Chapter 9

9.1 Three-Dimensional Coordinate Systems

- Working in three dimensions:
  - How do we represent points in space?
  - What is the right-hand rule?
  - What are the x-, y-, and z-coordinate planes? How can a point $P(a, b, c)$ be projected to each of the coordinate planes?
  - What is $\mathbb{R}^3$?
- Identifying basic surfaces in $\mathbb{R}^3$:
  - What surface in $\mathbb{R}^3$ is represented by the equation $z = 3$?
  - What surface in $\mathbb{R}^3$ is represented by the equation $y = 5$?
  - Why do we have to be careful about the context of our equations?
  - What does the equation $x^2 + y^2 = 1$ represent as a surface in $\mathbb{R}^3$?
  - Describe and sketch the surface in $\mathbb{R}^3$ represented by the equation $y = x$.
- Distances in three dimensions:
  - What is the distance formula in three dimensions?
  - Find the distance from the point $P(2, -1, 7)$ to the point $Q(1, -3, 5)$.
- Spheres:
  - Find an equation of a sphere with radius r and center $C(h, k.l)$.
  - Show that $x^2 + y^2 + z^2 + 4x - 6y + 2x + 6 = 0$ is the equation of a sphere, and find its center and radius.
  - What region in $\mathbb{R}^3$ is represented by the following inequalities?

$$1 \leq x^2 + y^2 + z^2 \leq 4 \qquad z \leq 0$$

9.2 Vectors

- Introduction to vectors:
  - What is a vector? What are some examples of vectors?
  - What are the various ways to represent vectors?
  - Suppose a particle moves along a line segment from point A to point B. What is the displacement vector from A to B?
- Vector operations:
  - How do we add vectors? How to draw the sum of vectors?
  - Given vectors $\vec{u}$ and $\vec{v}$, why is $\vec{u}+\vec{v}=\vec{v}+\vec{u}$?
  - How do we scale vectors?
  - How do we subtract vectors? How to draw the difference of vectors?
- Coordinate representation of vectors:
  - How to represent vectors using coordinates?
- Vectors between two points:
  - Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, what is the vector $\vec{AB}$?
  - Find the vector represented by the directed line segment with initial point $A(2, -3, 4)$ and terminal point $B(-2, 1, 1)$.
- Lengths of vectors:
  - How to compute the magnitude of a vector $\vec{v}$?
- Vector operations using components:
  - How do we add vectors using coordinates?
  - How do we scale vectors using coordinates?
  - If $\vec{a}=\langle 4, 0, 3\rangle$ and $\vec{b}=\langle -2, 1, 5\rangle$, find $|\vec{a}|$ and the vectors $\vec{a}+\vec{b}, \vec{a}-\vec{b}, 3\vec{b}$ and $2\vec{a}+5\vec{b}$.
- Properties of vector operations:
  - Let $\vec{a}, \vec{b}$, and $\vec{c}$ are vectors in V_n and c and d are scalars. Summarize the properties of these vectors.
- Unit vectors and $\vec{i}$, $\vec{j}$, and $\vec{k}$:
  - What are the vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$?
  - Show that any vector in V_3 can be expressed in terms of the vectors $\vec{i}$, $\vec{j}$, $\vec{k}$.
  - If $\vec{a}=\vec{i}+2\vec{j}-3\vec{k}$ and $\vec{b}=4\vec{i}+7\vec{k}$, express the vector $2\vec{a}+3\vec{b}$ in terms of $\vec{i}, \vec{j}$, and $\vec{k}$.
  - What is a unit vector? If $\vec{a}\neq 0$, what is a unit vector that has the same direction as $\vec{a}$?
  - Find the unit vector in the direction of the vector $2\vec{i}-\vec{j}-2\vec{k}$.
  - Know how to find vector sums and magnitudes in various contexts (e.g. force diagrams).

9.3 The Dot Product

- Work and the dot product:
  - What is the dot product of two nonzero vectors $\vec{a}$ and $\vec{b}$?
  - What is the work done by a constant force F in moving an object through a distance d?
  - How can we reinterpret work in terms of the dot product?
  - Find $\vec{a} \cdot \vec{b}$ if $\vec{a}$ and $\vec{b}$ have lengths 4 and 6, and the angle between them is $\pi/3$.
  - A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of $35°$ above the horizontal. Find the work done by the force.
- Orthogonal vectors:
  - What does it mean for two vectors to be orthogonal?
  - Prove that two vectors $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\vec{a} \cdot \vec{b} = 0$.
- The dot product in component form:
  - What is the dot product in terms of components?
  - Compute the following dot products:
    1. $\langle 2, 4\rangle \cdot \langle 3, -1\rangle$
    2. $\langle -1, 7, 4\rangle \cdot \langle 6, 2, -\frac{1}{2}\rangle$
    3. $(\vec{i} + 2\vec{j} - 3\vec{k}) \cdot (2\vec{j} - \vec{k})$
  - Show that $2\vec{i} + 2\vec{j} - \vec{k}$ is perpendicular to $5\vec{i} - 4\vec{j} + 2\vec{k}$.
  - Find the angle between the vectors $\vec{a} = \langle 2, 2, -1\rangle$ and $\vec{b} = \langle 5, -3, 2\rangle$.
  - A force is given by a vector $F = 3\vec{i} + 4\vec{j} + 5\vec{k}$ and moves a particle from the point $P(2, 1, 0)$ to the point $Q(4, 6, 2)$. Find the work done.
- Properties of the dot product:
  - If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors in V_3 and c is a scalar, what are $\vec{a} \cdot \vec{a}$, $\vec{a} \cdot \vec{b}$, $\vec{a} \cdot (\vec{b} + \vec{c})$, $(c\vec{a}) \cdot \vec{b}$, and $\vec{0} \cdot \vec{a}$?
- Projections
  - What is the vector projection of $\vec{b}$ onto $\vec{a}$
  - What is the scalar projection of $\vec{b}$ onto $\vec{a}$
  - Find the scalar projection and vector projection of $\vec{b} = \langle 1, 1, 2\rangle$ onto $\vec{a} = \langle -2, 3, 1\rangle$.

9.4 The Cross Product

- Torque and the cross product:
  - What is torque?
  - If $\vec{a}$ and $\vec{b}$ are nonzero three-dimensional vectors, what is the cross product of $\vec{a}$ and $\vec{b}$?
  - Show that two nonvectors $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} \times \vec{b} = 0$.
  - A bolt is tightened by applying a 40-N force to a 0.25-m wrench, as shown below. Find the magnitude of the torque about the center of the bolt.
- Cross products of $\vec{i}, \vec{j}, \vec{k}$:
  - Find $\vec{i} \times \vec{j}$ and $\vec{j} \times \vec{i}$.
  - Summarize the cross products of $\vec{i}$, $\vec{j}$, and $\vec{k}$.
- Properties of the cross product:
  - Is $\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$?
  - Is $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$?
  - What properties of the cross product hold?
- The cross product using components:
  - How to calculate $\vec{a} \times \vec{b}$ using components?
  - How can we use determinants to simplify the cross product calculation?
  - Use determinants to calculate the cross product of $\vec{a} = \langle 1, 3, 4 \rangle$ and $\vec{b} = \langle 2, 7, -5 \rangle$.
  - Find a vector perpendicular to the plane that passes through the points $P(1, 4, 6)$, $Q(-2, 5, -1)$, $R(1, -1, 1)$.
- Geometry of the cross product:
  - Give a geometric interpretation of the length of the cross product.
  - Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.
- Scalar triple product:
  - What is the scalar triple product of the vectors $\vec{a}, \vec{b}$, and $\vec{c}$?
  - Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$, and $\vec{c} = \langle c_1, c_2, c_3 \rangle$. Show that

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

  - Use the scalar triple product to show that the vectors $\vec{a} = \langle 1, 4, -7 \rangle$, $\vec{b} = \langle 2, -1, 4 \rangle$, and $\vec{c} = \langle 0, -9, 18 \rangle$ lie in the same plane.

9.5 Equations of Lines and Planes

- Lines:
  - How do we describe a line in three-dimensional space?
  - How can we describe a line in three-dimensional space parametrically?
  - Are the vector equation and parametric equations of a line unique?
  - What are the symmetric equations a line L?
  - Find a vector equation and parametric equations for the line that passes through the point (5,1,3) and is parallel to the vector $\vec{i} + 4\vec{j} - 2\vec{k}$. Find two other points on the line.
  - Find parametric equations and symmetric equations of the line that passes through the points $A(2,4,-3)$ and $B(3,-1,1)$. At what point does this line intersect the xy-plane?
- Line Segments:
  - How to describe the line segment AB between the points $A(2,4,-3)$ and $B(3,-1,1)$?
  - How to describe the line segment from vector $\vec{r_0}$ to $\vec{r_1}$?
- Planes:
  - How to describe a plane in space?
  - How to describe a plane using a scalar equation?
  - Find an equation of the plane through the point $(2,4,-1)$ with normal vector $\vec{n} = \langle 2,3,4\rangle$. Find the intercepts and sketch the plane.
  - How to rewrite the equation of a plane using a linear equation?
  - Find an equation of the plane that passes through the points $P(1,3,2)$, $Q(3,-1,6)$ and $R(5,2,0)$.
  - Find the point at which the line with parametric equations $x = 2+3t$, $y = -4t$, $z = 5+t$ intersects the plane $4x + 5y - 2z = 18$.
- Angle between planes:
  - How can we determine if two planes are parallel?
  - If two planes are not parallel, what is the angle between the two planes?
  - Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$. Find symmetric equations for the line of intersection L of these two planes.
- Distances:
  - Find a formula for the distance from a point $P_1(x_1, y_1, z_1)$ to the plane $ax+by+cz+d = 0$.
  - Find the distance between the parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.
- Skew lines:
  - Show that the lines L_1 and L_2 with parametric equations

    $$\begin{array}{lll} x = 1+t & y = -2+3t & z = 4-t \\ x = 2s & y = 3+s & z = -3+4s \end{array}$$

    are skew lines.
  - Find the distance betweeen the skew lines L_1 and L_2 above.

9.6 Functions and Surfaces

- Functions of two variables:
  - What is a function of two variables?
  - If $f(x, y) = 4x^2 + y^2$, what are the domain and range of $f(x, y)$?
  - If $f(x, y) = \dfrac{\sqrt{x + y + 1}}{x - 1}$, evaluate $f(3, 2)$ and find and sketch the domain.
  - If $f(x, y) = x \ln(y^2 - x)$, evaluate $f(3, 2)$ and find and sketch the domain.
- Graphs of functions:
  - If f is a function of two variables with domain D, what is the graph of f?
  - Sketch the graph of the function $f(x, y) = 6 - 3x - 2y$.
  - Sketch the graph of the function $f(x, y) = x^2$.
  - How can we use cross-sections to help us sketch graphs of functions of two variables?
  - Use traces to sketch the graph of the function $f(x, y) = 4x^2 + y^2$.
  - Sketch the graph of $f(x, y) = y^2 - x^2$.
- Quadric surfaces:
  - What is a quadric surface?
  - Sketch the quadric surface with equation $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$.
  - Is the ellipsoid $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$ the graph of a function?
  - What are the six basic types of quadric surfaces? What are their equations in standard form?
  - Classify the quadric surface $x^2 + 2z^2 - 6x - y + 10 = 0$.

9.7 Cylindrical and Spherical Coordinates

- Cylindrical coordinates:
  - How do we convert between polar and Cartesian coordinates in two dimensions?
  - What is the cylindrical coordinate system?
  - Plot the point with cylindrical coordinates $(2, 2\pi/3, 1)$ and find its rectangular coordinates.
  - Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.
  - When are cylindrical coordinates useful? What is the equation of a cylinder in cylindrical coordinates?
  - Describe the surface whose equation in cylindrical coordinates is $z = r$.
  - Find an equation in cylindrical coordinates for the ellipsoid $4x^2 + 4y^2 + z^2 = 1$.
- Spherical coordinates:
  - What is the spherical coordinate system?
  - When are spherical coordinates useful? What is the equation of a sphere in spherical coordinates?
  - Find equations for surfaces using spherical coordinates.
  - What is the relationship between rectangular and spherical coordinates?
  - The point $(2, \pi/4, \pi/3)$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.
  - The point $(0, 2\sqrt{3}, -2)$ is given in rectangular coordinates. Find spherical coordinates for this point.
  - Find an equation in spherical coordinates for the hyperboloid of two sheets with equation $x^2 - y^2 - z^2 = 1$.
  - Find a rectangular equation for the surface whose spherical equation is $\rho = \sin\theta \sin\phi$.