The Kadison-Kastler distance and the Irrational Rotation Algebras

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The Kadison-Kastler distance $d(A, B)$ is defined for operator algebras $A, B$ which are both subalgebras of some other algebra $C$. Usually we take $C = B(\mathcal{H})$. The Kadison-Kastler distance of von Neumann algebras has been studied, but in this talk, unless otherwise specified, $A$ and $B$ will be $C^*$-algebras.

Write $A_1$ for the unit ball of $A$, and similarly for $B$. Then the Kadison-Kastler distance is defined by

$$d(A, B) = \inf\{\gamma \geq 0 : \forall a \in A_1, \exists b \in B_1 \ni \|a - b\| < \gamma \text{ and } \forall b \in B_1, \exists a \in A_1 \ni \|a - b\| < \gamma\}.$$ leave on board!

Some known results:

- If $A$ is an injective von Neumann algebra and $d(A, B)$ is small then $A, B$ are isomorphic. Moreover, the isomorphism is implemented by a unitary (Phillips 1974; Christensen 1980).

- If $d(A, B) < 1/4$, then $A$ unital $\Leftrightarrow B$ unital.
• If \( d(A, B) < 1/101 \), then \( A \) nuclear \( \iff \) \( B \) nuclear (Christensen 1980).

• There are non-separable nuclear \( C^* \)-algebras with \( d(A, B) \) arbitrarily small but \( A, B \) are not isomorphic (Choi & Christensen 1983).

• If \( A \) is separable and nuclear and \( d(A, B) < 1/420,000 \), then \( A \) and \( B \) are isomorphic (Christensen, Sinclair, Smith, White, Winter, 2012).

• If \( d(A \otimes M_n, B \otimes M_n) < 1/3 \) for all \( n \in \mathbb{N} \), then \( K_0(A) \cong K_0(B) \) (Khoshkam 1984).

I was hoping to be able to use this last result in my research, since my goal is to find such an isomorphism on \( K \)-theory for twisted group(oid) \( C^* \)-algebras. However, the hypothesis fails — pretty spectacularly — for one of the most basic examples of twisted group \( C^* \)-algebras, the rotation algebras. So I’ll explain what the irrational rotation algebras are and show that \( d(A_0, A_\theta) > 1 \) for any \( \theta \in \mathbb{R} \setminus \mathbb{Q} \).

There are lots of ways to think of the rotation algebras \( A_\theta \) for \( \theta \in \mathbb{R} \):

• \( A_\theta \) is the universal \( C^* \)-algebra generated by two unitaries \( u, v \) such that
  \[ uv = e^{2\pi i \theta} vu. \]

• \( A_\theta = C(\mathbb{T}) \rtimes \mathbb{Z} \) is the crossed product generated by the action of \( \mathbb{Z} \) on \( \mathbb{T} \) corresponding to “rotation by \( \theta \)” of the circle \( \mathbb{T} \).

• \( A_\theta = C^*(\mathbb{Z}^2, c_\theta) \) is a twisted group \( C^* \)-algebra.

I’ll use this last presentation since it’s the one that ties the rotation algebras to my research interests.

So, what’s a twisted group \( C^* \)-algebra? Two columns: one for general theory and one for example.

Def: Let \( G \) be a locally compact Hausdorff topological group. We say that a function \( c : G \times G \to \mathbb{T} \) is a 2-cocycle (or just a cocycle) if for all \( x, y, z \in G \) we have
\[
c(x, yz)c(y, z) = c(xy, z)c(x, y).
\]
In our case of the rotation algebras, where $G = \mathbb{Z}^2$, fix $\theta \in \mathbb{R}$ and set $\rho = e^{2\pi i \theta}$. Define
\[
c_{\theta}((m, n), (p, q)) := \rho^{mq-np}.
\]
**Check that $c_{\theta}$ is a cocycle, possibly just checking for the exponent.**

Once we have a cocycle, that allows us to define a twisted convolution multiplication on $C_c(G)$:
\[
f \ast_c g(x) = \int_G f(xy)g(y^{-1})c(xy, y^{-1}) \, dy
\]
and an involution
\[
f^*(x) = f(x^{-1})c(x, x^{-1}).
\]
One has to check that the multiplication is associative, but this isn’t too hard.

Now, one can check that if $g \in L^2(G)$ then $f \ast_c g \in L^2(G)$ for any $f \in C_c(G)$, and moreover, that
\[
\|f \ast_c g\|_2 \leq \|f\|_1 \|g\|_2,
\]
so we can think of $C_c(G)$ as operators in $B(L^2(G))$.

**DEF:** Given a cocycle on a LCH group $G$, we define the **reduced twisted group $C^*$-algebra** $C^*_r(G, c)$ to be the norm completion of $C_c(G)$ in $B(L^2(G))$.

One can also define the **full twisted group $C^*$-algebra** as the completion of $C_c(G)$ in the norm
\[
\|f\| := \sup\{\|\pi(f)\| : \pi \text{ is an } L^1\text{-norm-decreasing } \ast\text{-representation of } C_c(G, c).\}
\]
In the case of the rotation algebras, $C^*(\mathbb{Z}^2, c_{\theta}) = C^*_r(\mathbb{Z}^2, c_{\theta})$, so I will usually drop the $r$.

To convince you that $C^*(\mathbb{Z}^2, c_{\theta})$ is really the universal algebra $A_{\theta}$, I should at least show you what the unitaries $u$ and $v$ are:
\[
u(m, n) = \begin{cases} 1, & (m, n) = (1, 0) \\ 0, & \text{else} \end{cases}
\]
\[
v(m, n) = \begin{cases} 1, & (m, n) = (0, 1) \\ 0, & \text{else} \end{cases}
\]
You can check that they’re unitaries (ie, \( u^* u = uu^* = vv^* = v^* v \) is the identity, which in this case is \( \delta_{0,0} \)). Let’s see that \( uv = \rho vu \). Remember that multiplication here is twisted convolution, so

\[
\begin{align*}
  u \ast_{\theta} v(m, n) &= \sum_{j, k \in \mathbb{Z}} u(m + j, n + k)v(-j, -k)\rho^{(m+j)(-k)+j(n+k)} \\
  &= \rho^{1/2} \cdot \begin{cases} 1, & (m, n) = (1, 1) \\
  0, & \text{else} \end{cases},
\end{align*}
\]

since in order to have \( v(-j, -k) \) nonzero we need \( j = 0, k = -1 \), and then we must have \( n = m = 1 \) in order to have \( u(m + j, n + k) \) nonzero.

Similarly,

\[
\begin{align*}
  v \ast u(m, n) &= \sum_{j, k \in \mathbb{Z}} v(m + j, n + k)u(-j, -k)\rho^{(m+j)(-k)+j(n+k)} \\
  &= \rho^{-1/2} \cdot \begin{cases} 1, & (m, n) = (1, 1) \\
  0, & \text{else} \end{cases},
\end{align*}
\]

since we need \( j = -1, k = 0 \) in order to have \( u(-j, -k) \) nonzero, and then we need \( m = n = 1 \) in order to have \( v(m + j, n + k) \) nonzero. Thus \( u \ast v = \rho v \ast u \) as claimed.

If you’re willing to believe that \( C^*(\mathbb{Z}^2, c_\theta) = A_\theta \) — that is, that \( u, v \) as defined above generate \( C^*(\mathbb{Z}^2, c_\theta) \) — then it follows by definition that any element in \( C^*(\mathbb{Z}^2, c_\theta) \) can be approximated arbitrarily closely in norm by an element of the form

\[
w = \sum_{m, n \in N} a_{mn}u^m v^n
\]

for \( N \) a finite subset of \( \mathbb{Z} \) and each \( a_{mn} \in \mathbb{C} \).

Let’s make some sense of this expression, at least in the case when \( \theta = 0 \) so that we don’t have to worry about the cocycle. We’ll decorate all our functions with a subscript 0 to remind us that we’re in \( A_0 \). Note that

\[
v_0 \ast v_0(j, k) = \sum_{m, n \in \mathbb{Z}} v_0(m + j, n + k)v_0(-m, -n) = \delta_{0,2}.
\]

Discuss? Similarly,

\[
u_0 \ast u_0(j, k) = \delta_{2,0}.
\]
Using “American induction,” we see that
\[ u_0^m = \delta_{m,0}, \quad v_0^n = \delta_{0,n}, \]
and since
\[ \delta_{m,0} \ast \delta_{0,n}(j,k) = \sum_{p,q \in \mathbb{Z}} \delta_{m,0}(j + p, q + k)\delta_{0,n}(-p, -q) = \delta_{m,n}(j,k), \]
it follows that
\[ w_0 = \sum_{m,n \in \mathbb{N}} a_{mn} u_0^m v_0^n = \sum_{m,n \in \mathbb{N}} a_{mn} \delta_{m,n}. \]

**Prop:** Let \( \theta \in \mathbb{R} \setminus \mathbb{Q} \) be irrational. Then
\[ d(C^*(\mathbb{Z}^2, c_0), C^*(\mathbb{Z}^2, c_\theta)) \geq 1. \]

**Proof:** We will show that for \( w = w_0 \in C^*(\mathbb{Z}^2, c_0) \) as above, and \( v = v_\theta \in C^*(\mathbb{Z}^2, c_\theta) \), then
\[ \| w - v \|_{B(L^2(\mathbb{Z}^2))} \geq 1. \]
This will prove the proposition: we have found an element (namely \( v \)) in \( C^*(\mathbb{Z}^2, c_\theta) \) such that no element \( w \in C^*(\mathbb{Z}^2, c_0) \) approximates \( v \) in norm with any better bound than 1, so the Kadison-Kastler distance must be at least 1.

First, observe that an ONB for \( L^2(\mathbb{Z}^2) \) is given by the indicator functions \( \delta_{m,n} \), so
\[ \| w_0 - v_\theta \|^2 \geq \| (w_0 - v_\theta) \ast \delta_{m,n} \|^2_2. \]
Observe that
\[ v_\theta \ast \delta_{m,n}(p,q) = \sum_{j,k \in \mathbb{Z}} v(p + j, k + q)\delta_{m,n}(-j, -k)\rho^{\frac{\rho}{4} - \frac{pk}{2}} = \delta_{m,n+1}(p,q)\rho^{\frac{2-m}{2}}, \]
while
\[ w_0 \ast \delta_{m,n}(p,q) = \sum_{j,k \in \mathbb{Z}} w_0(p + j, k + q)\delta_{m,n}(-j, -k) \]
\[ = w_0(p - m, q - n) \]
\[ = a_{p-m,q-n}. \]
Thus, for all \( m, n \in \mathbb{Z} \), we have

\[
\|w_0 - v_\theta\|^2 \geq \|w_0 \ast \delta_{m,n} - v_\theta \ast \delta_{m,n}\|^2 \\
\geq \sup_{p,q \in \mathbb{Z}} |w_0 \ast \delta_{m,n}(p,q) - v_\theta \ast \delta_{m,n}(p,q)|^2 \\
= \sup_{p,q \in \mathbb{Z}} \left| a_{p-m,q-n} - \delta_{m,n+1}(p,q) \rho^{-m} \right| \\
\geq |a_{0,1} - \rho^{-m}|,
\]

where the second line follows because \( \delta_{j,k} \) form an orthonormal basis for \( L^2(\mathbb{Z}^2) \), so for any \( f \in L^2(\mathbb{Z}^2) \) we have

\[
\|f\|^2 = \sum_{j,k \in \mathbb{Z}} |\langle f, \delta_{j,k} \rangle| = \sum_{j,k} \left| \sum_{m,n} f(m,n) \delta_{j,k}(m,n) \right| = \sum_{j,k} |f(j,k)| \geq \sup_{j,k} |f(j,k)|.
\]

This finishes the proof: If \( \theta \) is irrational, then as \( m \) ranges over \( \mathbb{Z} \), \( \rho^{-m} \) ranges densely over the unit circle, so no matter what value we pick for \( a_{0,1} \) we will always have some \( m \) for which

\[
|a_{0,1} - \rho^{-m}| \geq 1.
\]

In other words,

\[
\forall \ w_0 \in C^*(\mathbb{Z}^2,c_0), \ \|w_0 - v_\theta\| \geq 1,
\]

and so \( d(C^*(\mathbb{Z}^2,c_\theta), C^*(\mathbb{Z}^2,c_0)) \geq 1. \ \Box. \)