e^{-2U(\rho)}$$

$$\sim \begin{cases} \dfrac{3}{\hat R^2}\left(1 + \dfrac{2m_{T_0}}{\tilde\rho}\right) & \text{as } \rho \to \infty, \\ \dfrac{3}{\hat R^2}\left(\dfrac{1+\gamma}{1-\gamma}\right)^{8/\sqrt7}\left(1 + \dfrac{2m_{T_0}}{\tilde\rho}\right) & \text{as } \rho \to -\infty. \end{cases} \tag{23}$$

On each time-slice of constant $\tilde t$ the line element induced by $\hat G_{T_0}$ is $d\tilde\rho^2 + \tilde r^2(\tilde\rho)\, d\Omega^2$ (see Eq. (16c)). The fact that in the numerical solution $\tilde r(\tilde\rho)$ (consequently, also $\bar r(\bar\rho)$) has a positive minimum value and is asymptotically infinite at $\pm\infty$ tells that in the universe described approximately by $\hat G_{T_0}$ and exactly by $\hat G$ there is an ever present Einstein–Rosen 'tunnel' connecting two regions of asymptotically Euclidean topology [7].[2] Is this a one-way tunnel, or can it accommodate two-way traffic? The answer lies in the behavior of u. If $|u(\tilde\rho)| \geq 1$ somewhere, the traffic is one-way only and there is a blackhole in the vicinity. If $|u(\tilde\rho)| < 1$ everywhere, then traffic is two-way and there is no blackhole. Figure 4 says, "Two-way traffic, no blackhole." The existence of this two-way tunnel with no blackhole is generic for $\gamma < 1$, inasmuch as $\tilde r(\tilde\rho)$ has a positive minimum value (see Appendix) and, in view of (22) and the monotonicity of $U(\rho)$ ($U' = m\hat R/r^2 > 0$), $u^2(\tilde\rho)$ rises monotonically with decreasing ρ from $u^2(\infty) = 0$ to $u^2(-\infty) = 1 - [(1-\gamma)/(1+\gamma)]^{8/\sqrt7} < 1$.

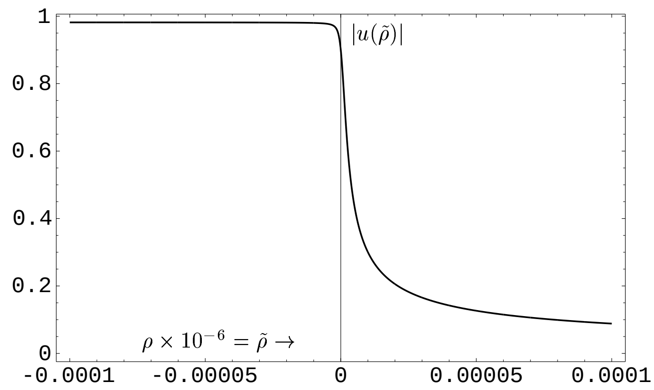

FIG. 4. Plot of the test-particle free-fall speed $|u(\tilde\rho)|$, for $T_0 = 0$, on the interval $-100 < \rho < 100$.

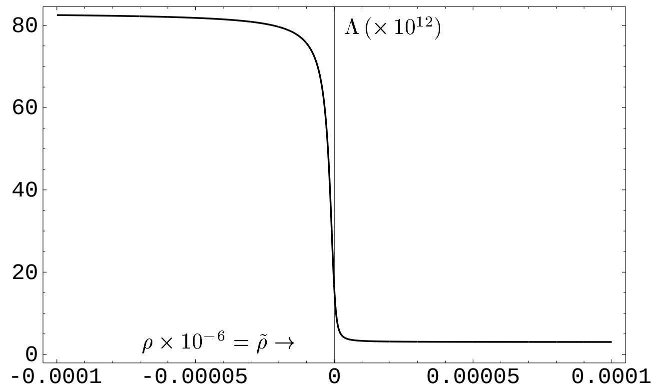

FIG. 5. Plot of the 'residual cosmological (non)constant' Λ versus $\tilde\rho$, for $T_0 = 0$, on the interval $-100 < \rho < 100$ ($\Lambda := 3/\phi^2 = 3e^{-2U(\rho)}$).

An immediate consequence of (21) and (22) is that $\hat G_{T_0}$ is asymptotic to the Schwarzschild metric of (active) mass parameter m_{T_0} as $\rho \to \infty$. Additionally, from $\phi = e^{U(\rho)} \sim e^{U(\infty)} = \hat R$ and $\Lambda = 3/\phi^2 \sim 3/\hat R^2$ as $\rho \to \infty$, we see that, far from the center of gravitation in the positive ρ direction, the residual radius of curvature ϕ and the residual cosmological (non)constant Λ are asymptotic to the radius of curvature and the cosmological constant of the de Sitter universe.

The vector field $\partial_{\tilde t} + u(\tilde\rho)\,\partial_{\tilde\rho}$ is geodesic for $\hat G_{T_0}$; it is the velocity field of a cloud of test particles free-falling downward from rest at ∞. The speed $|u(\tilde\rho)|$ of such a free-falling particle increases monotonically with decreasing $\tilde\rho$ right through the tunnel, out the other side, and on to $-\infty$. This entails that the particle, once past the narrowest part, the 'throat', of the tunnel, behaves as if pushed away from it — that the gravitating center is repulsive on the other, low side of the throat. Moreover, the repulsion is stronger than the attraction, by a ratio of mass parameters equal to $[(1+\gamma)/(1-\gamma)]^{4/\sqrt7}$, which ratio increases to ∞ as $m \to m_{\text{crit}}$ (see Appendix). Because, however, $|u(-\infty)| < 1$, an observer free-falling from rest at ∞ never reaches light speed. With a sufficient means of propulsion the observer could, at any point, turn back and join a cohort of test particles free-falling upward to rest at ∞, flowing with the geodesic velocity field $[1 + u^2(\tilde\rho)][1 - u^2(\tilde\rho)]^{-1}\,\partial_{\tilde t} - u(\tilde\rho)\,\partial_{\tilde\rho}$. If the propulsion failed, he could at least shine a light whose photons would eventually arrive at ∞, redshifted by an amount that is the greater the closer $|u(-\infty)|$ is to 1, which, in view of (22), is the closer that γ is to 1, thus the closer that m is to m_{crit}. A topological hole in space gravitating in such a way is to an observer on the high side a 'darkhole', as dark as you like, but never black. To an observer on the low side it would be a 'brighthole', emanating blue-shifted light that came through the tunnel from the high side.[3]

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[2] Einstein and Rosen spoke of a 'bridge', but 'tunnel' seems to describe the topology better.

[3] For the full cosmological metric $\hat G$ there would come into play the phenomenon, first seen in the de Sitter universe, that

In Newtonian terms, the cloud of test particles falling with velocity $\partial_{\tilde{t}} + u(\tilde{\rho})\,\partial_{\tilde{\rho}}$ would have 'specific kinetic energy' (kinetic energy per unit of inertial mass) $\mathcal{KE} = \frac{1}{2}u^2(\tilde{\rho})$. On the other hand $-\frac{1}{2}u^2(\tilde{\rho})$ can be identified as the Newtonian 'specific gravitational potential' $\mathcal{V}$, in the sense that $-\partial\mathcal{V}/\partial\tilde{\rho} = -m_{T_0}/\tilde{r}^2(\tilde{\rho})$, which is, one can see, the acceleration of the test particles in the cloud. It then is automatic that $\mathcal{KE} + \mathcal{V} = 0$.

The shape-mirroring between the graphs of $|u(\tilde{\rho})|$ and Λ in Figs. 4 and 5 is not accidental, inasmuch as $\Lambda = 3/\hat{R}^2\big(1 - u^2(\tilde{\rho})\big) = 3/\hat{R}^2\big(1 - 2|\mathcal{V}|\big)$. This relationship says that, in the space-time described by the metric $\hat{G}$, not only is the analog of the cosmological constant not constant, it is determined by the specific gravitational potential $\mathcal{V}$, and is smallest where $|\mathcal{V}|$ is smallest, largest where $|\mathcal{V}|$ is largest.

A point to notice is that there is no upper bound on the mass parameter m. The inequality $m < m_{\text{crit}} := \big(2/\sqrt{7}\big)\big(r_0/\hat{R}\big)$ merely correlates m and the hole-sizing parameter r_0; m can grow to any size, but r_0 must grow along with it. Moreover, no matter how small or how large the positive mass m (or, equivalently, the asymptotic mass parameter m_{T_0}), there can be darkholes of that strength that are wide, with a slow flow, because $r_0 \gg \big(\sqrt{7}/2\big)\,m\hat{R}$ (so that $\gamma \approx 0$ and $|u(-\infty)| \ll 1$), and darkholes that are narrow, with a fast flow, because $r_0 \approx \big(\sqrt{7}/2\big)\,m\hat{R}$ (so that $\gamma \approx 1$ and $|u(-\infty)| \approx 1$). The actual size of the hole is determined by the minimum value $\tilde{r}_{\min}$ of $\tilde{r}$, the area of its smallest spherical cross section being $4\pi\tilde{r}^2_{\min}$. This minimum radius is proportional to r_0, the relation being such that $\tilde{r}_{\min}$ rises monotonically from $e^{T_0/\hat{R}}\,r_0/\hat{R}$ to $\big(2/\sqrt{7}\big)\big(1+\sqrt{7}/2\big)^{1+2/\sqrt{7}}\,e^{T_0/\hat{R}}\,r_0/\hat{R} \approx 3.32\,e^{T_0/\hat{R}}\,r_0/\hat{R}$ as m_{T_0} is increased from 0 to $e^{T_0/\hat{R}}\,m_{\text{crit}}$ (concurrently, the value of $\tilde{\rho}$ at which the minimum occurs increases from 0 to $\big[2/\sqrt{7} + \ln\big(4/\sqrt{7}\big)\big]\,e^{T_0/\hat{R}}\,r_0/\hat{R} \approx 1.17\,e^{T_0/\hat{R}}\,r_0/\hat{R}$; see Appendix). Thus the radius of the hole is of the order of magnitude of $e^{T_0/\hat{R}}\,r_0/\hat{R}$ for all admissible values of its asymptotic mass parameter m_{T_0}. That $\tilde{r}_{\min}$ grows larger with m_{T_0} one can 'explain' as follows: to make room for the increasing gravitational flux, the tunnel from the dark side to the bright side expands as if the space it is made of possessed elasticity in the directions transversal to the flow. As well, the bright side can be said to grow infinitely more 'roomy' in comparison to the dark side as $m_{T_0} \to e^{T_0/\hat{R}}\,m_{\text{crit}}$, a consequence of the asymptotic behavior of $\tilde{r}(\tilde{\rho})/\tilde{\rho}$ for $\rho \to -\infty$ (and $\gamma \to 1$) displayed in (21).

Further to be noticed is that although m was restricted to nonnegative values, this restriction is not dictated by the mathematics. Nothing essential is lost by retaining it, however, for every solution of Eqs. (14) with $m < 0$ has a companion solution with $m > 0$ such that their respective human time-scale metrics $\hat{G}_{T_0}$ are mirror images of one another under an isometry that reverses the sense of ρ and maps the asymptotic regions $\rho \sim \pm\infty$ of one of the space-time manifolds to the opposite asymptotic regions of the other (see Appendix). The darkhole and the brighthole make an indivisible, organic whole, not affected by any pretense that its 'mass' is negative.

Finally, consider the question of 'negative energy' and 'exotic matter'. To a high-side observer at a reasonable distance from the center the darkhole is just a normal gravitational attractor, able to exhibit all of the *visible* features of a blackhole. To a low-side observer the brighthole is repulsive, and thus popularly termed 'exotic'. Is the energy density therefore positive on the dark side and negative on the bright side? In a strict sense the question is not meaningful, inasmuch as Einstein's assumed relationship between energy and geometry has been explicitly disallowed here, with no substitute put in its place. If one admits, however, that the Einstein tensor $\hat{E}$, which is conserved (that is, has covariant divergence zero), represents in some fashion a connection between energy and geometry, then examination of $\hat{E}$ is in order. That tensor decomposes naturally into three parts, as follows:

$$\hat{E} = \hat{E}_{\text{expansion}} + \hat{E}_{\text{gravity}} + \hat{E}_{\text{space}}. \tag{24}$$

In terms of the orthonormal coframe system $\{\hat{\omega}^\mu\}$, proper to the class of (generally noninertial) observers at rest in the coordinate system $[\![\rho, \vartheta, \varphi]\!]$, defined by

$$\hat{\omega}^0 := e^{U(\rho)}\,dT/\hat{R}, \tag{25a}$$
$$\hat{\omega}^1 := e^{T/\hat{R}}e^{-U(\rho)}\,d\rho, \tag{25b}$$
$$\hat{\omega}^2 := e^{T/\hat{R}}e^{-U(\rho)}r(\rho)\,d\vartheta, \tag{25c}$$
$$\hat{\omega}^3 := e^{T/\hat{R}}e^{-U(\rho)}r(\rho)(\sin\vartheta)\,d\varphi, \tag{25d}$$

such that

$$\hat{G} = \hat{\omega}^0\otimes\hat{\omega}^0 - \hat{\omega}^1\otimes\hat{\omega}^1 - \hat{\omega}^2\otimes\hat{\omega}^2 - \hat{\omega}^3\otimes\hat{\omega}^3, \tag{26}$$

these parts are expressed by (see Appendix)

$$\hat{E}_{\text{expansion}} = \Lambda\hat{G} + e^{-T/\hat{R}}\frac{2m\hat{R}}{r^2(\rho)}\left[\hat{\omega}^0\otimes\hat{\omega}^1 + \hat{\omega}^1\otimes\hat{\omega}^0\right], \tag{27}$$

$$\hat{E}_{\text{gravity}} = e^{-2T/\hat{R}}e^{2U(\rho)}\frac{3m^2\hat{R}^2}{r^4(\rho)} \times\left[\hat{\omega}^0\otimes\hat{\omega}^0 + \hat{\omega}^1\otimes\hat{\omega}^1 - \hat{\omega}^2\otimes\hat{\omega}^2 - \hat{\omega}^3\otimes\hat{\omega}^3\right], \tag{28}$$

$$\hat{E}_{\text{space}} = e^{-2T/\hat{R}}e^{2U(\rho)}\frac{4{r_0}^2 + 7m^2\hat{R}^2}{8r_0r^3(\rho)} \times\left[-2(\hat{\omega}^1\otimes\hat{\omega}^1) + \hat{\omega}^2\otimes\hat{\omega}^2 + \hat{\omega}^3\otimes\hat{\omega}^3\right]. \tag{29}$$

all nontachyonic test particles come asymptotically to rest at a point in space as $T \to \infty$ [6].

Each of these parts is, one can show, *individually* conserved in the same sense that their sum $\hat{E}$ is conserved; thus each may be taken as descriptive of a particular, separate aspect of the space-time. The part $\hat{E}_{\text{expansion}}$ arises primarily from the exponential expansion property of the metric, with some modification owed to the presence of the gravitational attractor–repeller. To the extent that energy can be said to reside in that expansion, its density as it appears to the observers at rest would presumably be the coefficient of $\hat{\omega}^0 \otimes \hat{\omega}^0$ in $\hat{E}_{\text{expansion}}$, which is $3e^{-2U(\rho)}$ $(= \Lambda)$, a positive quantity. By way of comparison, the Einstein tensor $\hat{E}_{T_0}$ of the metric $\hat{G}_{T_0}$ has no counterpart to $\hat{E}_{\text{expansion}}$; it reduces to $\hat{E}_{\text{gravity}} + \hat{E}_{\text{space}}$, but with $e^{T_0/\hat{R}}$ in place of $e^{T/\hat{R}}$, and $e^{-2T_0/\hat{R}}$ in place of $e^{-2T/\hat{R}}$.

There is a clear separation of the primary sources of the energies, momenta, stresses, strains, and pressures that the tensors $\hat{E}_{\text{gravity}}$ and $\hat{E}_{\text{space}}$ presumably display. For $\hat{E}_{\text{gravity}}$ that source is the gravity of the attractor–repeller: $\hat{E}_{\text{gravity}}$ is proportional to the square of the mass parameter m (and inversely proportional to $r^4(\rho)$). For $\hat{E}_{\text{space}}$ the primary source is the curvature of space: the three components of $\hat{E}_{\text{space}}$ are the parts of the sectional curvatures of the metric $d\rho^2 + r^2(\rho)\, d\Omega^2$ that are inversely proportional to $r^3(\rho)$, modified by the factors $e^{-2T/\hat{R}}$ and $e^{2U(\rho)}$.

If the coefficient of $\hat{\omega}^0 \otimes \hat{\omega}^0$ in $\hat{E}_{\text{gravity}}$ is taken to be the energy density of the gravitational field of the attractor–repeller, then that energy density is positive everywhere, on the repulsive side as well as on the attractive side. Moreover, it remains so for all observers moving subluminally, there being no Lorentz boost from the coframe system $\{\hat{\omega}^\mu\}$ to a moving coframe system in which the 00 component of $\hat{E}_{\text{gravity}}$ is not positive (a property shared by the 00 component of $\hat{E}_{\text{expansion}}$).

It is instructive to study the $m = 0$ case. The metric reduces to

$$\hat{G} = dT^2 - e^{2T/\hat{R}}\big[d\bar{\rho}^2 + \bar{r}^2(\bar{\rho})\, d\Omega^2\big], \tag{30}$$

where now $\bar{\rho} = \rho/\hat{R}$ and $\bar{r}(\bar{\rho}) = r(\rho)/\hat{R}$. There is no center of attraction or repulsion, there is just the tunnel connecting the two asymptotically Euclidean regions. An observer can sit at rest wherever and for so long as he pleases and experience as a (nominally) gravitational effect only the ongoing cosmic expansion of the space around him. The Einstein tensor reduces to $\hat{E}_{\text{expansion}} + \hat{E}_{\text{space}}$, with $\hat{E}_{\text{expansion}} = (3/\hat{R}^2)\hat{G}$ and

$$\begin{aligned}\hat{E}_{\text{space}} = e^{-2T/\hat{R}} &\frac{\bar{r}_0}{2\bar{r}^3(\bar{\rho})} \\ &\times \big[-2(\hat{\omega}^1 \otimes \hat{\omega}^1) + \hat{\omega}^2 \otimes \hat{\omega}^2 + \hat{\omega}^3 \otimes \hat{\omega}^3\big],\end{aligned} \tag{31}$$

where $\bar{r}_0 := r_0/\hat{R}$. The only nonzero energy density present is the $3/\hat{R}^2$ contributed by $\hat{E}_{\text{expansion}}$. An alternate way of expressing $\hat{E}_{\text{space}}$ is

$$\begin{aligned}\hat{E}_{\text{space}} = e^{-2T/\hat{R}} \big[&-\kappa_{\vartheta\varphi}(\hat{\omega}^1 \otimes \hat{\omega}^1) \\ &- \kappa_{\rho\varphi}(\hat{\omega}^2 \otimes \hat{\omega}^2) - \kappa_{\rho\vartheta}(\hat{\omega}^3 \otimes \hat{\omega}^3)\big],\end{aligned} \tag{32}$$

where $\kappa_{\vartheta\varphi}$, $\kappa_{\rho\varphi}$, and $\kappa_{\rho\vartheta}$ are the sectional curvatures of the spatial metric $d\bar{\rho}^2 + \bar{r}^2(\bar{\rho})\, d\Omega^2$ referred to the tangent subspaces spanned by $\{\partial_\vartheta, \partial_\varphi\}$, $\{\partial_\rho, \partial_\varphi\}$, and $\{\partial_\rho, \partial_\vartheta\}$, respectively, given by

$$\kappa_{\vartheta\varphi} = \frac{1 - (\bar{r}')^2(\bar{\rho})}{\bar{r}^2(\bar{\rho})} = \frac{\bar{r}_0}{\bar{r}^3(\bar{\rho})} \tag{33}$$

and

$$\kappa_{\rho\varphi} = \kappa_{\rho\vartheta} = -\frac{\bar{r}''(\bar{\rho})}{\bar{r}(\bar{\rho})} = -\frac{\bar{r}_0}{2\bar{r}^3(\bar{\rho})}. \tag{34}$$

Thus the components of $\hat{E}_{\text{space}}$ are just the curvatures of space diluted by the factor $e^{-2T/\hat{R}}$ induced by the ongoing cosmic expansion. If there is energy bound up in these components, it has nothing to do with any gravity in the sense of attraction or repulsion, but only to do with stresses and strains associated with the curvature of space (*not* space-time). It exists and contributes to the *inertial* mass of the tunnel, but it does not gravitate, so it has no *active* gravitational mass equivalent. Its manifestation as inertial mass could be thought of as the resistance presented by these stresses and strains to the deformations of space that would be required if the tunnel were to move. That two of these stresses and strains are associated with negative sectional curvatures should cause no alarm, especially in light of the fact that the field equations that produced $\hat{G}$ are *vacuum* field equations, deriving as they do from the action principle $\delta \int \big(\hat{\Psi} - \hat{\Psi}_\infty\big)\, d\hat{V} = 0$, which is no less geometrical in concept than the action principle $\delta \int \Psi\, dV = 0$ that yields the Einstein vacuum field equations. To hold that such curvatures are rare and are to be found only in exotic circumstances, to hold, in other words, that Nature abhors a negatively curved vacuum, is to presume to know more about Nature than Nature knows about itself.

Taken together, these considerations suggest that some energy can be associated with gravity and some cannot, thus that not all energy 'produces' gravity (a consequence of which might be that the 'cosmological constant problem' [9] does not exist). Do they support in any way the widely held belief that there are 'exotic' relationships between energy and geometry that justify calling the energy 'negative'? No! They do not. Their lesson is clear: Energy relates to geometry as it will — not as some uninvited adjectives say it must.

Note. The metric $\hat{G}_{T_0}$ and the space-time it describes are in all qualitative aspects identical with those derived and extensively analyzed under Case III in my 1973 paper Ether flow through a drainhole: a particle model in general relativity [8].[4] What I there called a 'drainhole'

[4] A comparison of Ref. [8] with the present paper should take into account that the Ricci and Einstein tensors of [8] are the negatives of those used here.

I would in the present context call a darkhole, or, more accurately, a 'darkhole–brighthole'. The flowless (that is, the massless, nongravitating) drainhole, whose metric has the form $dt^2 - \left[d\rho^2 + \left(\rho^2 + n^2\right) d\Omega^2\right]$, was later reinvented and put on exhibit by Morris and Thorne as an example of a 'traversible wormhole' [10,11].

APPENDIX

The definitional conventions used for the curvature scalar Ψ, Ricci tensor Φ, and Riemann tensor Θ of a metric G are the following, in which $\omega^\mu = dx^\mu$, $e_\mu = \partial/\partial x^\mu$, and $(\ )_{.\mu} := e_\mu(\ ) = \partial(\ )/\partial x^\mu$:

$$\begin{aligned}
G &= \omega^\kappa \otimes g_{\kappa\lambda}\, \omega^\lambda, && \text{(A1)}\\
G^{-1} &= e_\kappa \otimes g^{\kappa\lambda}\, e_\lambda, && \text{(A2)}\\
\Psi &= \Phi_\kappa{}^\kappa = \Phi_{\kappa\lambda} g^{\lambda\kappa}, && \text{(A3)}\\
\Phi &= \omega^\kappa \otimes \Phi_{\kappa\lambda}\, \omega^\lambda = \omega^\kappa \otimes \Theta_\kappa{}^\mu{}_{\lambda\mu}\, \omega^\lambda, && \text{(A4)}\\
\Theta &= \omega^\kappa \otimes 2\left(d_\wedge \omega_\kappa{}^\mu - \omega_\kappa{}^\pi \wedge \omega_\pi{}^\mu\right) \otimes e_\mu && \text{(A5)}\\
&= \omega^\kappa \otimes \Theta_\kappa{}^\mu{}_{\lambda\nu}\, \omega^\nu \otimes \omega^\lambda \otimes e_\mu, && \text{(A6)}\\
\Theta_\kappa{}^\mu{}_{\lambda\nu} &= \Gamma_\kappa{}^\mu{}_{\nu.\lambda} - \Gamma_\kappa{}^\mu{}_{\lambda.\nu}\\
&\quad + \Gamma_\kappa{}^\pi{}_\nu \Gamma_\pi{}^\mu{}_\lambda - \Gamma_\kappa{}^\pi{}_\lambda \Gamma_\pi{}^\mu{}_\nu, && \text{(A7)}
\end{aligned}$$

where, with $\mathbf{d}$ denoting the torsionless covariant differentiation consistent with G, the connection 1-forms $\omega_\kappa{}^\mu$ and connection coefficients $\Gamma_\kappa{}^\mu{}_\lambda$ are determined by $\mathbf{d}e_\kappa = \omega_\kappa{}^\mu \otimes e_\mu = \Gamma_\kappa{}^\mu{}_\lambda\, \omega^\lambda \otimes e_\mu = \left\{{}_\kappa{}^\mu{}_\lambda\right\} \omega^\lambda \otimes e_\mu = \frac{1}{2}(g_{\nu\lambda.\kappa} + g_{\kappa\nu.\lambda} - g_{\kappa\lambda.\nu}) g^{\nu\mu}\, \omega^\lambda \otimes e_\mu$.

The second of Eqs. (14a) implies that $\left[r\left(r'^2\right) - r - 7m^2\hat{R}^2/4r\right]' = 0$, thus that

$$\begin{aligned}
r'^2 &= 1 - \frac{c}{r} + \frac{7}{4}\frac{m^2\hat{R}^2}{r^2} && \text{(A8)}\\
&= \frac{(r - r_0)(r - \gamma^2 r_0)}{r^2}, && \text{(A9)}
\end{aligned}$$

where $c = \left(r_0^2 + 7m^2\hat{R}^2/4\right)/r_0$, as determined by the initial conditions $r(0) = r_0$ and $r'(0) = 0$, and $\gamma := m/m_{\text{crit}} < 1$, with $m_{\text{crit}} := \left(2/\sqrt{7}\right)(r_0/\hat{R})$. (Solutions with $\gamma \geq 1$ exist, but are not considered here.) The latter of Eqs. (14a) becomes

$$r'' = \frac{r_0}{2r^3}\left[(1 + \gamma^2)(r - r_0) + (1 - \gamma^2) r_0\right], \qquad \text{(A10)}$$

from which follows that $r''(0) = (1 - \gamma^2)/2r_0 > 0$, thus that r has the minimum value r_0 at 0, and that $\operatorname{sgn}(r'(\rho)) = \operatorname{sgn}(\rho)$. Equation (A9) implies that $rr' = \operatorname{sgn}(r')\sqrt{(r - r_0)(r - \gamma^2 r_0)}$, which in turn implies that

$$\int_0^\rho \frac{r}{\sqrt{(r - r_0)(r - \gamma^2 r_0)}}\, dr = \int_0^\rho \operatorname{sgn}(r'(\lambda))\, d\lambda. \quad \text{(A11)}$$

Computation of these integrals yields Eq. (17).

Determination of $U(\rho)$ proceeds as follows:

$$\begin{aligned}
U(\rho) - U(\infty) &= \int_\infty^\rho U'(\lambda)\, d\lambda = \int_\infty^\rho \frac{m\hat{R}}{r^2(\lambda)}\, d\lambda\\
&= m\hat{R}\int_\infty^\rho \frac{\operatorname{sgn}(r')}{r\sqrt{(r - r_0)(r - \gamma^2 r_0)}}\, dr && \text{(A12)}\\
&= m\hat{R}\begin{cases} \displaystyle\int_\infty^\rho \frac{1}{r\sqrt{(r - r_0)(r - \gamma^2 r_0)}}\, dr & \text{if } \rho \geq 0,\\[2ex] \displaystyle\int_\infty^0 \frac{1}{r\sqrt{(r - r_0)(r - \gamma^2 r_0)}}\, dr & \\[2ex] \displaystyle -\int_0^\rho \frac{1}{r\sqrt{(r - r_0)(r - \gamma^2 r_0)}}\, dr & \text{if } \rho \leq 0, \end{cases} && \text{(A13)}\\
&= \frac{2m\hat{R}}{\gamma\, r_0}\begin{cases} -\ln\left(\dfrac{\sqrt{r(\rho) - \gamma^2 r_0} - \gamma\sqrt{r(\rho) - r_0}}{(1 - \gamma)\sqrt{r(\rho)}}\right) \\ \qquad \text{if } \rho \geq 0,\\[2ex] \ln\left(\dfrac{\sqrt{r(\rho) - \gamma^2 r_0} - \gamma\sqrt{r(\rho) - r_0}}{(1 + \gamma)\sqrt{r(\rho)}}\right) \\ \qquad \text{if } \rho \leq 0, \end{cases} && \text{(A14)}\\
&= \frac{4}{\sqrt{7}}\ln\left(\frac{\sqrt{r(\rho) - \gamma^2 r_0} + \operatorname{sgn}(\rho)\gamma\sqrt{r(\rho) - r_0}}{(1 + \gamma)\sqrt{r(\rho)}}\right). && \text{(A15)}
\end{aligned}$$

Upon replacement of $U(-\infty)$ by $\ln\hat{R}$, Eq. (18) follows.

From $\tilde{r}(\tilde{\rho}) := e^{T_0/\hat{R}}\,\bar{r}(\bar{\rho}) := e^{T_0/\hat{R}} e^{-U(\rho)} r(\rho)$ and $\tilde{\rho} := e^{T_0/\hat{R}}\bar{\rho} := e^{T_0/\hat{R}}\rho/\hat{R}$ one sees that $\tilde{r}'(\tilde{\rho}) = \bar{r}'(\bar{\rho}) = \hat{R}\, e^{-U(\rho)}\left[r'(\rho) - r(\rho)U'(\rho)\right]$, thus that $\tilde{r}'(\tilde{\rho}^*) = 0$ if and only if $r'(\rho^*)/r(\rho^*) = U'(\rho^*)$, where $\rho^* := \hat{R}\, e^{-T_0/\hat{R}}\tilde{\rho}^*$. Because $U' = m\hat{R}/r^2$, this condition is equivalent to $r^2(\rho^*) r'^2(\rho^*) = m^2\hat{R}^2 = 4\gamma^2 r_0^2/7$, which in view of Eq. (A9) is equivalent to

$$r^2(\rho^*) - (1 + \gamma^2) r_0 r(\rho^*) + \tfrac{3}{7}\gamma^2 r_0^2 = 0. \qquad \text{(A16)}$$

This, together with $r \geq r_0$, entails that

$$r(\rho^*) = \frac{1}{2}\left[(1 + \gamma^2) + \sqrt{(1 - \gamma^2)^2 + \frac{16}{7}\gamma^2}\right] r_0. \quad \text{(A17)}$$

As γ goes from 0 to 1, $r(\rho^*)$ increases steadily from r_0 to $\left(1 + 2/\sqrt{7}\right) r_0$.

Equations (17) and (A17) imply

$$\begin{aligned}
\rho^* = \Bigg[&\frac{2}{\sqrt{7}}\gamma + \frac{1}{2}\left(1 + \gamma^2\right)\\
&\times \ln\left(\frac{\sqrt{(1 - \gamma^2)^2 + \frac{16}{7}\gamma^2} + \frac{4}{\sqrt{7}}\gamma}{1 + \gamma^2}\right)\Bigg] r_0. \quad \text{(A18)}
\end{aligned}$$

From this it follows that $\tilde{\rho}^*$ increases steadily from 0 to $\left[2/\sqrt{7} + \ln\left(4/\sqrt{7}\right)\right] e^{T_0/\hat{R}}\, r_0/\hat{R}$ as γ goes from 0 to 1.

By combining $\tilde{r}(\tilde{\rho}) = e^{T_0/\hat{R}} e^{-U(\rho)} r(\rho)$ with Eqs. (A15) and (A17), one finds that the minimum radius $\tilde{r}(\tilde{\rho}^*)$

($=: \tilde{r}_{\rm min}$) increases monotonically from $e^{T_0/\hat{R}}r_0/\hat{R}$ to $(2/\sqrt{7})(1+\sqrt{7}/2)^{1+2/\sqrt{7}}e^{T_0/\hat{R}}r_0/\hat{R}$ as γ goes from 0 to 1, thus as m is increased from 0 to $m_{\rm crit}$, and m_{T_0} from 0 to $e^{T_0/\hat{R}}m_{\rm crit}$.

Applying the coordinate change $T := \hat{R}\left[t - \frac{1}{2}U(\rho)\right]$ alters Eq. (13) to

$$\hat{G} = \frac{e^{2U(\rho)}}{\hat{R}^2}\,dT^2 - e^{2T/\hat{R}}\,e^{-2U(\rho)}\left[d\rho^2 + r^2(\rho)\,d\Omega^2\right], \tag{A19}$$

which makes

$$\hat{G}_{T_0} = \frac{e^{2U(\rho)}}{\hat{R}^2}\,dT^2 - e^{2T_0/\hat{R}}\,e^{-2U(\rho)}\left[d\rho^2 + r^2(\rho)\,d\Omega^2\right]. \tag{A20}$$

Consider now a second solution of Eqs. (14) in the form of a metric $\check{G}$, defined on the same manifold $\hat{\mathcal{M}}$ that $\hat{G}$ is defined on, but with respect to its own coordinate system $[\![\check{t},\check{\rho},\vartheta,\varphi]\!]$, and having its own parameters $\check{m}$ and $\check{r}_0$ ($\hat{R}$, ϑ, and φ being the same as for $\hat{G}$), with $0 \geq \check{m} > \check{m}_{\rm crit} := -(2/\sqrt{7})(\check{r}_0/\hat{R})$. Just as for $\hat{G}$, the coordinate change $\check{T} := \hat{R}\left[\check{t} - \frac{1}{2}\check{U}(\check{\rho})\right]$ and the condition $\check{U}' = -2\check{V}$ make

$$\check{G} = \frac{e^{2\check{U}(\check{\rho})}}{\hat{R}^2}\,d\check{T}^2 - e^{2\check{T}/\hat{R}}\,e^{-2\check{U}(\check{\rho})}\left[d\check{\rho}^2 + \check{r}^2(\check{\rho})\,d\Omega^2\right] \tag{A21}$$

and

$$\check{G}_{\check{T}_0} = \frac{e^{2\check{U}(\check{\rho})}}{\hat{R}^2}\,d\check{T}^2 - e^{2\check{T}_0/\hat{R}}\,e^{-2\check{U}(\check{\rho})}\left[d\check{\rho}^2 + \check{r}^2(\check{\rho})\,d\Omega^2\right]. \tag{A22}$$

Next suppose that $\check{T}$ and T are related by

$$\check{T} = T\left(\frac{1-\gamma}{1+\gamma}\right)^{4/\sqrt{7}}, \tag{A23}$$

and $\check{\rho}$ and ρ, by

$$\check{\rho} = -\rho\left(\frac{1+\gamma}{1-\gamma}\right)^{4/\sqrt{7}}, \tag{A24}$$

and let $\mathcal{F}$ be the diffeomorphism of $\hat{\mathcal{M}}$ that maps the point P with coordinates $[\![T,\rho,\vartheta,\varphi]\!]$ to the point $\mathcal{F}(P)$ with coordinates $[\![\check{T},\check{\rho},\vartheta,\varphi]\!]$, that is, $\mathcal{F} = \check{X}^{-1}X$, where $X\colon \hat{\mathcal{M}} \to \mathbb{R}^4$ is the coordinate system $[\![T,\rho,\vartheta,\varphi]\!]$ and $\check{X}\colon \hat{\mathcal{M}} \to \mathbb{R}^4$ is the coordinate system $[\![\check{T},\check{\rho},\vartheta,\varphi]\!]$. For $\mathcal{F}$ to be an isometry with respect to $\hat{G}_{T_0}$ and $\check{G}_{\check{T}_0}$ it is necessary and sufficient that the pullback of $\check{G}_{\check{T}_0}$ by $\mathcal{F}$ be equal to $\hat{G}_{T_0}$, that is, $\check{G}_{\check{T}_0}(\mathcal{F})(d\mathcal{F})(d\mathcal{F}) = \hat{G}_{T_0}$. This will be true if and only if the expression

$$\frac{e^{2\check{U}(\check{\rho})}}{\hat{R}^2}\left(\frac{1-\gamma}{1+\gamma}\right)^{8/\sqrt{7}} dT^2 - e^{2\check{T}_0/\hat{R}}\,e^{-2\check{U}(\check{\rho})}\left[\left(\frac{1+\gamma}{1-\gamma}\right)^{8/\sqrt{7}} d\rho^2 + \check{r}^2(\check{\rho})\,d\Omega^2\right] \tag{A25}$$

for $\check{G}_{\check{T}_0}$ derived from Eq. (A22) agrees with the expression of $\hat{G}_{T_0}$ in Eq. (A20). This in turn will be true if and only if $\check{T}_0 = T_0$,

$$\check{U}(\check{\rho}) := U(\rho) + \ln\left(\frac{1+\gamma}{1-\gamma}\right)^{4/\sqrt{7}}, \tag{A26}$$

and

$$\check{r}(\check{\rho}) := \left(\frac{1+\gamma}{1-\gamma}\right)^{4/\sqrt{7}} r(\rho). \tag{A27}$$

(Note that $\check{r}(\check{\rho})/\check{\rho} = -r(\rho)/\rho \to -1$ as $\rho \to \infty$, thus as $\check{\rho} \to -\infty$.) Are these consistent with the supposition that $\check{U}$ and $\check{r}$ satisfy Eqs. (14)? One has that

$$\check{U}'(\check{\rho}) = \frac{d\rho}{d\check{\rho}}U'(\rho) = \frac{d\rho}{d\check{\rho}}\frac{m\hat{R}}{r^2(\rho)} = -\left(\frac{1+\gamma}{1-\gamma}\right)^{4/\sqrt{7}}\frac{m\hat{R}}{\check{r}^2(\check{\rho})}, \tag{A28}$$

thus that $\check{U}'(\check{\rho}) = (\check{m}\hat{R})/\check{r}^2(\check{\rho})$ provided only that

$$\check{m} = -m\left(\frac{1+\gamma}{1-\gamma}\right)^{4/\sqrt{7}}. \tag{A29}$$

It is straightforward to check that this same condition guarantees that $\check{r}$ will satisfy the second of Eqs. (14a). Satisfaction of Eqs. (14b) demands only the further stipulation that $\check{r}_0 = r_0\left[(1+\gamma)/(1-\gamma)\right]^{4/\sqrt{7}}$.

From these calculations the following inferences may be drawn:

1. For every space-time metric $\hat{G}$ of the assumed form (Eq. (13)) that satisfies the initial-value problem of Eqs. (14) with a positive mass parameter m there is one with a negative mass parameter $\check{m}$ whose human time-scale approximant space-time metric $\check{G}_{T_0}$ is, at each era T_0 of cosmic time, isometric to $\hat{G}_{T_0}$ — and vice versa. Consequently, there is on the human time-scale no useful distinction to be made between the metrics with $m > 0$ and those with $m < 0$.

2. Each human time-scale approximant $\hat{G}_{T_0}$ of the metric $\hat{G}$ is self-isometric under an isometry that reverses the direction of increase of ρ (cf. Eq. (A24)), therefore maps the asymptotic region $\rho \sim \infty$ onto the opposite asymptotic region $\rho \sim -\infty$. Moreover, the details of that isometry make clear that, whereas $\hat{G}_{T_0}$ is asymptotic as $\rho \to \infty$ to a Schwarzschild metric with positive mass parameter m_{T_0} $(:= me^{T_0/\hat{R}})$, it is asymptotic as $\rho \to -\infty$ to a Schwarzschild metric with negative mass parameter $\check{m}_{T_0}$ $(:= \check{m}e^{T_0/\hat{R}})$ such that

$$\frac{-\check{m}_{T_0}}{m_{T_0}} = \frac{-\check{m}}{m} = \left(\frac{1+\gamma}{1-\gamma}\right)^{4/\sqrt{7}} > 1. \tag{A30}$$

Introduction of the coordinate $\hat{T} := t + \int V(\rho)\, d\rho$ gives the metric of Eq. (13) the (tensor product) form

$$\hat{G} = e^{2U(\rho)}(d\hat{T} \otimes d\hat{T}\,) - e^{2\hat{T}} e^{-2\int V(\rho)\, d\rho} e^{-3U(\rho)} \times \left(\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3\right), \tag{A31}$$

where $\omega^1 := d\rho$, $\omega^2 := r(\rho)d\vartheta$, and $\omega^3 := r(\rho)(\sin\vartheta)d\varphi$. A standard calculation of the Einstein tensor $\hat{E}$, followed by application of Eqs. (14a), yields the following equation, provided that $C = 0$ is chosen when the antidifferentiation $\int V(\rho)\, d\rho = -\frac{1}{2}\int U'(\rho)\, d\rho = -\frac{1}{2}U(\rho) + C$ is performed:

$$\begin{aligned}\hat{E} &:= \hat{\Phi} - \tfrac{1}{2}\hat{\Psi}\hat{G} \\ &= 3e^{-2U(\rho)}\hat{G} + e^{-2\hat{T}}e^{4U(\rho)}\left[\frac{3}{4}\frac{m^2\hat{R}^2}{r^4(\rho)}\right]\left(d\hat{T} \otimes d\hat{T}\right) \\ &\quad + \frac{2m\hat{R}}{r^2(\rho)}\left(d\hat{T} \otimes \omega^1 + \omega^1 \otimes d\hat{T}\right) \\ &\quad + \left[-\frac{1 - r'^2(\rho)}{r^2(\rho)} - \frac{m^2\hat{R}^2}{r^4(\rho)}\right]\left(\omega^1 \otimes \omega^1\right) \\ &\quad + \left[\frac{1 - r'^2(\rho)}{2r^2(\rho)} + \frac{1}{8}\frac{m^2\hat{R}^2}{r^4(\rho)}\right]\left(\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3\right).\end{aligned} \tag{A32}$$

Using the definitions of Eqs. (25) in this equation, as well as Eq. (A8), one arrives at the decomposition of $\hat{E}$ expressed in Eqs. (24) and (27–29).

[1] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-time* (Cambridge Univ. Press, Cambridge, U.K., 1973), pp. 256–275.

[2] A. Einstein, Die Grundlage der allgemeinen Relativitätstheorie, Ann. der Physik **49**, 769–822 (1916), translated in *The Principle of Relativity* (Dover, New York), pp. 109–164.

[3] H. Bondi, Negative mass in general relativity, Rev. Mod. Phys. **29**, 423–428 (1957).

[4] L. B. Kreuzer, Experimental measurement of the equivalence of active and passive gravitational mass, Phys. Rev. **169**, 1007–1012 (1968).

[5] A. Einstein, *Essays in Physics* (Philosophical Library, New York, 1950), p. 39.

[6] E. Schrödinger, *Expanding Universes* (Cambridge Univ. Press, Cambridge, U.K., 1956), pp. 28–35.

[7] A. Einstein and N. Rosen, The particle problem in the general theory of relativity, Phys. Rev. **48**, 73–77 (1935).

[8] H. G. Ellis, Ether flow through a drainhole: a particle model in general relativity, J. Math. Phys. **14**, 104–118 (1973); Errata: **15**, p. 520 (1974).

[9] S. Weinberg, The cosmological constant problem, Rev. Mod. Phys. **61**, 1–23 (1989).

[10] M. S. Morris and K. S. Thorne, Wormholes in spacetime and their use for interstellar travel: a tool for teaching general relativity, Am. J. Phys. **56**, 395–412 (1988).

[11] G. Clément, The Ellis Geometry (Letter to the editor), Am. J. Phys. **57**, p. 967 (1989).

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