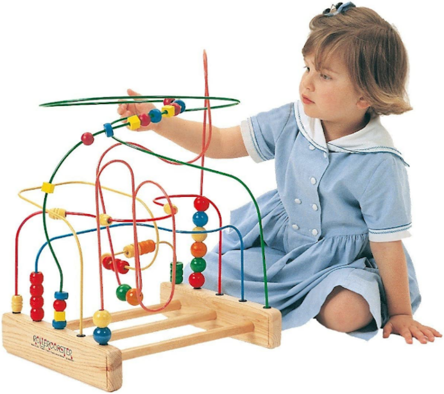
Coxeter groups and diagram algebras



Definition

A *Coxeter system* is a group with presentation

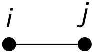
$$\langle s_1, \dots, s_n \mid (s_i s_j)^{M_{ij}} = 1 \rangle$$

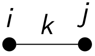
where M is a symmetric $n \times n$ matrix with $M_{ii} = 1$ and $M_{ij} \in \{2, 3, \dots\}$. n is called the *rank* of the group.

- When $i = j$, this means $s_i^2 = 1$ for all i, thus generators are idempotent and self-inverse
- When $M_{ij} = 2$, this means $(s_i s_j)^2 = 1 \Rightarrow s_i s_j = s_j s_i$, i.e. s_i and s_j commute
- When $M_{ij} \geq 3$, s_i and s_j are said to have a *braid relation*:
  - $(s_i s_j)^3 = 1$ implies $s_i s_j s_i = s_j s_i s_j$
  - $(s_i s_j)^4 = 1$ implies $s_i s_j s_i s_j = s_j s_i s_j s_i$

Instead of a matrix M we can use a edge-labeled graph on $\{1, ..., n\}$ where

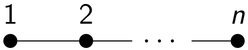
- 

means s_i, s_j commute
- 

means s_i, s_j have a 3-braid relation
- 

means s_i, s_j have a k-braid relation

Coxeter sytem of type A_n:



i.e., adjacent generators have a 3-braid relation and all other pairs commute.

We represent elements of a Coxeter group by *fully reduced* words in the generators.

Theorem (Matsumoto)

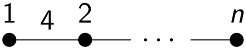
For any element $w \in W$ of a Coxeter group, all reduced words for W have the same length. Furthermore, any reduced word is related to any other by a sequence of braid relations.

In type A_4:

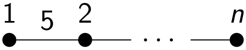
- $s_3 s_1 s_2 s_1 s_4 \to s_3 s_2 s_1 s_2 s_4 \to s_3 s_2 s_1 s_4 s_2 \to s_3 s_2 s_4 s_1 s_2 \to s_3 s_4 s_2 s_1 s_2 \to s_3 s_4 s_1 s_2 s_1$ are reduced words representing the same element.

The other Coxeter types we will be concerned with in this talk are closely related to A_n:

B_n:



H_n:



These look innocent, but blow up combinatorially:

- A_4 has 120 elements
- B_4 has 384 elements
- H_4 has 14,440 elements!

Given a Coxeter group $W = \langle s_1, \ldots, s_n \ : \ \ldots \rangle$, we can define the *Hecke algebra* of W:

Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ be the ring of Laurent polynomials over the integers. $\mathcal{A}$ is a commutative ring.

Now we will define an $\mathcal{A}$-algebra by a certain presentation.

Definition

The *Hecke algebra* of W is the (unital, associative) $\mathcal{A}$-algebra with presentation

$$H(W) = \langle T_1, \ldots, T_n \ : \ \text{(braid relations)}, (T_i - v)(T_i + v^{-1}) = 0 \rangle$$

Note that $(T_i - v)(T_i - v^{-1}) = 0 \Leftrightarrow T_i^2 = (v - v^{-1})T_i + 1$, so we can still "get rid of any squares".

It turns out $H(W)$ has a very simple free basis:

Theorem

$H(W)$ is free over the basis

$$\{T_w = T_{i_1} \dots T_{i_k} : s_{i_1} \dots s_{i_k} \text{ is any reduced expression for } w \in W\}$$

Indexed by the group elements of W.

i.e., in $H(A_4)$ we have $T_{232} = T_2 T_3 T_2 = T_3 T_2 T_3 = T_{323}$ (since $s_2 s_3 s_2 = s_3 s_2 s_3$ represent the same element in W).

Typical elements look like:

- $(3v^2 - v^{-1}) T_{121} = (3v^2 - v^{-1}) T_{212}$
- $v T_1 + v^{-1} T_{24} = v T_1 + v^{-1} T_{42}$
- $T_1(v + T_{24}) = T_1(v + T_2 T_4) = v T_1 + T_1 T_2 T_4 = v T_1 + T_{124}$

In general, $H(W)$ has many bases (all necessarily indexed by the elements of W). But one is of particular interest:

Theorem (Kazdahn, Lusztig)

There exists a unique basis $\{C_w : w \in W\}$ for H satisfying some precise technical properties called the *Kazdahn-Lusztig basis*.
When a C-basis element is expanded in terms of the T-basis, the coefficients (or "structure constants") are the *Kazdahn-Lusztig polynomials*:

$$C_w = T_w + \sum_{y<w} p_{y,w} T_y$$

In particular, $H(W)$ is *generated* by $C_i = T_i + v^{-1}$.

Warning

This basis is far more compliated than the T-basis! $C_{i_1 i_2 \dots i_k}$ is usually *very different* from $C_{i_1} C_{i_2} \dots C_{i_k}$! We cannot just do arithmetic/multiplication "trivially" like we could in the T-basis.

Now we will take a quotient of $H(W)$ by a certain ideal:

$$\mathcal{I}(W) = \langle C_{\text{long braids}} \rangle = \langle C_{ij\ldots i} \ : \ s_i s_j \ldots s_i \text{ is a long braid} \rangle$$

We'll call the resulting quotient $TL(W) = H(W)/\mathcal{I}(W)$ for reasons that will be explained shortly.

How can we figure out what $TL(W)$ looks like?

- Since the C_i generate $H(W)$, their equivalence classes $U_i = \pi(C_i)$ generate the quotient.
- If we can figure out what relations the U_i have we can present $TL(W)$ by generators and relations.

Let's do this "concretely" in type A_n, where adjacent numbers have a 3-braid relation, and all others commute. We have

$$\mathcal{I}(A_n) = \langle C_{i(i+1)i} \ : \ 1 \leq i < n \rangle = \langle C_{121}, C_{232}, \ldots \rangle$$

Next, recalling that $C_i = T_i + v^{-1}$, you may calculate:

$$C_i^2 = (T + v^{-1})^2 = (v + v^{-1})T_1 + 1 + v^{-2} = (v + v^{-1})C_i$$

You can verify by another straightforward calculation that $C_i C_k = C_k C_i$ for non-adjacent, i, k.

Finally, the following calculation requires knowing what C_{121} is, but if you knew you would compute (likewise for all indices):

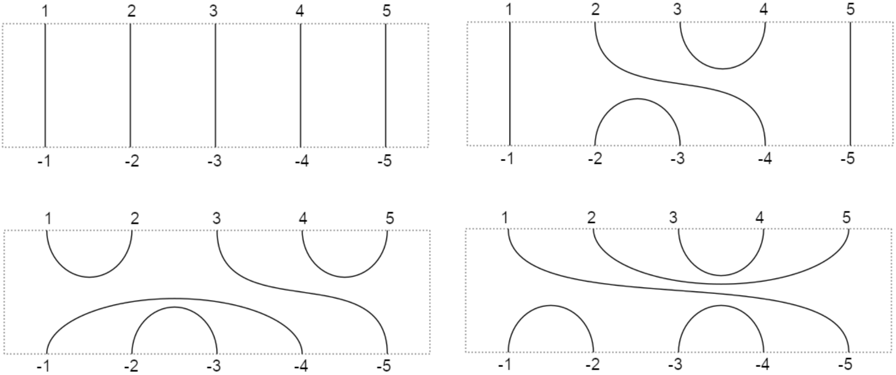
$$\begin{aligned} C_1 C_2 C_1 &= (T_1 + v^{-1})(T_2 + v^{-1})(T_1 + v^{-1}) \\ &= T_{121} + v^{-1} T_{21} + v^{-1} T_{12} + v^{-2} T_1 + v^{-2} T_2 + v^{-3} + T_1 + v^{-1} \\ &= C_{121} + C_1 \\ &\equiv C_1 \quad \text{(in the quotient)} \end{aligned}$$

So to summarize, we've "showed" that $TL(A_n)$ is abstractly presented by generators and relations in the following way:

$$\begin{aligned} \langle U_1, \ldots, U_n \ : \ & U_i^2 = (v + v^{-1})U_i, \\ & U_i U_k = U_k U_i \text{ for non-adjacent } i, k, \\ & U_i U_j U_i = U_i \text{ for adjacent } i, j \rangle \end{aligned}$$

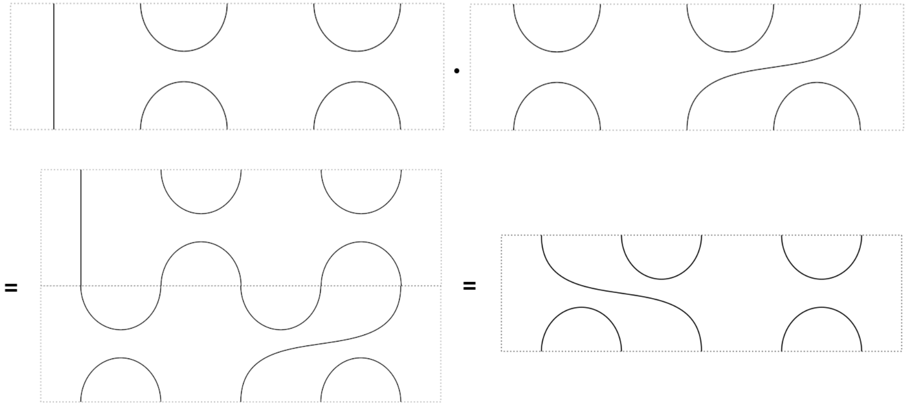
It turns out that this particular presentation reveals an isomorphism with something much more concrete...

Consider a *non-crossing pairing* of $n+1$ “north” nodes with $n+1$ “south” nodes. For example, when $n=4$:



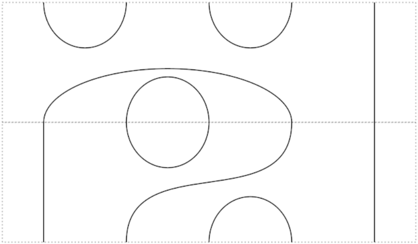
As suggested by the image, we also think of this as non-crossing 2-ary partition diagrams of the set $\{-(n+1), \ldots, 1, 1, \ldots, n+1\}$.

We can "multiply" these diagrams via vertical concatenation:

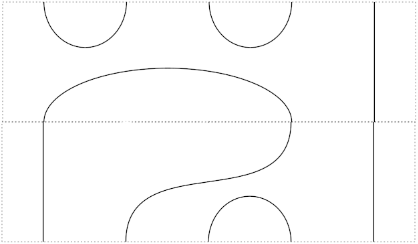


Form the free $\mathcal{A}$-algebra over these diagrams (i.e. formal $\mathcal{A}$-linear combinations of diagrams), and where multiplication is by vertical concatentation + "reduction rules". Right now, the only reduction rule is

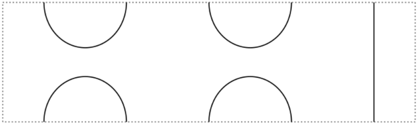
- any closed loops formed in a concatenation "come out" as a scalar multiplication by $\delta = v + v^{-1}$:



$= \delta$



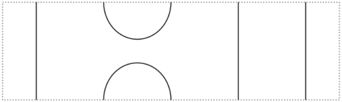
$= (v + v^{-1})$



This is the *Temperley Lieb Algebra*.

Claim: The following diagrams generate all the diagrams:

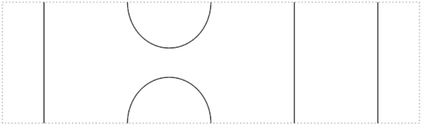
$U_1 =$ 

$U_2 =$ 

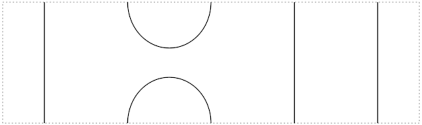
$U_3 =$ 

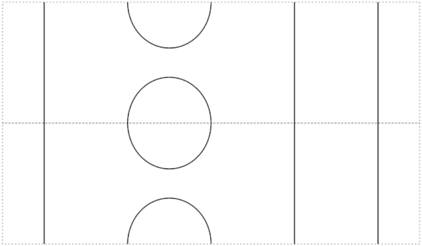
$U_4 =$ 

Furthermore, notice $U_i^2 = (v + v^{-1})U_i$

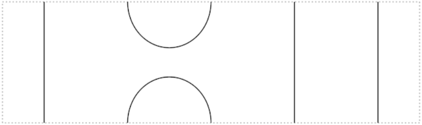


$\cdot$

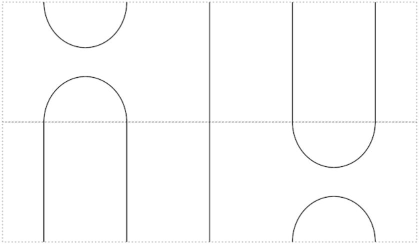




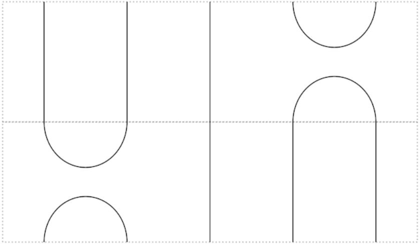
$= \delta$



And $U_i U_k = U_k U_i$ for non-adjacent, U_i, U_k:

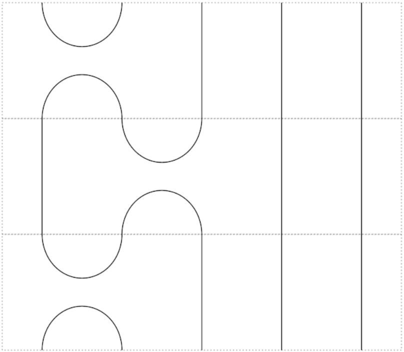


$=$



$=$

Finally, $U_i U_{(i+1)} U_i = U_i$ for adjacent i, j:



$=$

In fact, $TL(A_n)$ is isomorphic to the Temperley-Lieb algebra of diagrams with $n+1$ nodes, which has been well known for a while.

One can form the *Generalized Temperley-Lieb algebras* $TL(W)$ by the same construction as a quotient of the Hecke algebra.

Question: Is there a diagram algebra realization for the other Generalized TL algebras?

One is known for types A, B, H, D, E and $\widetilde{C_n}$.

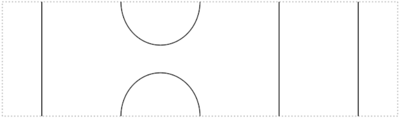
If you repeat the whole process for type B_n, you'll get the following presentation of $TL(B_n)$.

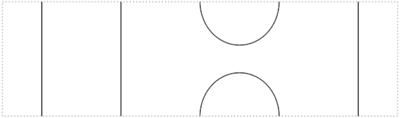
$$\begin{aligned}\langle U_1, \ldots, U_n \ : \ & U_i^2 = (v + v^{-1})U_i, \\ & U_i U_k = U_k U_i \text{ for non-adjacent } i, k, \\ & U_i U_j U_i = U_i \text{ for adjacent } i, j \text{ and } \{i, j\} \neq \{1, 2\}, \\ & U_1 U_2 U_1 U_2 = 2 U_1 U_2 \rangle\end{aligned}$$

This can be realized by "decorated" Temperley-Lieb diagrams, where arcs are allowed to carry a "decoration" (according to certain rules).

$TL(B_4)$ is generated by

$U_1 = 2$ 

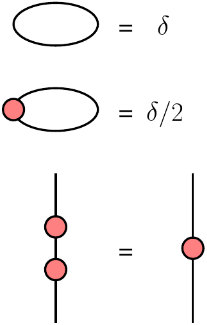
$U_2 =$ 

$U_3 =$ 

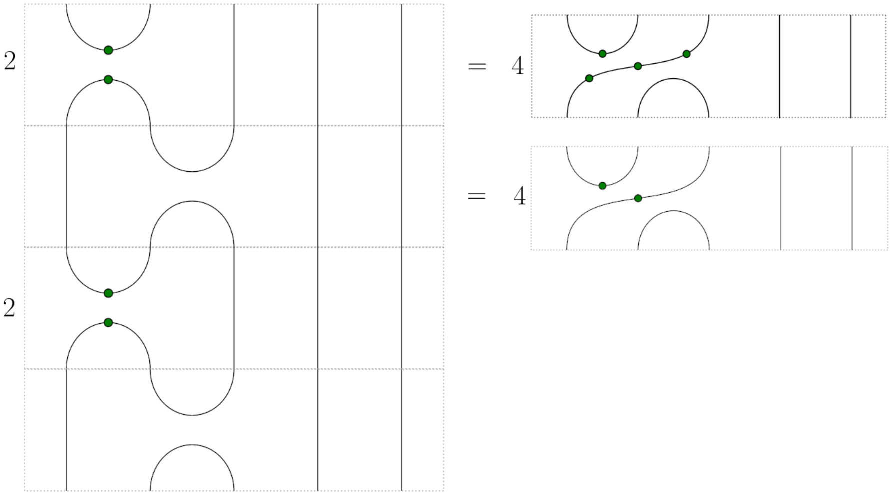
$U_4 =$ 

Multiplication is still concatenation + "reduction rules", but now there are more reduction rules:

- Replace any instance of 2 decorations with 1 decoration.
- An undecorated loop comes out as δ.
- A decorated loop comes out as $\delta/2$.



This time we'll only check the new relation $U_1 U_2 U_1 U_2 = 2 U_1 U_2$.



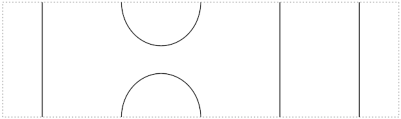
If you do this for type H_n, you'll get the following presentation for $TL(H_n)$:

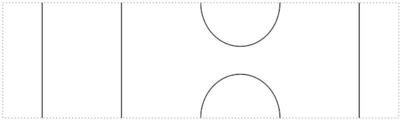
$$\begin{aligned}\langle U_1, \ldots, U_n \ : \ & U_i^2 = (v + v^{-1})U_i, \\ & U_i U_k = U_k U_i \text{ for non-adjacent } i, k, \\ & U_i U_j U_i = U_i \text{ for adjacent } i, j \text{ and } \{i, j\} \neq \{1, 2\}, \\ & U_1 U_2 U_1 U_2 U_1 = 3U_1 U_2 U_1 - U_1 \rangle\end{aligned}$$

The diagram realization is the same as the previous one with different reduction rules.

$TL(H_4)$ is generated by

$U_1 =$ 

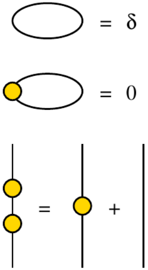
$U_2 =$ 

$U_3 =$ 

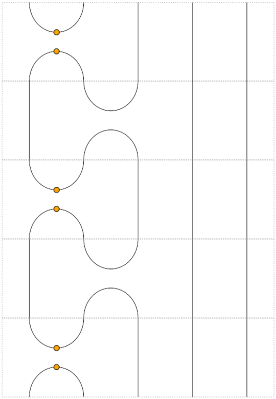
$U_4 =$ 

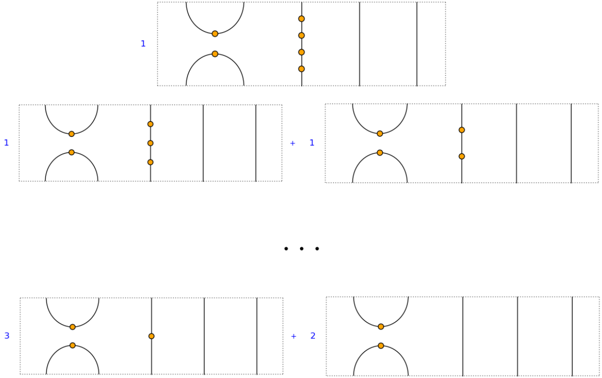
Reduction rules for H_n are

- If a diagram has an edge with 2 decorations, split it into two copies that have 1 and 0 decorations on that edge, respectively.
- An undecorated loop comes out as δ
- A decorated loop comes out as 0 (i.e., the whole diagram is gone)



This time, I'll just show a computation of $U_1U_2U_1U_2U_1$, and you can check™ that it's the same as what you get from $3U_1U_2U_1 - U_1$.





Ultimately, all such diagram realizations are of the same "scheme", where the basis elements are decorated Temperley-Lieb diagrams and multiplication is by concatenation + "decoration reduction rules".

Recall that $H(W)$ has bases indexed by the elements of W. It turns out:

Theorem

Bases of $TL(W)$ are indexed by the *fully-commutative* elements of W; $w \in W$ is *fully commutative* if any reduced word for w can be transformed into any other reduced word by using only *commutation* relations (no longer braid relations); equivalently, no reduced word for w contains a long braid.

In particular, any basis of $TL(W)$ is in bijection with $W_c =$ the fully commutative elements of W.

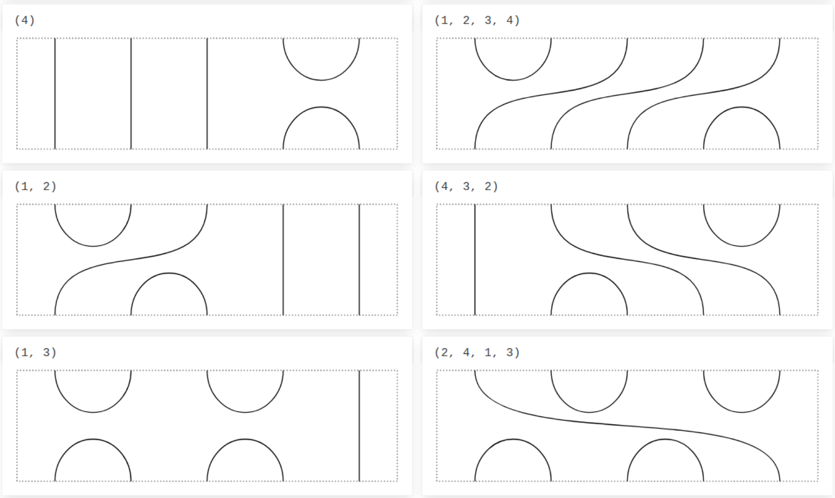
As far as bases of $TL(W)$ go, there are particular ones of interest. Certian algebras over $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ may have a *canonical basis* or *IC-basis*, a term coming from representation theory (but the existence or uniqueness is not guaranteed)

An *example* of such a canonical basis is the Kazdahn-Lusztig basis of the $H(W)$, but it is not the case that a canonical basis for $TL(W)$ can simply be obtained from the projection of the Kazdahn-Lusztig basis of $H(W)$.

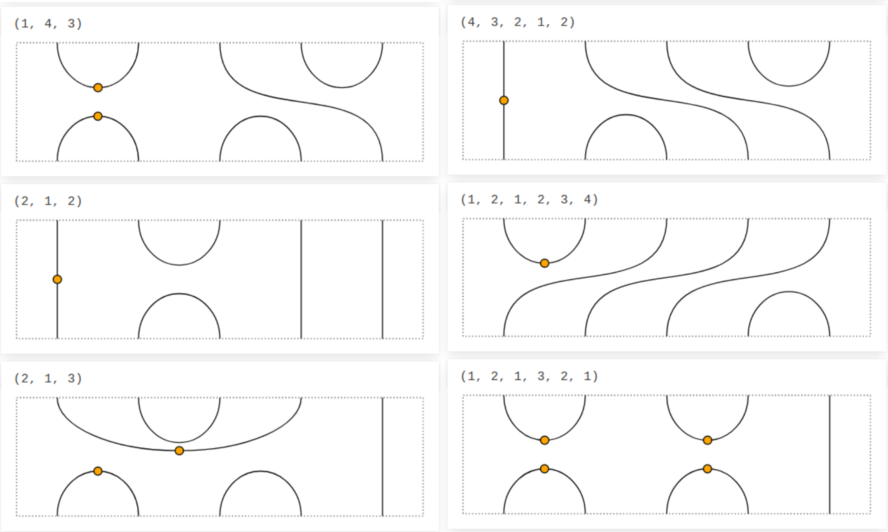
Question: Can the canonical basis be understood in terms of diagrams? If so, can we describe the bijection between fully commutative elements and their "canonical diagrams"?

In cases A, B, and H, the bijection is non-trivial but can be described combinatorially. In cases A and H, the canonical diagram basis can be described combinatorially as "admissible diagrams" satisfying certain restrictions. The same is true in case B, but it requires some additional creativity.

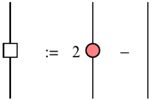
Some canonical diagrams in $TL(A_4)$ with their corresponding FC word: (https://math.colorado.edu/∼chme3268/diagrams)



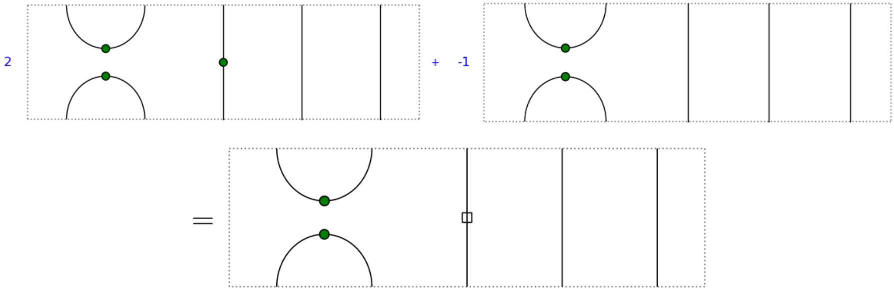
Some canonical diagrams in $TL(H_4)$:



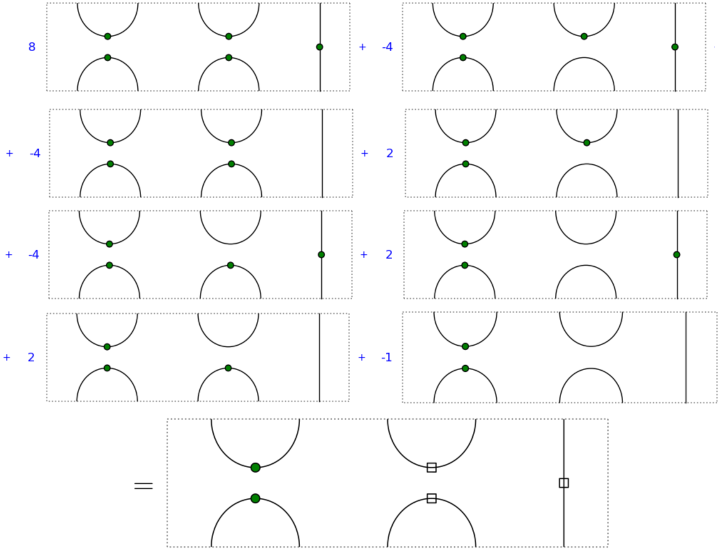
In type B, in order to describe the canonical basis, we need to invent a notational shorthand: the "square" decoration:



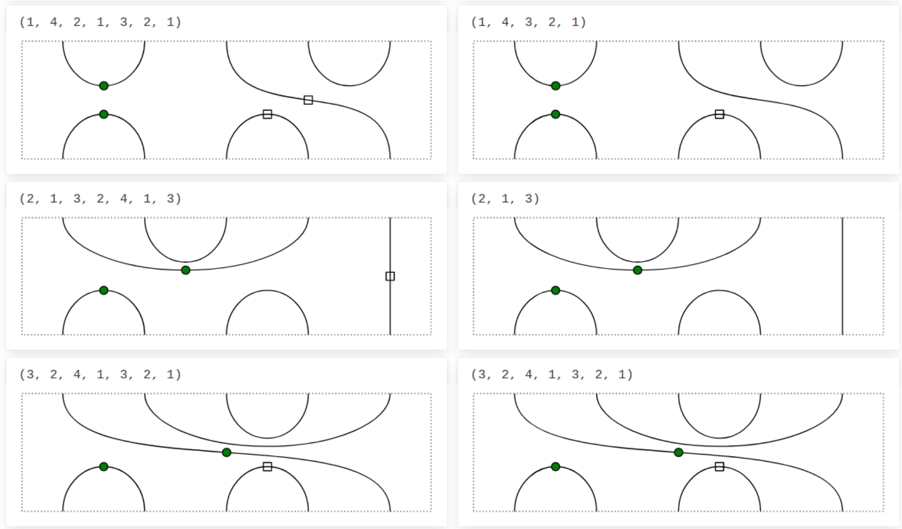
Like the reduction rule in type H, this takes place on "whole diagrams"; for example:



This notational device can compound to express unwieldy linear combinations as single diagrams:



Using the square we can much more easily describe the canonical diagrams in type B:



There is a certain involution on W_c = the fully commutative elements of W that has a very complicated definition.

Definition

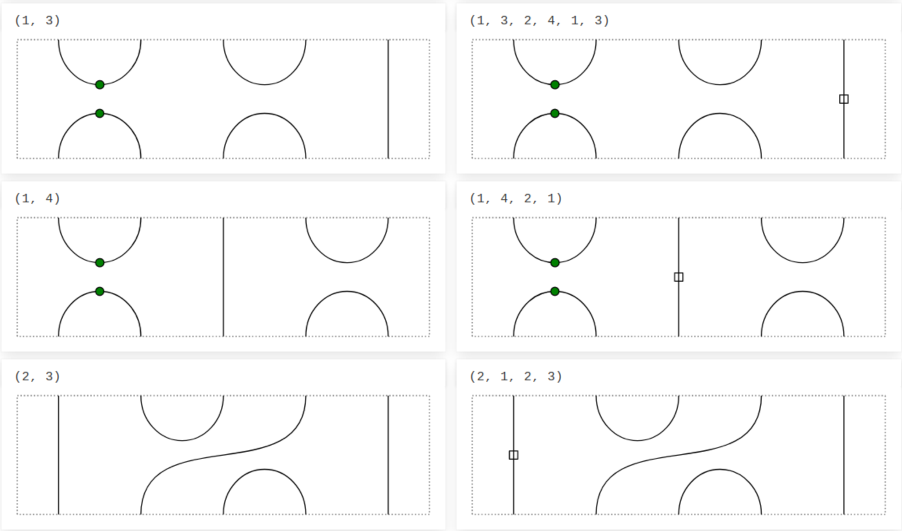
The *Mathas-Lusztig involution* $\lambda : W \to W$ can be defined on $w \in W$ as follows: Let T_{w_0} be the T-basis element in $H(W)$ corresponding to the longest element $w_0 \in W$. Compute $T_{w_0} C_w$ and expand the result in terms of the C-basis:

$$T_{w_0} C_w = \sum_{y \in W} \alpha_{y,w} C_y$$

There exists a unique element y in the same *left Kazdahn-Lusztig cell* as w such that $\alpha_{y,w} \neq 0$, and $\lambda(w) = y$.

Can this involution be better understood under the bijection with diagrams?

One conjecture supported with evidence is that in type B, when the ML involution is *not* the identity (it may often be), it appears to have an interpretation as "toggling a square" on the diagram:



Just for fun, I've included the decorations and reduction rules for type $\widetilde{C}$, but we won't talk about it.