Syntax and Semantics

or: Galois Theory but I don't mention fields

Syntax

- Defined by some selection of well-formed strings of symbols called **formulas** or **sentences**
- Usually given via an **inductive definition**
  - Minimal sets including **axioms** and closed under **inference rules**

Semantics

- Supplies "meaning" to syntax
- A **model** consists of a **structure** and/or a **valuation**
- The **truth** of a formula or sentence is given via a **recursive** definition

Classical Propositional Logic

Syntax

Fix a countable set of propositional variables $A = \{p, q, r, \ldots\}$ and define **formulas**:

$$\varphi ::= p \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \neg\varphi$$

Semantics

In *classical* logic, a model is simply given by a **valuation** $v : A \to 2$.

Given a valuation v, we write $v \vDash \varphi$ for "φ is true under v", defined **recursively**:

$v \vDash p$	iff	$v(p) = 1$
$v \vDash \top$	iff	always
$v \vDash \bot$	iff	never
$v \vDash \varphi \wedge \psi$	iff	$v \vDash \varphi$ and $v \vDash \psi$
$v \vDash \varphi \vee \psi$	iff	$v \vDash \varphi$ or $v \vDash \psi$
$v \vDash \varphi \to \psi$	iff	$v \nvDash \varphi$, or $v \vDash \psi$ (*material implication*)

Notice this definition uniquely extends a valuation $A \to 2$ to a function $\mathsf{Form} \to 2$

A formula ψ is a *logical consequence* of Γ, written $\Gamma \models \varphi$, if:

- The truth of formulas in Γ *forces* the truth of φ *regardless* of the model.
- ($v \models \Gamma$ implies $v \models \varphi$ for all v)

We can characterize this *syntactically* with a proof **calculus** for classical logic. One such system (due to Frege) is:

- **Axioms**
  - $p \to (q \to p)$
  - $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$
  - $\neg\neg p \to p$
- **Inference Rules**
  - Modus Ponens: From φ and $\varphi \to \psi$, infer ψ
  - Uniform substitution: Replace propositional letters in φ with other formulas.

$\Gamma \vdash \varphi$ means there is a finite list of formulas, ending at φ, each of which is an axiom or a formula from Γ, or follows from earlier formula(s) via an inference rule.

Fundamental concepts connecting syntax and semantics

Soundness: If $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$

Completeness: If $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$

First Order Logic

Syntax

Fix a language $\mathcal{L}$ containing

- Function symbols ($+$, $-$, $\cdot$, exp, S)
- Relation symbols ($<$, $\leq$, $\equiv$, $\cong$, $\in$)
- Constant symbols ($0, 1, \pi, e$)

Fix a countable set $V = \{x, y, z, \ldots\}$ of *variables*.

$\mathcal{L}$-terms: $\quad t ::= x \mid c \mid f(t, t, \ldots, t)$ for all function symbols $f \in \mathcal{L}$

Atomic $\mathcal{L}$-formulas: Relation symbols or equality ("$=$") applied to/between terms.

$\mathcal{L}$-formuals: $\quad \varphi ::= \alpha \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \neg\varphi \mid \forall x\ \varphi \mid \exists x\ \varphi$

If every occurrence of the variable x occurs somewhere in the scope of a quantifier $\forall x$, it is *bound*; otherwise it's *free*.

A sentence is an $\mathcal{L}$-formula that has no free variables.

Semantics

A model is a **structure** $\mathbb{M}$ that consists of a set M along with an interpretation of the language:

- An actual function $f^{\mathbb{M}} : M^n \to M$ for each function symbol $f \in \mathcal{L}$
- An actual relation $R^{\mathbb{M}} \subseteq M^n$ for each relation symbol $R \in \mathcal{L}$
- An actual element $c^{\mathbb{M}} \in M$ for each constant symbol $c \in \mathcal{L}$

Each term $t(x_1, \ldots, x_n)$ extends to an evaluation function $t^{\mathbb{M}} : M^n \to M$

For an formula $\varphi(\overline{x})$ and values for the free variables $\overline{a} \in M$, define $\mathbb{M} \models \varphi(\overline{a})$ recursively:

$\mathbb{M} \models (t_1 = t_2)(\overline{a})$	iff	$t_1^{\mathbb{M}}(\overline{a}) = t_2^{\mathbb{M}}(\overline{a})$
$\mathbb{M} \models R(t_1, \ldots, t_n)(\overline{a})$	iff	$(t_1^{\mathbb{M}}(\overline{a}), \ldots, t_n^{\mathbb{M}}(\overline{a})) \in R^{\mathbb{M}}$
...propositional connectives		
$\mathbb{M} \models (\forall x\ \varphi(x, \overline{y}))(\overline{a})$	iff	for all $b \in M$, $\mathbb{M} \models \varphi(b, \overline{a})$
$\mathbb{M} \models (\exists x\ \varphi(x, \overline{y}))(\overline{a})$	iff	there is some $b \in M$ s.t. $\mathbb{M} \models \varphi(b, \overline{a})$

Note that *sentences* are definitively true or false in a model; write $\mathbb{M} \models \varphi$

Examples

Langauge of rings $\mathcal{L} = \{+, -, \cdot, 0, 1\}$

- $\mathbb{R} \vDash \exists x\ x \cdot x = 1 + 1$, $\mathbb{Q} \nvDash \exists x\ (x \cdot x = 1 + 1)$

$\mathcal{L} = \{<\}$

- $\mathbb{Z} \nvDash \forall x \forall y\ (x < y \to \exists z (x < z \land z < y))$, $\mathbb{Q}$ does.

Different models (structures) validate or falsify different sentences.

$\models$ is a relation between **structures** and **sentences**

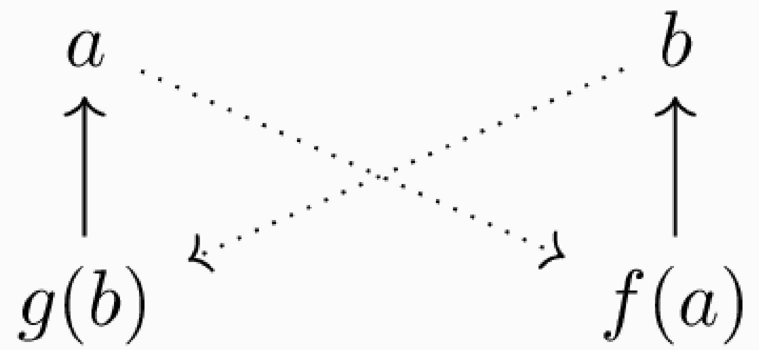
This relation *induces a Galois connection* between **classes of structures** and **sets of sentences**

Galois Connections

A **Galois connection** is a dual adjunction between two posets.

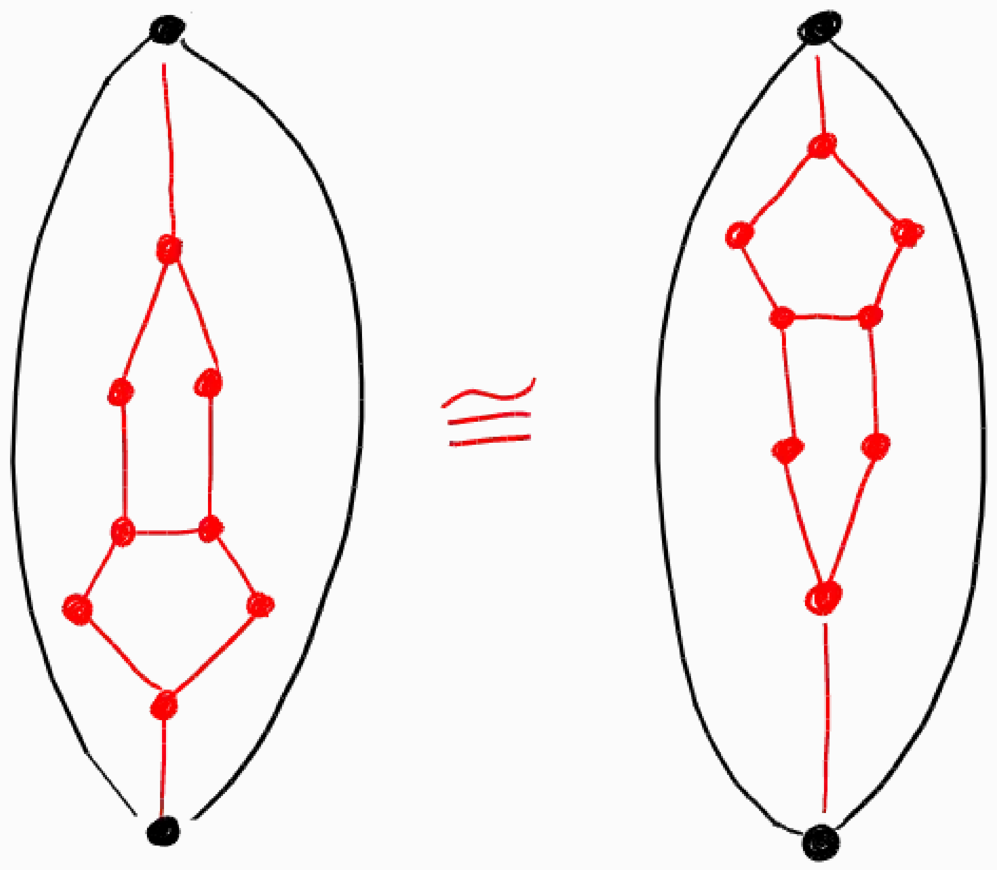
That is, given posets A and B

- A pair of maps $f : A \to B,\ g : B \to A$ so that
- f and g are **order-reversing** ($a \leq a'$ implies $f(a') \leq f(a)$)
- $f(a) \leq b$ if and only if $g(b) \leq a$ (natural iso of hom sets)
- Alternatively, $a \leq gf(a)$ and $b \leq fg(b)$ (unit/counit)



Facts about ~~adjoint functors~~ Galois connections

- Every adjunction restricts to an equivalence of full subcategories
  - Here, these are the elements that appear as the image of either function
  - So we get a dual isomorphism between $g[B] \subseteq A$ and $f[A] \subseteq B$
  - These are called the **stable elements (sets)**
- Every adjunction yields a monad on both categories given by the composition/double-dual
  - A monad on a poset is a **closure operator** $c : A \to A$
    - *extensive* $a \leq c(a)$
    - *monotone* $a \leq b \to c(a) \leq c(b)$
    - *idempotent* $cc(a) = c(a)$
- a is *closed* if $a = c(a)$
- The stable elements are exactly the closed elements.



Galois connections from relations

Take sets X, Y and a relation $R \subseteq X \times Y$

R induces a Galois connection between $\mathcal{P}(X)$ and $\mathcal{P}(Y)$.

$$f : \mathcal{P}(X) \to \mathcal{P}(Y) := U \mapsto \{y \in Y : uRy \;\forall u \in U\}$$

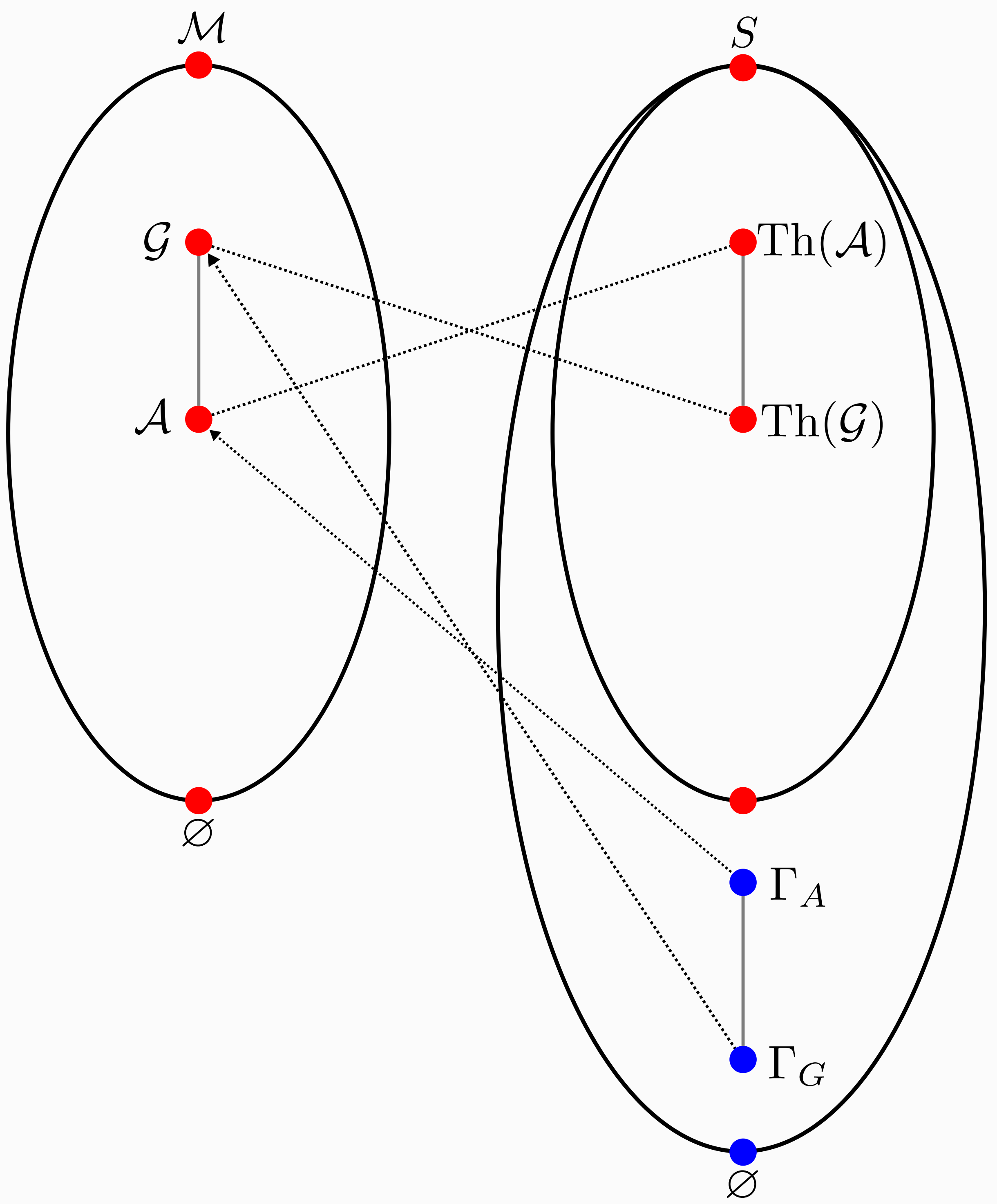
$$g : \mathcal{P}(Y) \to \mathcal{P}(X) := V \mapsto \{x \in X : xRv \;\forall v \in V\}$$

The stable elements of the connection are called **stable sets** = **closed sets**.

Galois connection of FOL

- $\mathcal{M}$ = class of $\mathcal{L}$-structures, S = set of $\mathcal{L}$-sentences
- $\models \subseteq \mathcal{M} \times S$
- $\mathrm{Th} : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(S)$ takes a subclass $\mathcal{K}$ of structures to its **theory**, the set of sentences true in all structures in $\mathcal{K}$
- $\mathrm{Mod} : \mathcal{P}(S) \to \mathcal{P}(\mathcal{M})$ takes a set Γ of sentences to its class of **models**, the structures which believe everything in Γ.
- A closed set on the semantic side is an **elementary class**
- A closed set on the syntactic side is a **theory**

Language of groups $\mathcal{L} = \{\cdot, ^{-1}, 1\}$



- $\Gamma_G = \{\forall x\ 1 \cdot x = x \cdot 1 = x,\ \ \forall x \forall y \forall z\ x \cdot (y \cdot z) = (x \cdot y) \cdot z,\ \ \forall x\ x \cdot x^{-1} = x^{-1} \cdot x = 1\}$
- $\Gamma_A = \Gamma_G \cup \{\forall x \forall y\ x \cdot y = y \cdot x\}$

"Features"

Elementary equivalence is wacky

- (language of ordered fields) $\mathrm{Mod}(\mathrm{Th}(\mathbb{R}))$ contains fields with infinitesimal elements ($\varepsilon^2 = 0$)
- (language of rings) $\mathrm{Mod}(\mathrm{Th}(\mathbb{N}))$ contains $2^{\aleph_0}$ non-isomorphic countable models that contain "infinitely large primes"
- Generally, any theory that has an infinite model has models of *any cardinality*

There are non-trivial examples of non-elementary (non-closed) classes

- (language of groups) The class of *torsion groups* is not elementary
- (language of ordered sets) The class of *well-ordered sets* is not elementary

- The goal of a **proof calculus** is to characterize closure on the syntactic side *internally*
- A proof calculus is **sound and complete** exactly when it meets this goal.
- We can in fact do this for FOL:

- **Axiom schema**
  - Propositional tautologies
  - $\varphi(t) \to \exists x\ \varphi(x)$ for any term t
  - $\forall x\ \varphi(x) \to \varphi(t)$ for any term t
  - Assert $=$ is an equivalence relation and equal terms can be freely substituted/exchanged for each other
- **Inference Rules**
  - Modus Ponens: from φ and $\varphi \to \psi$ infer ψ
  - Q1: From $\varphi \to \psi$ where x is not free in φ, infer $\varphi \to \forall x\ \psi$
  - Q2: From $\varphi \to \psi$ where x is not free in ψ, infer $\exists x\ \varphi \to \psi$

- Sometimes, we can characterize closure on the semantic side as well.

When the signature $\mathcal{L}$ contains only function symbols (no relations), an $\mathcal{L}$-structure is called an **algebra**

An **equation** or **identity** is a universally quantified sentence asserting equality of terms (e.g., group axioms)

Galois Connection of Universal Algebra

- $\mathcal{A}$ = class of $\mathcal{L}$-algebras, E = set of $\mathcal{L}$-equations
- $\vDash \subseteq \mathcal{A} \times E$
- $\mathrm{EqTh} : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(E)$ takes a class $\mathcal{K}$ of algebras to its **equational theory**, the set of equations true in $\mathcal{K}$.
- $\mathrm{Mod} : \mathcal{P}(E) \to \mathcal{P}(\mathcal{A})$ takes a set Γ of equations to its class of **models**, the algebras which believe everything in Γ.
- A closed set on the semantic side is an **variety** or **equational class**
  - Closure is usually denoted $\mathcal{V}(-)$ for "variety generated by"
- A closed set on the syntactic side is an **equational theory**

Closure on the syntactic side simplifies dramatically; there is a sound and complete **equational calculus** that reflects how we reason with equations:

- **Axioms**:
  - $t = t$ for all terms t
- **Inference Rules**:
  - From $s = t$ infer $t = s$
  - From $r = s$ and $s = t$ infer $r = t$
  - From $q = r$ and $s = t$ infer $q[s/x] = r[t/x]$ where x is a variable

But we can also characterize closure on the semantic side, by a famous theorem of Birkhoff:

(HSP Theorem) For any class $\mathcal{K}$ of algebras, $\mathcal{V}(\mathcal{K}) = HSP(\mathcal{K})$, where

- P denotes "products of"
- S denotes "subalgebras of"
- H denotes "homomorphic images of" (a.k.a. quotients)

Modal Logic

Syntax

Take a countable set of propositional variables $A = \{p, q, r, \ldots\}$ and define **formulas**:

$$\varphi ::= p \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \neg\varphi \mid \Box\varphi \mid \Diamond\varphi$$

Semantics

Evaluation happens inside a **structure** called a **frame**, $\mathfrak{F} = (W, R)$. W is a set of *worlds* and $R \subseteq W \times W$ is a binary relation representing *accessibility*. Of course we also need a **valuation** $v : A \rightarrow \mathcal{P}(W)$

$v(p)$ is meant to represents the set of worlds where p holds.

Given a **frame** $\mathfrak{F}$ along with a valuation v, we wish to define "φ is true at world x": $(\mathfrak{F}, v), x \vDash \varphi$

Evaluation of propositional connectives at a particular world happens in the exact same recursive way (classically, via truth tables)

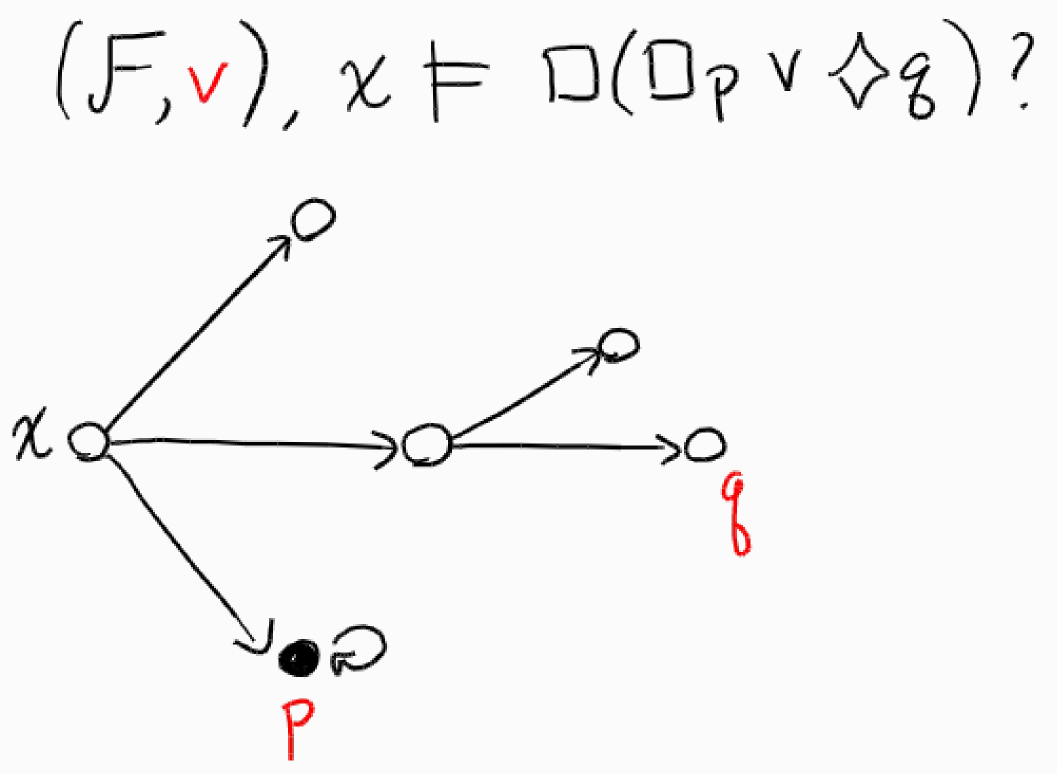
We wish to interpret the modalities as

$\Box\varphi$ means "in all accessible worlds, φ holds"

$\Diamond\varphi$ means "there is some accessible world where φ holds"

Formally, define $(\mathfrak{F}, v), x \vDash \varphi$ as

$(\mathfrak{F}, v), x \vDash p$	iff	$x \in v(p)$
... propositional connectives		
$(\mathfrak{F}, v), x \vDash \Box\varphi$	iff	$\forall y$ such that xRy, $(\mathfrak{F}, v), y \vDash \varphi$
$(\mathfrak{F}, v), x \vDash \Diamond\varphi$	iff	$\exists y$ such that xRy and $(\mathfrak{F}, v), y \vDash \varphi$



Frame Semantics

A frame $\mathfrak{F}$ *validates* a formula φ if φ holds at every world in $\mathfrak{F}$ *regardless* of the valuation.

$\mathfrak{F} \vDash \varphi \;:\; \mathfrak{F}$ validates φ

What does it say about $\mathfrak{F}$ if it validates the formula $p \to \Diamond p$?

$\mathfrak{F} \vDash p \to \Diamond p$ if and only if R is **reflexive**

What does it say about $\mathfrak{F}$ if it validates $\Diamond\Diamond p \to \Diamond p$?

$\mathfrak{F} \vDash \Diamond\Diamond p \to \Diamond p$ if and only if R is **transitive**

Frame semantics is about how modal formulas control the characteristics of the frames that validate them.

Galois connection of Frame Semantics

- $\mathcal{F}$ = class of frames, S = set of modal formulas
- $\models \subseteq \mathcal{F} \times S$
- $\mathrm{Log} : \mathcal{P}(\mathcal{F}) \to \mathcal{P}(S)$ takes a subclass $\mathcal{K}$ of frames to its **logic**, the set of formulas valid on all frames in $\mathcal{K}$
- $\mathrm{Fr} : \mathcal{P}(S) \to \mathcal{P}(\mathcal{F})$ takes a set Γ of formulas to the class of **frames** which validate everything in Γ.
- A closed set on the semantic side is called **modally definable**
- A closed set on the syntactic side is ... ?

Normal Modal Logics

Let's define a proof calulus for modal logic:

- **Axioms**:
  - Propositional axioms
  - $\Box(p \to q) \to (\Box p \to \Box q)$
- **Inference Rules**
  - Modus ponens, Substitution
  - Necessitation: From φ, infer $\Box\varphi$

The closure $N(-)$ of a set of sentences under this deduction system is called a **normal modal logic**

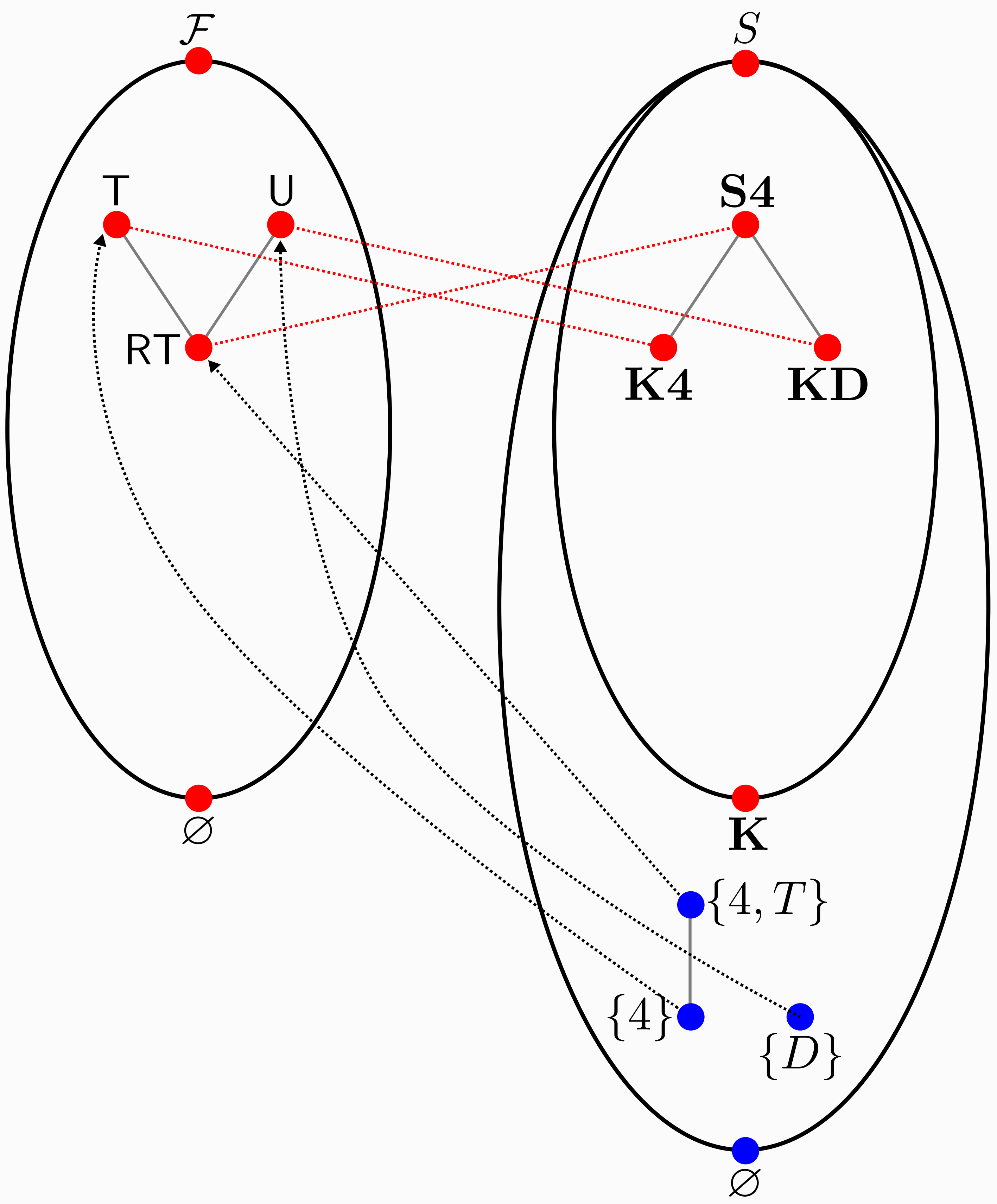
It is a very reasonable candidate for characterizing the closure of the Galois connection.

Let

- $4 := \Diamond\Diamond p \to \Diamond p$
- $T := p \to \Diamond p$
- $D := \Box p \to \Diamond p$

Our proof calculus "works well" for these axioms:

Axioms	Frm(−)	Log(Frm(−))
$\{\varnothing\}$	All frames	$\mathbf{K} = N(\varnothing)$
$\{4\}$	Transitives frames	$\mathbf{K4} = N(4)$
$\{4, T\}$	Transitive and reflexive frames	$\mathbf{S4} = N(4, T)$
$\{D\}$	Unbounded frames	$\mathbf{KD} = N(D)$



Frame incompleteness

Even though $N(-)$ works for most of the classically studied systems, it does not work in general 😢

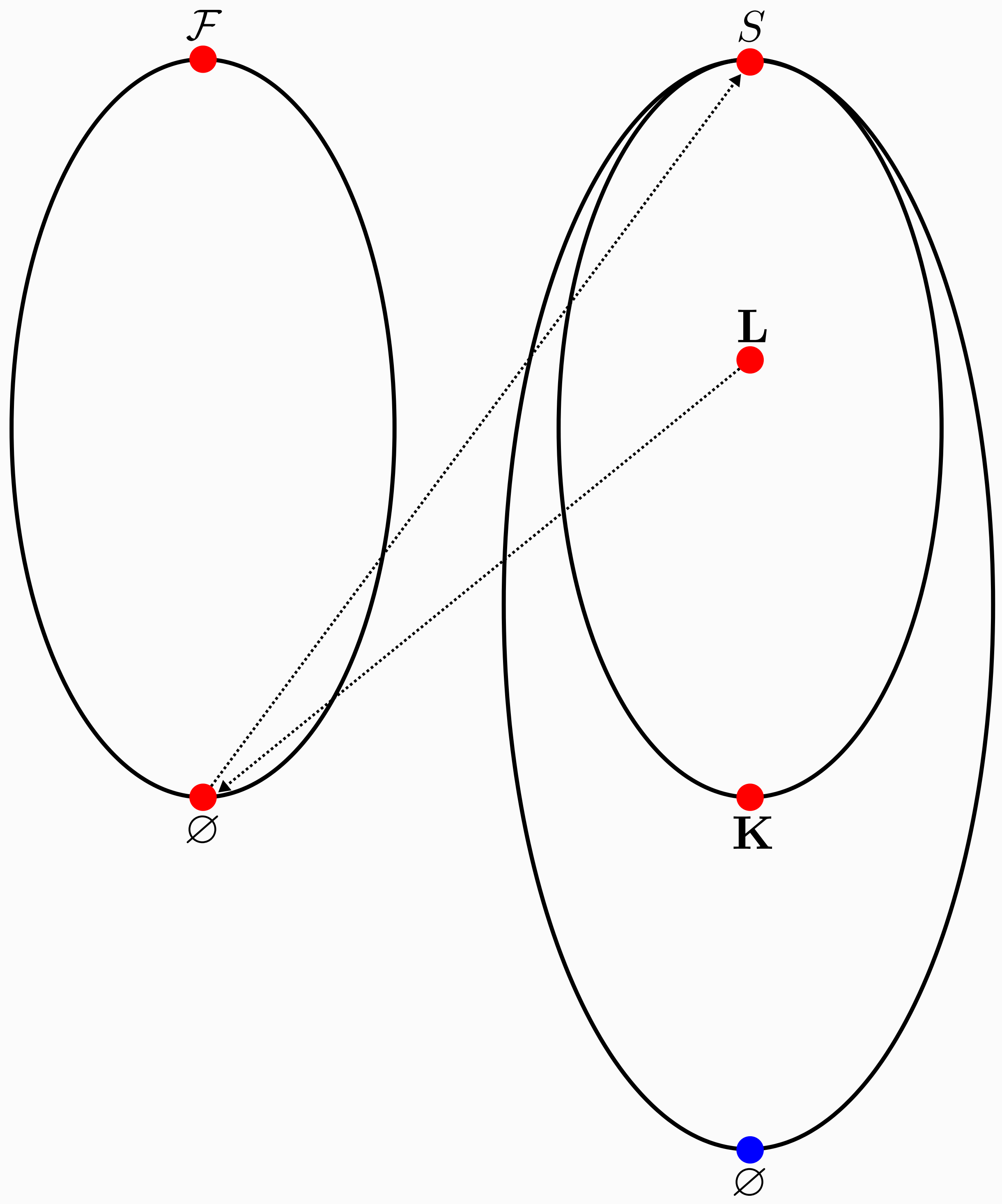
One can construct a logic L that is

- Proper (not inconsistent)
- A normal modal logic ($N(L) = L$, or closed according to N)
- But $\mathrm{Frm}(L) = \varnothing$

Logics like L are called **frame incomplete**.

"Most" (uncountably many) normal modal logics are frame incomplete.

Moreover, for general reasons, no proof calculus in the traditional sense can be sound and complete for frame semantics.



Adequate semantics

One way to fix this: Add topological structure on the semantic side:

A (descriptive) **general frame** is a structure $\mathfrak{F} = (W, \tau, R)$ where (W, τ) is a **Stone space** (compact, Hausdorff, basis of clopen sets) and R satisifies some conditions w.r.t. the topology.

When speaking of validity on general frames, we say $\mathfrak{F}$ validates φ if φ is true at every world under every **admissable valuation**, which may only assign propositional letters to *clopen sets*.

Remarkably, this "repairs" the situation so that $N(-)$ is a sound and complete proof calculus for general frame semantics.

A full account of why this works would be through the duality with **algebraic semantics**.

Topological semantics for modal logic

We could interpret modal logic as "talking about space"

Evaluation of truth happens inside a **structure** that is a **topological space** X. We also need a valuation $v : A \to \mathcal{P}(X)$

Given a space X and a valuation $v : A \to \mathcal{P}(X)$, we define "φ is true in X at the point x", $(X, v), x \vDash \varphi$

$(X, v), x \vDash p$	iff	$x \in v(p)$
... propositional connectives		
$(X, v), x \vDash \Box\varphi$	iff	$\exists$ open neighborhood U of x s.t. $\forall y \in U \ (X, v), y \vDash \varphi$
$(X, v), x \vDash \Diamond\varphi$	iff	$\forall$ open neighborhoods U of x, $\exists y \in U \ (X, v), y \vDash \varphi$

This extends the valuation $v : A \to \mathcal{P}(X)$ uniquely to a function $\mathrm{Form} \to \mathcal{P}(X)$.

If we think of this as assigning formulas to the set of points where they are true, then

- The propositional connectives correspond to boolean operations on these sets
- The modal operators correspond to closure and interior

$v(\neg\varphi)$	$=$	$v(\varphi)^c$
$v(\varphi \wedge \psi)$	$=$	$v(\varphi) \cap v(\psi)$
$v(\varphi \vee \psi)$	$=$	$v(\varphi) \cup v(\psi)$
$v(\Box\varphi)$	$=$	$\mathrm{Int}(v(\varphi))$
$v(\Diamond\varphi)$	$=$	$\mathrm{Cl}(v(\varphi))$

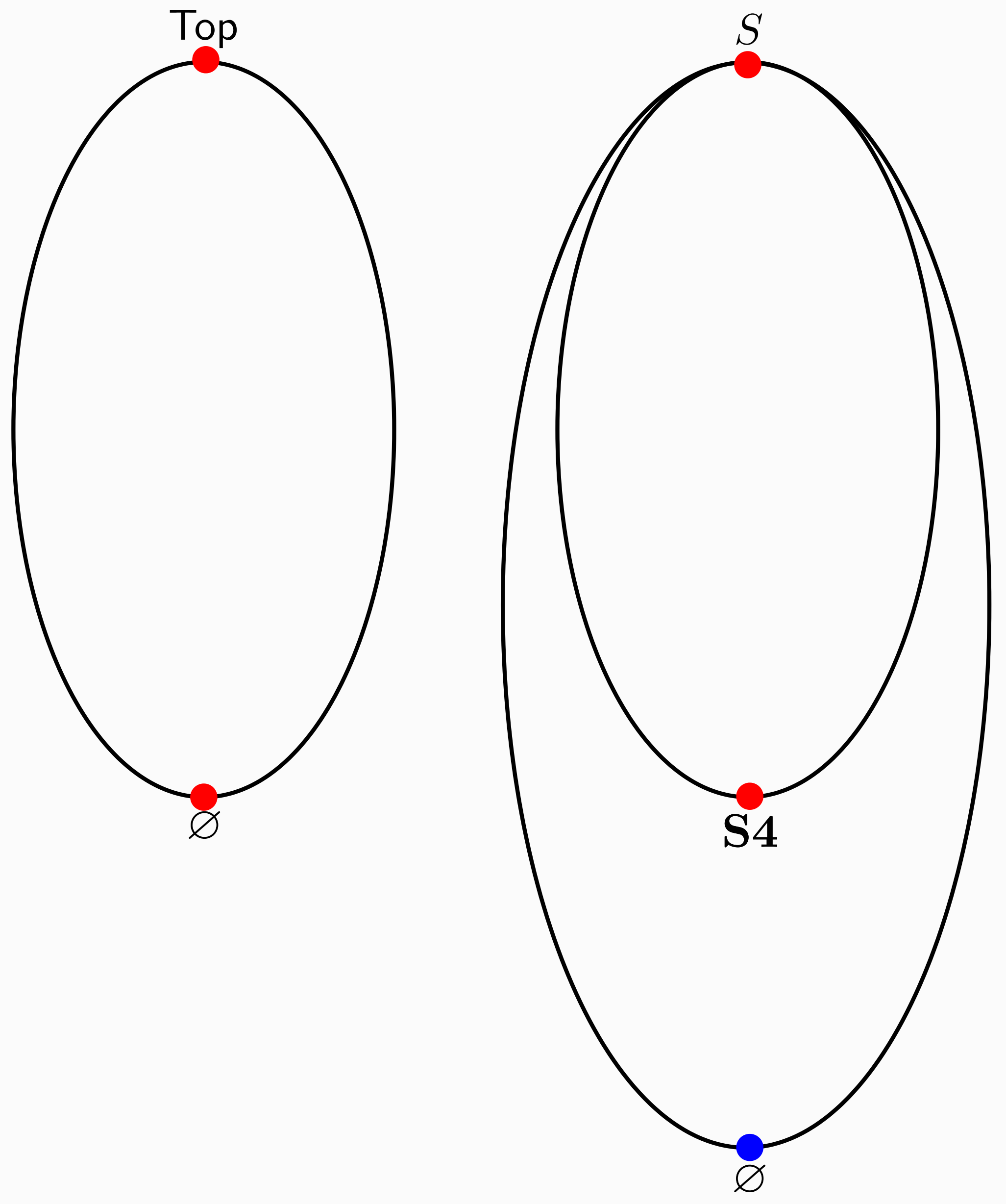
Again a formula φ is valid in X, $X \vDash \varphi$ if it is true at every point ($v(\varphi) = X$) regardless of the valuation.

Galois connection of topological semantics

An initial examination reveals that the axioms 4 and T are universally valid in all topological spaces.

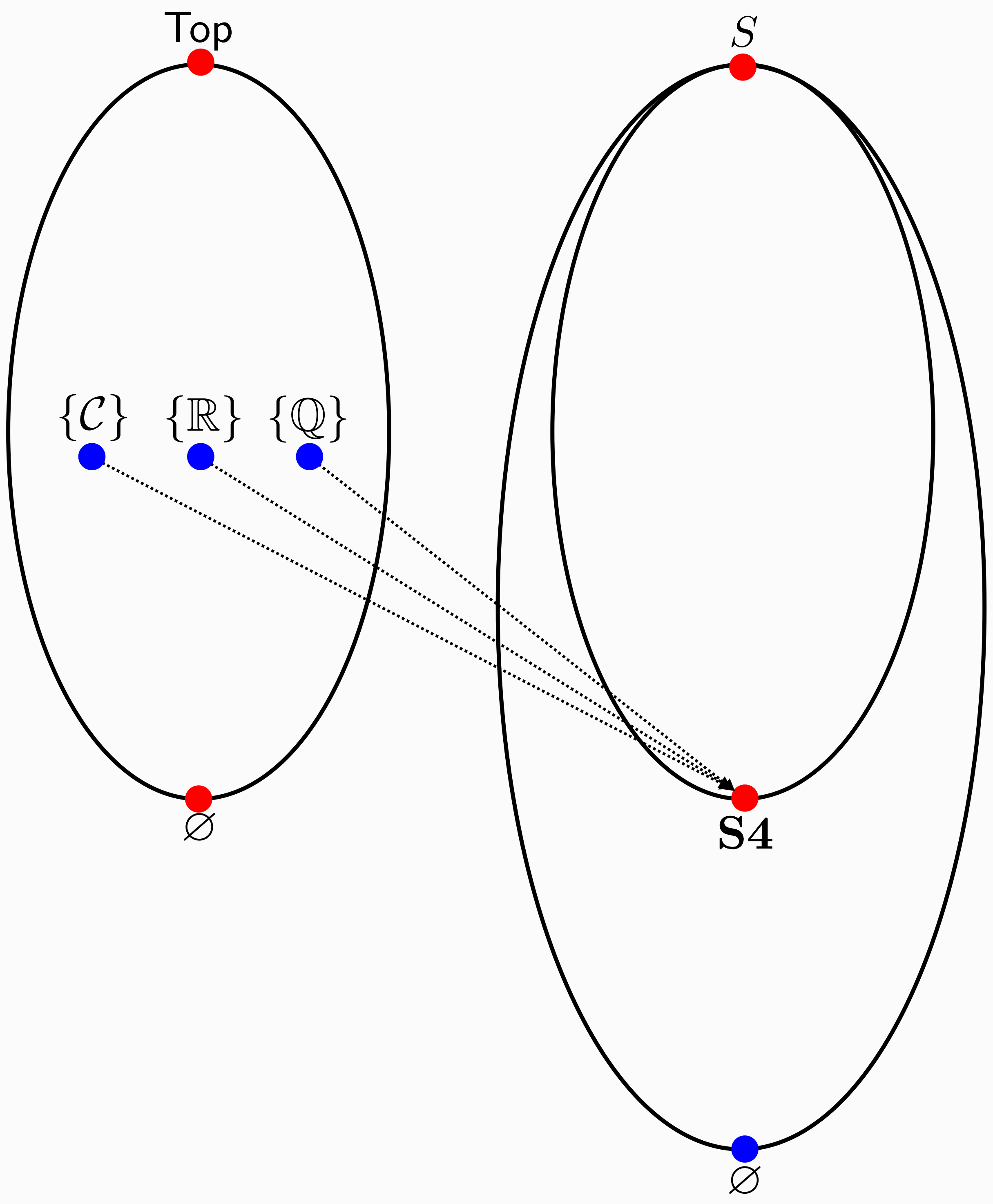
$4 := \Diamond\Diamond p \to \Diamond p$	closure is idempotent
$T := p \to \Diamond p$	any set is contained in its closure

And indeed the smallest closed set on the syntactic side, the logic of all topological spaces, is $\mathbf{S4}$



How low can you go?

An interesting result here is that **S4** is the logic of all topological spaces (**S4** is *complete* with respect to Top), but a famous result of Tarski and McKinsey shows that it is *complete* with respect to the real line. What does this mean?



- What does this say about this notion of semantics?