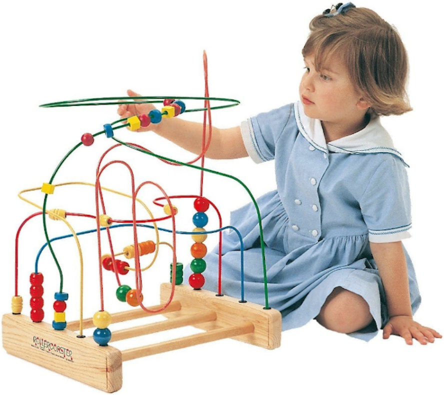
# Coxeter groups and diagram algebras



## Definition

A *Coxeter system* is a group with presentation

$$\langle s_1, \ldots, s_n \mid (s_i s_j)^{M_{ij}} = 1 \rangle$$

where $M$ is a symmetric $n \times n$ matrix with $M_{ii} = 1$ and $M_{ij} \in \{2, 3, \ldots\}$. $n$ is called the *rank* of the group.

- When $i = j$, this means $s_i^2 = 1$ for all $i$, thus generators are idempotent and self-inverse
- When $M_{ij} = 2$, this means $(s_i s_j)^2 = 1 \Rightarrow s_i s_j = s_j s_i$, i.e. $s_i$ and $s_j$ commute
- When $M_{ij} \geq 3$, $s_i$ and $s_j$ are said to have a *braid relation*:
  - $(s_i s_j)^3 = 1$ implies $s_i s_j s_i = s_j s_i s_j$
  - $(s_i s_j)^4 = 1$ implies $s_i s_j s_i s_j = s_j s_i s_j s_i$

Instead of a matrix $M$ we can use a edge-labeled graph on $\{1, ..., n\}$ where

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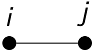
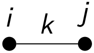
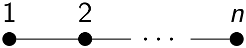
means $s_i, s_j$ commute
- 

means $s_i, s_j$ have a 3-braid relation
- 

means $s_i, s_j$ have a $k$-braid relation

Coxeter sytem of type $A_n$:




i.e., adjacent generators have a 3-braid relation and all other pairs commute.

We represent elements of a Coxeter group by *fully reduced* words in the generators.

**Theorem (Matsumoto)**

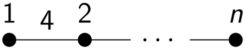
For any element $w \in W$ of a Coxeter group, all reduced words for $W$ have the same length. Furthermore, any reduced word is related to any other by a sequence of braid relations.

In type $A_4$:

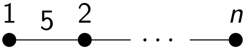
- $s_3 s_1 s_2 s_1 s_4 \to s_3 s_2 s_1 s_2 s_4 \to s_3 s_2 s_1 s_4 s_2 \to s_3 s_2 s_4 s_1 s_2 \to s_3 s_4 s_2 s_1 s_2 \to s_3 s_4 s_1 s_2 s_1$ are reduced words representing the same element.

The other Coxeter types we will be concerned with in this talk are closely related to $A_n$:

$B_n$:




$H_n$:




These look innocent, but blow up combinatorially:

- $A_4$ has 120 elements
- $B_4$ has 384 elements
- $H_4$ has 14,440 elements!

Given a Coxeter group $W = \langle s_1, \ldots, s_n \ : \ \ldots \rangle$, we can define the *Hecke algebra* of $W$:

Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ be the ring of Laurent polynomials over the integers. $\mathcal{A}$ is a commutative ring.

Now we will define an $\mathcal{A}$-algebra by a certain presentation.

## Definition

The *Hecke algebra* of $W$ is the (unital, associative) $\mathcal{A}$-algebra with presentation

$$H(W) = \langle T_1, \ldots, T_n \ : \ \text{(braid relations)}, (T_i - v)(T_i + v^{-1}) = 0 \rangle$$

Note that $(T_i - v)(T_i - v^{-1}) = 0 \Leftrightarrow T_i^2 = (v - v^{-1})T_i + 1$, so we can still "get rid of any squares".

It turns out $H(W)$ has a very simple free basis:

**Theorem**

$H(W)$ is free over the basis

$$\{T_w = T_{i_1} \dots T_{i_k} : s_{i_1} \dots s_{i_k} \text{ is any reduced expression for } w \in W\}$$

Indexed by the group elements of $W$.

i.e., in $H(A_4)$ we have $T_{232} = T_2 T_3 T_2 = T_3 T_2 T_3 = T_{323}$ (since $s_2 s_3 s_2 = s_3 s_2 s_3$ represent the same element in $W$).

Typical elements look like:

- $(3v^2 - v^{-1}) T_{121} = (3v^2 - v^{-1}) T_{212}$
- $v T_1 + v^{-1} T_{24} = v T_1 + v^{-1} T_{42}$
- $T_1(v + T_{24}) = T_1(v + T_2 T_4) = v T_1 + T_1 T_2 T_4 = v T_1 + T_{124}$

In general, $H(W)$ has many bases (all necessarily indexed by the elements of $W$). But one is of particular interest:

### Theorem (Kazdahn, Lusztig)

There exists a unique basis $\{C_w : w \in W\}$ for $H$ satisfying some precise technical properties called the *Kazdahn-Lusztig basis*.
When a $C$-basis element is expanded in terms of the $T$-basis, the coefficients (or "structure constants") are the *Kazdahn-Lusztig polynomials*:

$$C_w = T_w + \sum_{y<w} p_{y,w} T_y$$

In particular, $H(W)$ is *generated* by $C_i = T_i + v^{-1}$.

### Warning

This basis is far more compliated than the $T$-basis! $C_{i_1 i_2 \ldots i_k}$ is usually *very different* from $C_{i_1} C_{i_2} \ldots C_{i_k}$! We cannot just do arithmetic/multiplication "trivially" like we could in the $T$-basis.

Now we will take a quotient of $H(W)$ by a certain ideal:

$$\mathcal{I}(W) = \langle C_{\text{long braids}} \rangle = \langle C_{ij\ldots i} \ : \ s_i s_j \ldots s_i \text{ is a long braid} \rangle$$

We'll call the resulting quotient $TL(W) = H(W)/\mathcal{I}(W)$ for reasons that will be explained shortly.

How can we figure out what $TL(W)$ looks like?

- Since the $C_i$ generate $H(W)$, their equivalence classes $U_i = \pi(C_i)$ generate the quotient.
- If we can figure out what relations the $U_i$ have we can present $TL(W)$ by generators and relations.

Let's do this "concretely" in type $A_n$, where adjacent numbers have a 3-braid relation, and all others commute. We have

$$\mathcal{I}(A_n) = \langle C_{i(i+1)i} \ : \ 1 \leq i < n \rangle = \langle C_{121}, C_{232}, \ldots \rangle$$

Next, recalling that $C_i = T_i + v^{-1}$, you may calculate:

$$C_i^2 = (T + v^{-1})^2 = (v + v^{-1})T_1 + 1 + v^{-2} = (v + v^{-1})C_i$$

You can verify by another straightforward calculation that $C_i C_k = C_k C_i$ for non-adjacent, $i, k$.

Finally, the following calculation requires knowing what $C_{121}$ is, but if you knew you would compute (likewise for all indices):

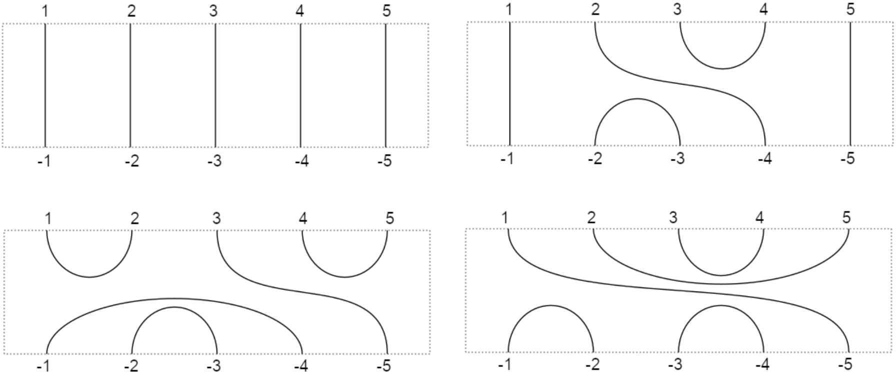
$$\begin{aligned} C_1 C_2 C_1 &= (T_1 + v^{-1})(T_2 + v^{-1})(T_1 + v^{-1}) \\ &= T_{121} + v^{-1} T_{21} + v^{-1} T_{12} + v^{-2} T_1 + v^{-2} T_2 + v^{-3} + T_1 + v^{-1} \\ &= C_{121} + C_1 \\ &\equiv C_1 \quad \text{(in the quotient)} \end{aligned}$$

So to summarize, we've "showed" that $TL(A_n)$ is abstractly presented by generators and relations in the following way:

$$\begin{aligned} \langle U_1, \ldots, U_n \ : \ & U_i^2 = (v + v^{-1})U_i, \\ & U_i U_k = U_k U_i \text{ for non-adjacent } i, k, \\ & U_i U_j U_i = U_i \text{ for adjacent } i, j \rangle \end{aligned}$$

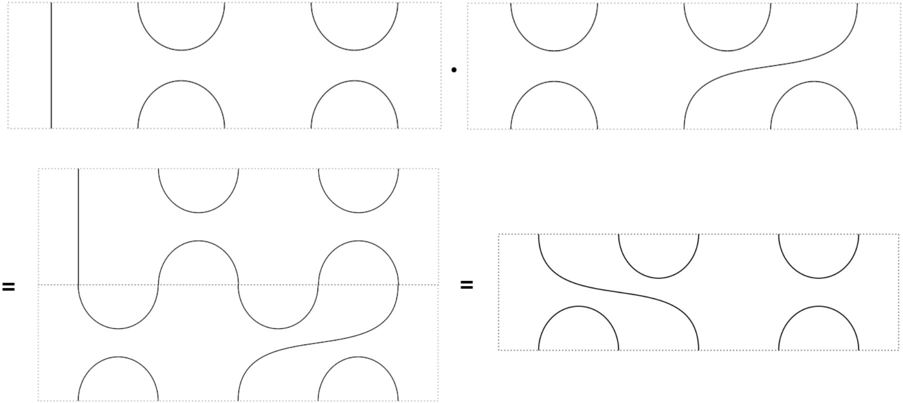
It turns out that this particular presentation reveals an isomorphism with something much more concrete...

Consider a *non-crossing pairing* of $n+1$ “north” nodes with $n+1$ “south” nodes. For example, when $n = 4$:



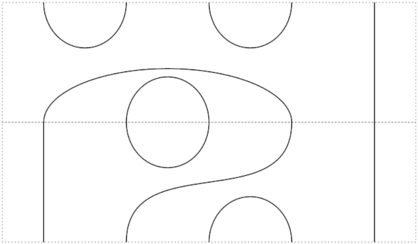

As suggested by the image, we also think of this as non-crossing 2-ary partition diagrams of the set $\{-(n+1), \ldots, 1, 1, \ldots, n+1\}$.

We can "multiply" these diagrams via vertical concatenation:

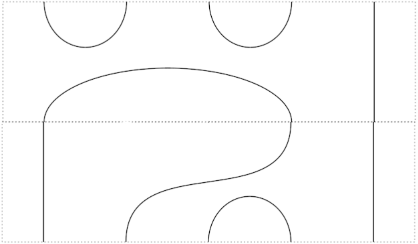


Form the free $\mathcal{A}$-algebra over these diagrams (i.e. formal $\mathcal{A}$-linear combinations of diagrams), and where multiplication is by vertical concatentation + “reduction rules”. Right now, the only reduction rule is

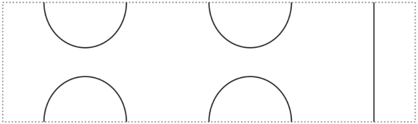
- any closed loops formed in a concatenation “come out” as a scalar multiplication by $\delta = v + v^{-1}$:



$= \delta$



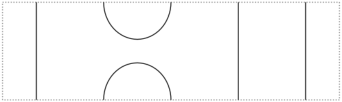
$= (v + v^{-1})$



This is the *Temperley Lieb Algebra*.

Claim: The following diagrams generate all the diagrams:

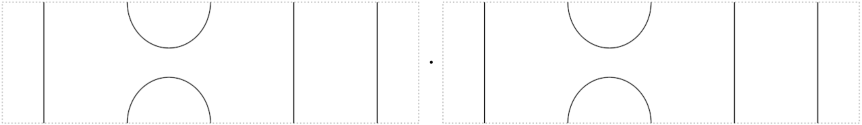
$U_1 =$ 

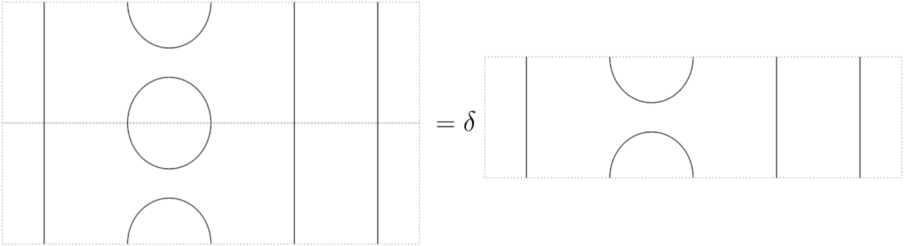
$U_2 =$ 

$U_3 =$ 

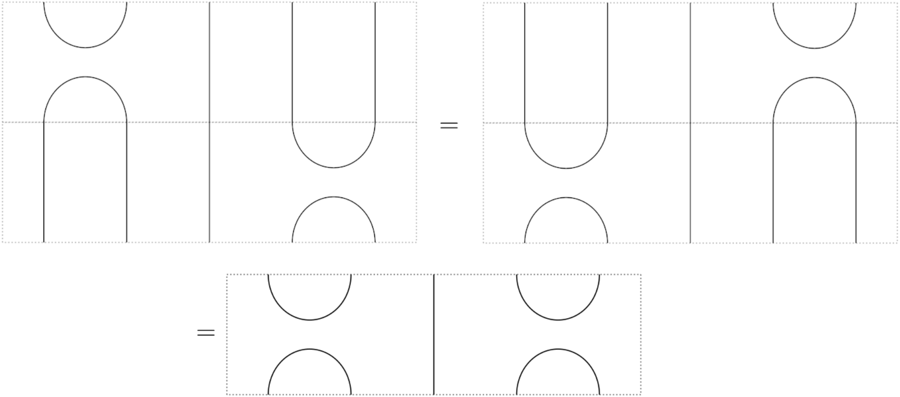
$U_4 =$ 

Furthermore, notice $U_i^2 = (v + v^{-1})U_i$

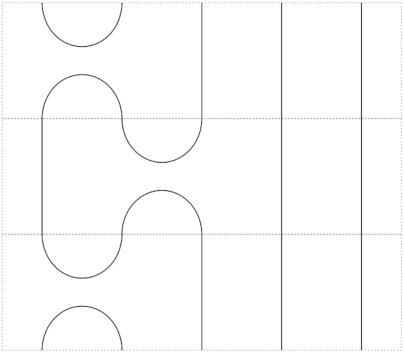




And $U_i U_k = U_k U_i$ for non-adjacent, $U_i, U_k$:



Finally, $U_i U_{(i+1)} U_i = U_i$ for adjacent $i, j$:



$=$

In fact, $TL(A_n)$ is isomorphic to the Temperley-Lieb algebra of diagrams with $n+1$ nodes, which has been well known for a while.

One can form the *Generalized Temperley-Lieb algebras* $TL(W)$ by the same construction as a quotient of the Hecke algebra.

**Question**: Is there a diagram algebra realization for the other Generalized TL algebras?

One is known for types $A, B, H, D, E$ and $\widetilde{C_n}$.

If you repeat the whole process for type $B_n$, you'll get the following presentation of $TL(B_n)$.

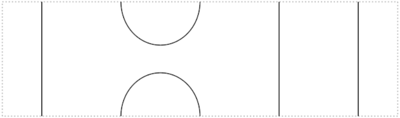
$$\begin{aligned}\langle U_1, \ldots, U_n \ : \ & U_i^2 = (v + v^{-1})U_i, \\ & U_i U_k = U_k U_i \text{ for non-adjacent } i, k, \\ & U_i U_j U_i = U_i \text{ for adjacent } i, j \text{ and } \{i, j\} \neq \{1, 2\}, \\ & U_1 U_2 U_1 U_2 = 2 U_1 U_2 \rangle\end{aligned}$$

This can be realized by "decorated" Temperley-Lieb diagrams, where arcs are allowed to carry a "decoration" (according to certain rules).

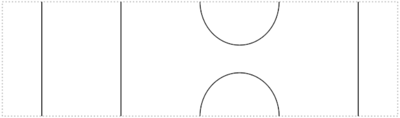
## $TL(B_4)$ is generated by

$U_1 = 2$

$U_2 =$



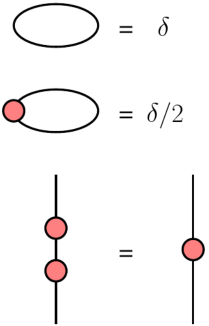
$U_3 =$



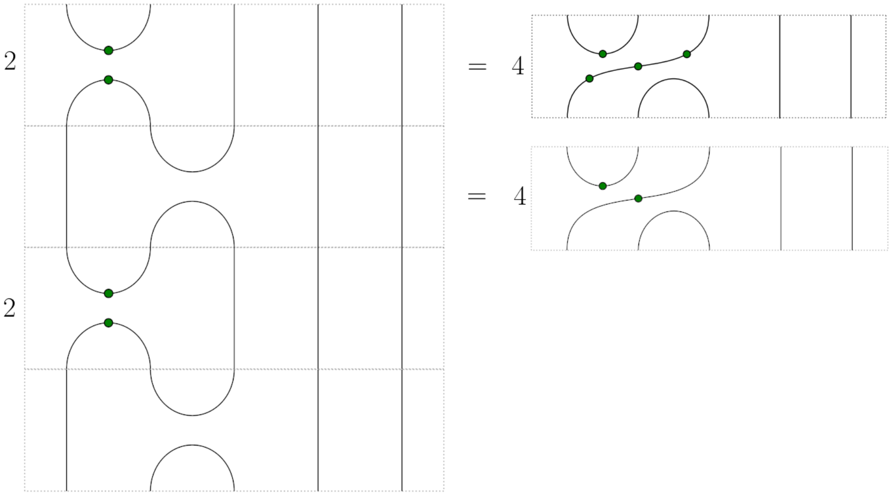
$U_4 =$

Multiplication is still concatenation + "reduction rules", but now there are more reduction rules:

- Replace any instance of 2 decorations with 1 decoration.
- An undecorated loop comes out as $\delta$.
- A decorated loop comes out as $\delta/2$.



This time we'll only check the new relation $U_1 U_2 U_1 U_2 = 2 U_1 U_2$.



If you do this for type $H_n$, you'll get the following presentation for $TL(H_n)$:

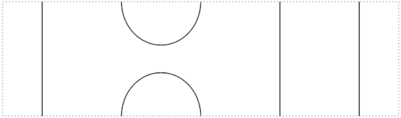
$$\begin{aligned}\langle U_1, \ldots, U_n \ :\ & U_i^2 = (v + v^{-1})U_i, \\ & U_i U_k = U_k U_i \text{ for non-adjacent } i, k, \\ & U_i U_j U_i = U_i \text{ for adjacent } i, j \text{ and } \{i, j\} \neq \{1, 2\}, \\ & U_1 U_2 U_1 U_2 U_1 = 3U_1 U_2 U_1 - U_1\rangle\end{aligned}$$

The diagram realization is the same as the previous one with different reduction rules.

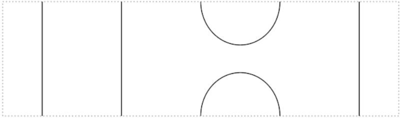
$TL(H_4)$ is generated by

$U_1 =$

$U_2 =$



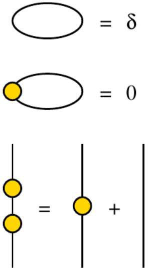
$U_3 =$



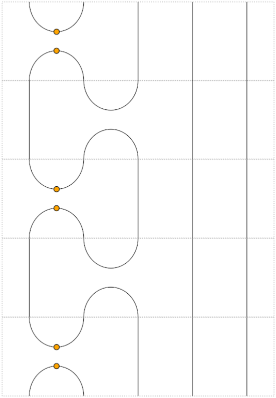
$U_4 =$

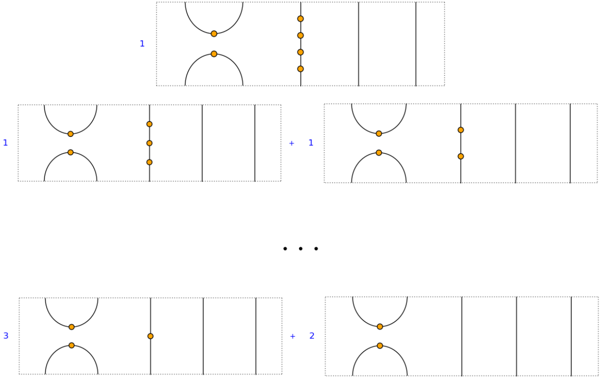
Reduction rules for $H_n$ are

- If a diagram has an edge with 2 decorations, split it into two copies that have 1 and 0 decorations on that edge, respectively.
- An undecorated loop comes out as $\delta$
- A decorated loop comes out as 0 (i.e., the whole diagram is gone)



This time, I'll just show a computation of $U_1 U_2 U_1 U_2 U_1$, and you can check™ that it's the same as what you get from $3U_1 U_2 U_1 - U_1$.





Ultimately, all such diagram realizations are of the same "scheme", where the basis elements are decorated Temperley-Lieb diagrams and multiplication is by concatenation + "decoration reduction rules".

Recall that $H(W)$ has bases indexed by the elements of $W$. It turns out:

**Theorem**

Bases of $TL(W)$ are indexed by the *fully-commutative* elements of $W$; $w \in W$ is *fully commutative* if any reduced word for $w$ can be transformed into any other reduced word by using only *commutation* relations (no longer braid relations); equivalently, no reduced word for $w$ contains a long braid.

In particular, any basis of $TL(W)$ is in bijection with $W_c =$ the fully commutative elements of $W$.

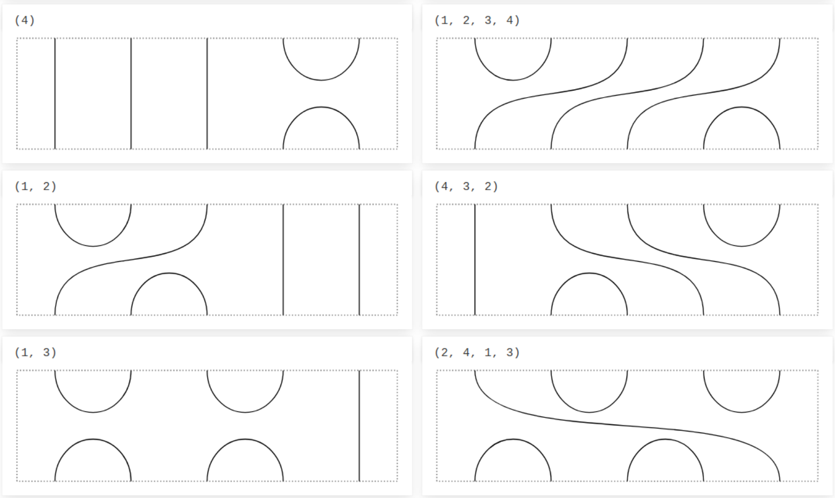
As far as bases of $TL(W)$ go, there are particular ones of interest. Certian algebras over $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ may have a *canonical basis* or *IC-basis*, a term coming from representation theory (but the existence or uniqueness is not guaranteed)

An *example* of such a canonical basis is the Kazdahn-Lusztig basis of the $H(W)$, but it is not the case that a canonical basis for $TL(W)$ can simply be obtained from the projection of the Kazdahn-Lusztig basis of $H(W)$.

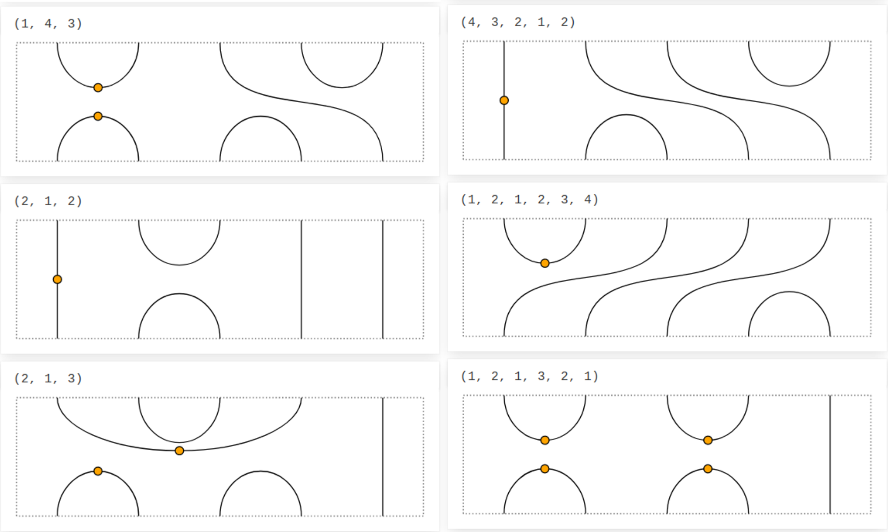
**Question**: Can the canonical basis be understood in terms of diagrams? If so, can we describe the bijection between fully commutative elements and their "canonical diagrams"?

In cases $A$, $B$, and $H$, the bijection is non-trivial but can be described combinatorially. In cases $A$ and $H$, the canonical diagram basis can be described combinatorially as "admissible diagrams" satisfying certain restrictions. The same is true in case $B$, but it requires some additional creativity.

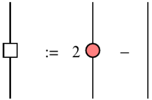
Some canonical diagrams in $TL(A_4)$ with their corresponding FC word: (https://math.colorado.edu/~chme3268/diagrams)



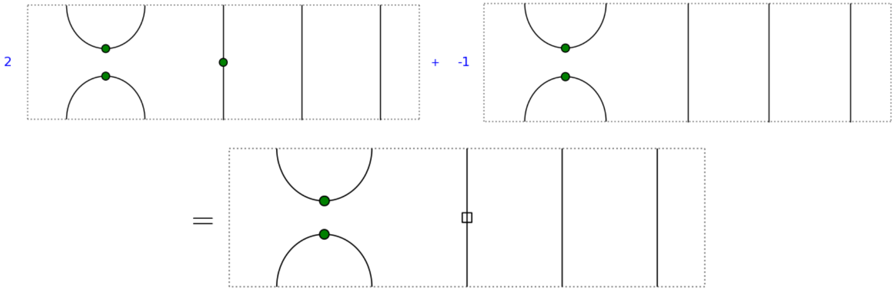
Some canonical diagrams in $TL(H_4)$:



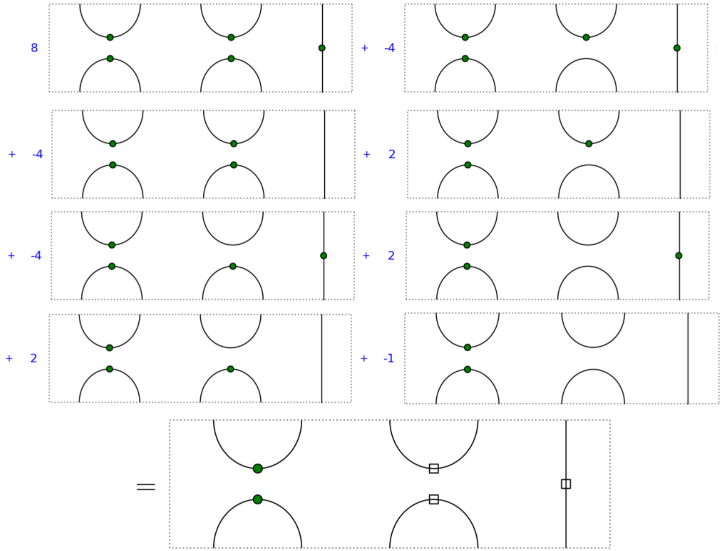
In type $B$, in order to describe the canonical basis, we need to invent a notational shorthand: the "square" decoration:



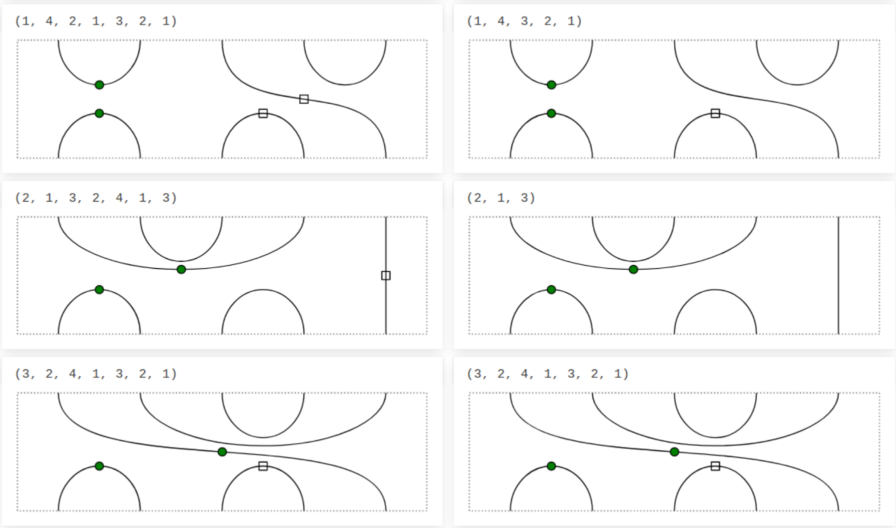

Like the reduction rule in type $H$, this takes place on "whole diagrams"; for example:



This notational device can compound to express unwieldy linear combinations as single diagrams:



Using the square we can much more easily describe the canonical diagrams in type $B$:



There is a certain involution on $W_c$ = the fully commutative elements of $W$ that has a very complicated definition.

## Definition

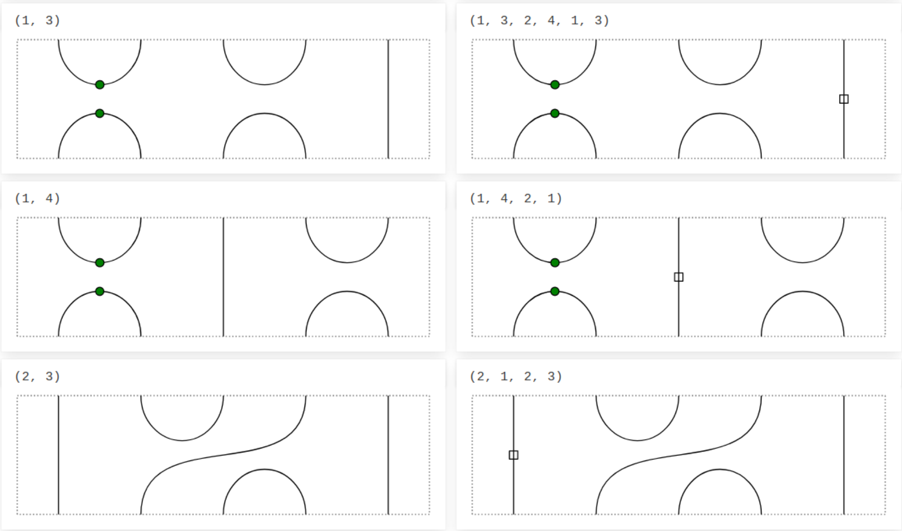
The *Mathas-Lusztig involution* $\lambda : W \to W$ can be defined on $w \in W$ as follows: Let $T_{w_0}$ be the $T$-basis element in $H(W)$ corresponding to the longest element $w_0 \in W$. Compute $T_{w_0} C_w$ and expand the result in terms of the $C$-basis:

$$T_{w_0} C_w = \sum_{y \in W} \alpha_{y,w} C_y$$

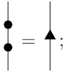
There exists a unique element $y$ in the same *left Kazdahn-Lusztig cell* as $w$ such that $\alpha_{y,w} \neq 0$, and $\lambda(w) = y$.

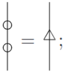
Can this involution be better understood under the bijection with diagrams?

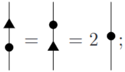
One conjecture supported with evidence is that in type $B$, when the ML involution is *not* the identity (it may often be), it appears to have an interpretation as "toggling a square" on the diagram:

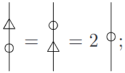


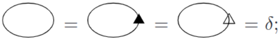
Just for fun, I've included the decorations and reduction rules for type $\widetilde{C}$, but we won't talk about it.

(1) 

(2) 

(3) 


(4) 


(5) 

Dana C. Ernst. “A diagrammatic representation of an affine $C$ Temperley–Lieb algebra”. In: *arXiv:0905.4457 [math]* (May 2009). arXiv: 0905.4457 version: 1. URL: http://arxiv.org/abs/0905.4457 (visited on 09/24/2020).

R. M. Green. “Cellular algebras arising from Hecke algebras of type H_n”. In: *arXiv:q-alg/9712019* (Dec. 1997). arXiv: q-alg/9712019 version: 1. URL: http://arxiv.org/abs/q-alg/9712019 (visited on 09/24/2020).

R. M. Green. “Decorated tangles and canonical bases”. In: *arXiv:math/0108076* (Aug. 2001). arXiv: math/0108076 version: 1. URL: http://arxiv.org/abs/math/0108076 (visited on 09/24/2020).

R. M. Green. “Generalized Temperley-Lieb algebras and decorated tangles”. In: *arXiv:q-alg/9712018* (Dec. 1997). arXiv: q-alg/9712018 version: 1. URL: http://arxiv.org/abs/q-alg/9712018 (visited on 09/24/2020).

R. M. Green and J. Losonczy. "Canonical Bases for Hecke Algebra Quotients". In: *arXiv:math/9904045* (Apr. 1999). arXiv: math/9904045 version: 1. URL: http://arxiv.org/abs/math/9904045 (visited on 09/24/2020).