FINAL EXAM PRACTICE PROBLEMS

(9.1) Find a vector function that parametrizes the line through $(4, -1, 2)$ and $(1, 1, 5)$.

(9.2) Find a vector function that parametrizes the line through $(-2, 2, 4)$ that is perpendicular to the plane $2x - y + 5z = 12$.

(9.3) Find an equation of the plane through $(3, -1, 1)$, $(4, 0, 2)$, and $(6, 3, 1)$.

(9.4) Calculate $\text{proj}_{\mathbf{a}}\, \mathbf{b}$ if $\mathbf{a} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

(10.1) Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 16$ and the plane $x + z = 5$.

(10.2) A particle's position at time t is given by $\mathbf{r}(t) = \langle t, \cos t, \sin t\rangle$. Determine the distance traveled by the particle between $t = 0$ and $t = 1$.

(11.1) Sketch several level curves of the function $f(x, y) = \sqrt{4x^2 + y^2}$.

(11.2) Evaluate the limit $\lim_{(x,y)\to(0,0)} \dfrac{2xy}{x^2 + 2y^2}$ or show that it does not exist.

(11.3) Find (a) the tangent plane and (b) a normal vector to the surface $z = 3x^2 - y^2 + 2x$ at the point $(1, -2, 1)$.

(11.4) Find (a) the tangent plane and (b) a normal vector to the surface $\mathbf{r}(u, v) = \langle u + v, u^2, v^2\rangle$ at the point $(3, 4, 1)$.

(11.5) Find the directional derivative of $f(x, y) = x^2 e^{-y}$ at the point $(-2, 0)$ in the direction of the point $(2, -3)$.

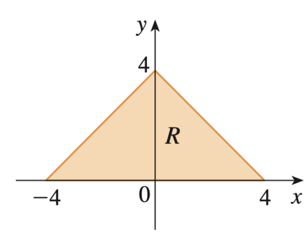
(11.6) Find the local maximum and minimum values and saddle points of the function $f(x, y) = 3xy - x^2y - xy^2$.

(11.7) Let $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$, $\mathbf{g}(x, y) = \langle x + y, 3x - y, 2x + y\rangle$, and $\mathbf{w}(x, y, z) = \langle 2x, 2y\rangle$. Which of the following compositions are well-defined?

$$\mathbf{r}\circ\mathbf{g}, \quad \mathbf{r}\circ\mathbf{w}, \quad \mathbf{g}\circ\mathbf{r}, \quad \mathbf{g}\circ\mathbf{w}, \quad \mathbf{w}\circ\mathbf{r}, \quad \mathbf{w}\circ\mathbf{g}$$

If a composition is well-defined, state its domain and codomain.

(12.1) Write $\iint_R f(x, y)\, dA$ as an iterated integral, where R is the region shown and f is an arbitrary continuous function on R.



(12.2) Rewrite the integral $\int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx$ as an iterated integral in the order $dx\,dy\,dz$.

(12.3) Use the transformation $u = x - y$ and $v = x + y$ to evaluate $\iint_R \frac{x-y}{x+y}\,dA$ where R is the square with vertices $(0,2)$, $(1,1)$, $(2,2)$, and $(1,3)$.

(12.4) Evaluate $\iiint_B (x^2+y^2+z^2)^2\,dV$, where B is the ball with center the origin and radius 5.

(13.1) Show that $\mathbf{F}(x,y) = \langle 4x^3y^2 - 2xy^3, 2x^4y - 3x^2y^2 + 4y^3\rangle$ is a conservative vector field and use this fact to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ along the curve $\mathbf{r}(t) = \langle t + \sin \pi t, 2t + \cos \pi t\rangle$.

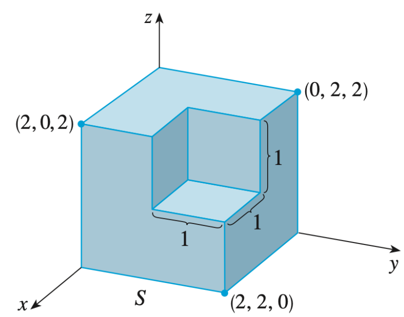
(13.2) Use Green's Theorem to evaluate $\int_C x^2y\,dx - xy^2\,dy$, where C is the circle $x^2+y^2=4$ with the counterclockwise orientation.

(13.3) Evaluate the surface integral $\iint_S z\,dS$, where S is the part of the paraboloid $z = x^2+y^2$ that lies under the plane $z = 4$.

(13.4) Evaluate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = \langle xz, -2y, 3x\rangle$ and S is the sphere $x^2+y^2+z^2 = 4$ with outward orientation.

(13.5) Use Stokes' Theorem to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = \langle xy, yz, zx\rangle$, and C is the triangle with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, oriented counterclockwise as viewed from above.

(13.6) Find $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = \langle x,y,z\rangle$ and S is the outwardly oriented surface shown in the figure (the boundary surface of a cube with a unit corner cube removed).



Answers

(9.1) $\mathbf{r}(t) = \langle 4-3t, -1+2t, 2+3t\rangle$
(9.2) $\mathbf{r}(t) = \langle -2+2t, 2-t, 4+5t\rangle$
(9.3) $-4x+3y+z=-14$
(9.4) $\langle -\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}\rangle$
(10.1) $\mathbf{r}(t) = \langle 4\cos t, 4\sin t, 5-4\cos t\rangle$
(10.2) $\sqrt{2}$
(11.1) The level curves form a family of concentric ellipses.
(11.2) The limit does not exist.
(11.3) (a) $z = 8x+4y+1$ (b) $\langle 8,4,-1\rangle$
(11.4) (a) $4x-y-2z=6$ (b) $\langle 4,-1,-2\rangle$
(11.5) $-\frac{4}{5}$.
(11.6) Maximum $f(1,1)=1$;
Saddle points $(0,0), (0,3), (3,0)$.
(11.7) Well-defined: $\mathbf{g}\circ\mathbf{w}:\mathbb{R}^3\to\mathbb{R}^3$,
$\mathbf{w}\circ\mathbf{r}:\mathbb{R}^1\to\mathbb{R}^2$, $\mathbf{w}\circ\mathbf{g}:\mathbb{R}^2\to\mathbb{R}^2$.
Not well-defined: $\mathbf{r}\circ\mathbf{g}, \mathbf{r}\circ\mathbf{w}$, $\mathbf{g}\circ\mathbf{r}$.
(12.1) $\int_0^4\int_{y-4}^{4-y} f(x,y)\,dx\,dy$
(12.2) $\int_0^1\int_0^{1-z}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dy\,dz$
(12.3) $-\ln 2$
(12.4) $312,500\pi/7$
(13.1) 0
(13.2) -8π
(13.3) $(\pi/60)(391\sqrt{17}+1)$
(13.4) $-64\pi/3$
(13.5) $-\frac{1}{2}$
(13.6) 21

(11.4) Find (a) the tangent plane and (b) a normal vector to the surface $\mathbf{r}(u,v) = \langle u+v, u^2, v^2\rangle$ at the point $(3,4,1)$.

$\vec{r}_u = \langle 1, 2u, 0\rangle$

$\vec{r}_v = \langle 1, 0, 2v\rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 2u & 0 \\ 1 & 0 & 2v \end{vmatrix} = \langle 4uv, -2v, -2u\rangle$$

At $(3,4,1)$, $u=2$, $v=1$

$\vec{n} = \langle 8, -2, -4\rangle$

divide by 2 (not necessary): $\vec{n} = \langle 4, -1, -2\rangle$

$\Rightarrow \quad 4(x-3) - 1(y-4) - 2(z-1) = 0$

$4x - y - 2z = 6$

(11.5) Find the directional derivative of $f(x,y) = x^2e^{-y}$ at the point $\overset{P}{(-2,0)}$ in the direction of the point $\underset{Q}{(2,-3)}$.

$$\vec{PQ} = \langle 4, -3 \rangle$$

$$\vec{u} = \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{\langle 4, -3 \rangle}{5} = \left\langle \frac{4}{5}, \frac{-3}{5} \right\rangle$$

$$\nabla f = \langle 2xe^{-y}, -x^2e^{-y} \rangle$$

$$\nabla f(-2,0) = \langle -4, -4 \rangle$$

$$D_u f(-2,0) = \nabla f(-2,0) \cdot \vec{u}$$

$$= \langle -4, -4 \rangle \cdot \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle$$

$$= \frac{-16}{5} + \frac{12}{5}$$

$$= \frac{-4}{5}$$

(11.7) Let $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$, $\mathbf{g}(x, y) = \langle x + y, 3x - y, 2x + y\rangle$, and $\mathbf{w}(x, y, z) = \langle 2x, 2y\rangle$. Which of the following compositions are well-defined?

$$\mathbf{r} \circ \mathbf{g}, \quad \mathbf{r} \circ \mathbf{w}, \quad \mathbf{g} \circ \mathbf{r}, \quad \mathbf{g} \circ \mathbf{w}, \quad \mathbf{w} \circ \mathbf{r}, \quad \mathbf{w} \circ \mathbf{g}$$

If a composition is well-defined, state its domain and codomain.

We have
$$\left\{ \begin{array}{l} \vec{r} : \mathbb{R} \to \mathbb{R}^3 \\ \vec{g} : \mathbb{R}^2 \to \mathbb{R}^3 \\ \vec{w} : \mathbb{R}^3 \to \mathbb{R}^2 \end{array} \right.$$

$\vec{r} \circ \vec{g} : \mathbb{R}^2 \xrightarrow{g} \mathbb{R}^3 \quad \mathbb{R} \xrightarrow{\vec{r}} \mathbb{R}^3$ ✗

$\vec{r} \circ \vec{w} : \mathbb{R}^3 \xrightarrow{w} \mathbb{R}^2 \quad \mathbb{R} \to \mathbb{R}^3$ ✗

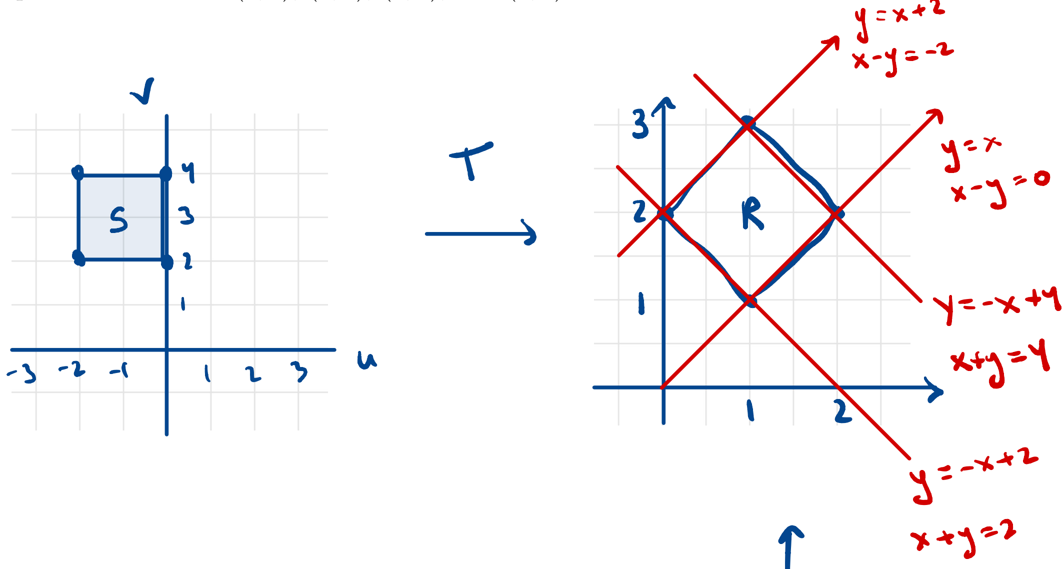
$\vec{g} \circ \vec{r} : \mathbb{R} \xrightarrow{r} \mathbb{R}^3 \quad \mathbb{R}^2 \xrightarrow{g} \mathbb{R}^3$ ✗

$\vec{g} \circ \vec{w} : \mathbb{R}^3 \xrightarrow{w} \mathbb{R}^2 \xrightarrow{g} \mathbb{R}^3$ ✓

$\vec{w} \circ \vec{r} : \mathbb{R} \xrightarrow{r} \mathbb{R}^3 \xrightarrow{w} \mathbb{R}^2$ ✓

$\vec{w} \circ \vec{g} : \mathbb{R}^2 \xrightarrow{g} \mathbb{R}^3 \xrightarrow{w} \mathbb{R}^2$ ✓

(12.3) Use the transformation $u = x - y$ and $v = x + y$ to evaluate $\iint_R \frac{x-y}{x+y}\, dA$ where R is the square with vertices $(0, 2)$, $(1, 1)$, $(2, 2)$, and $(1, 3)$.



The sides are

$x+y=2$, $x+y=4$

$x-y=0$, $x+y=-2$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= 2$$

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}$$

$$\iint_R \frac{x-y}{x+y}\,dA = \iint_S \frac{u}{v}\cdot\left|\frac{1}{2}\right| dA$$

$$= \int_2^4 \int_{-2}^0 \frac{u}{v}\cdot\frac{1}{2}\,du\,dv$$

$$= \frac{1}{2}\int_2^4 \left[\frac{1}{v}\cdot\frac{1}{2}u^2\right]_{u=-2}^{u=0} dv$$

$$= \frac{1}{2}\int_2^4 -\frac{1}{v}\cdot 2\,dv$$

$$= -\int_2^4 \frac{1}{v}\,dv$$

$$= -\left[\ln v\right]_{v=2}^{v=4}$$

$$= -\left[\ln 4 - \ln 2\right]$$

$$= -\left[2\ln 2 - \ln 2\right]$$

$$= -\ln 2$$

(12.4) Evaluate $\iiint_B (x^2+y^2+z^2)^2\,dV$, where B is the ball with center the origin and radius 5.

$$\iiint_B (\rho^2)^2 \cdot \rho^2 \sin\phi \; d\rho \, d\theta \, d\phi$$

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{5} \rho^6 \sin\phi \; d\rho \, d\theta \, d\phi$$

$$\int_0^{\pi} \sin\phi \, d\phi \cdot \int_0^{2\pi} d\theta \cdot \int_0^{5} \rho^6 \, d\rho$$

$$\left[-\cos\phi\right]_0^{\pi} \cdot \left[\theta\right]_0^{2\pi} \cdot \left[\frac{1}{7}\cdot\rho^7\right]_0^{5}$$

$$2 \cdot 2\pi \cdot \frac{1}{7} \cdot 5^7$$

$$\frac{312500}{7}\pi$$

this should say $0 \le t \le 1$

(13.1) Show that $\mathbf{F}(x,y) = \langle 4x^3y^2 - 2xy^3, 2x^4y - 3x^2y^2 + 4y^3 \rangle$ is a conservative vector field and use this fact to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the curve $\mathbf{r}(t) = \langle t + \sin \pi t, 2t + \cos \pi t \rangle$.

$$\frac{\partial Q}{\partial x} = 8x^3y - 6xy^2 \qquad \frac{\partial P}{\partial y} = 8x^3y - 6xy^2$$

$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, so $\vec{F} = \langle P, Q \rangle$ is conservative.

$\vec{r}(1) = \langle 1, 1 \rangle$, $\vec{r}(0) = \langle 0, 1 \rangle$

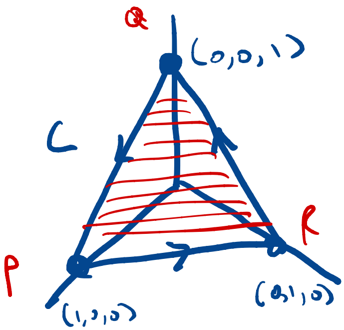
Could find potential function, but can also change the path: let C be $\vec{r}(t) = \langle t, 1 \rangle$ for $0 \le t \le 1$ instead. So $\vec{r}\,'(t) = \langle 1, 0 \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^1 \langle 4t^3 - 2t, 2t^4 - 3t^2 + 4 \rangle \cdot \langle 1, 0 \rangle \, dt$$

$$= \int_0^1 4t^3 - 2t \, dt = \left[t^4 - t^2 \right]_{t=0}^{t=1} = 0$$

(13.5) Use Stokes' Theorem to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = \langle xy, yz, zx\rangle$, and C is the triangle with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, oriented counterclockwise as viewed from above.



$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot dS$$

What is S? S is part of the plane passing through (1,0,0), (0,1,0) and (0,0,1). Find this plane:

$\vec{PQ} = \langle -1, 0, 1\rangle$, $\vec{PR} = \langle -1, 1, 0\rangle$

$$\vec{PQ}\times\vec{PR} = \begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{vmatrix} = \langle -1, -1, -1\rangle$$

We want positive normal to this plane, so $\vec{n} = \langle 1, 1, 1\rangle$

the plane has equation: $1(x-0) + 1(y-0) + 1(z-1) = 0$

$$x + y + z = 1$$

So a parametrization for S is

$$\vec{r}(x,y) = \langle x, y, 1-x-y\rangle \qquad \begin{matrix} 0 \le y \le 1-x \\ 0 \le x \le 1 \end{matrix}$$

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & yz & zx \end{vmatrix}$$

$$= \langle -y, -z, -x \rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot d\vec{S}$$

$$= \iint_D \langle -y, -z, -x \rangle \cdot \langle 1, 1, 1 \rangle \, dA$$

$$1 - y - x$$

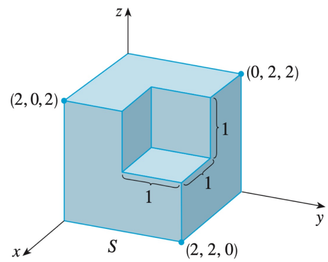
$$= \iint_D -y - z - x \quad dy\,dx$$

$$= \iint_D -y - (1 - y - x) - x \; dy\,dx$$

$$= \iint_D -1 \; dy\,dx$$

$$= -\text{Area}(D) = -\frac{1}{2} \cdot b \cdot h = -\frac{1}{2} \cdot 1 \cdot 1 = -\frac{1}{2}$$

(13.6) Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$ and S is the outwardly oriented surface shown in the figure (the boundary surface of a cube with a unit corner cube removed).



By the divergence theorem,

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \; dV$$

$$= \iiint_E 3 \; dV$$

$$= 3 \cdot \iiint_E 1 \; dV$$

$$= 3 \cdot \operatorname{Vol}(E)$$

$$= 3 \cdot (8-1)$$

$$= 21$$