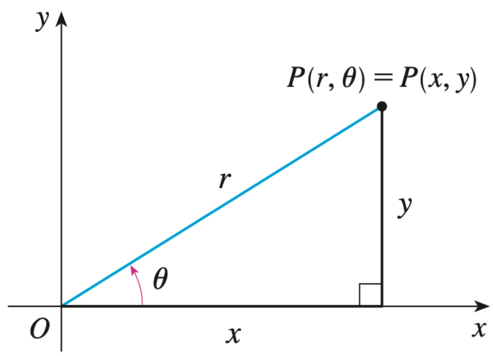
Lecture Notes
Math 2400 - Calculus III
Spring 2024

Name: Champ

9.7 Cylindrical and Spherical Coordinates

Question. In two dimensions, how do we convert between polar and Cartesian coordinates?



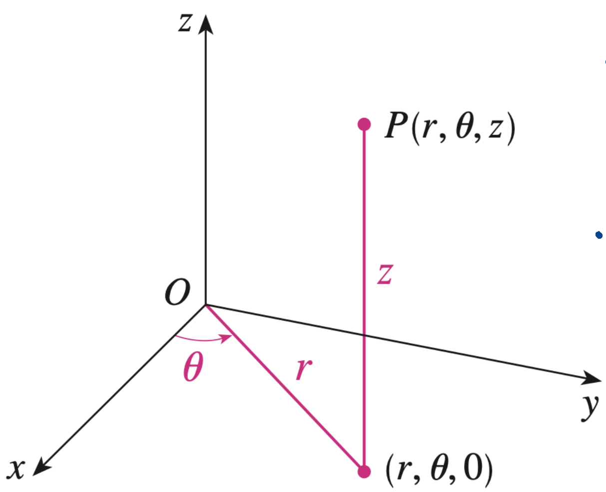
Given (r, θ)

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Given (x, y)

$$r^2 = x^2 + y^2$$
$$\tan\theta = \frac{y}{x}$$

Definition. What is the cylindrical coordinate system?



- A point P in space is a triple (r, θ, z)
- Use polar coordinates in the xy-plane and then z is the height.

Given (r, θ, z)

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Given (x, y, z)

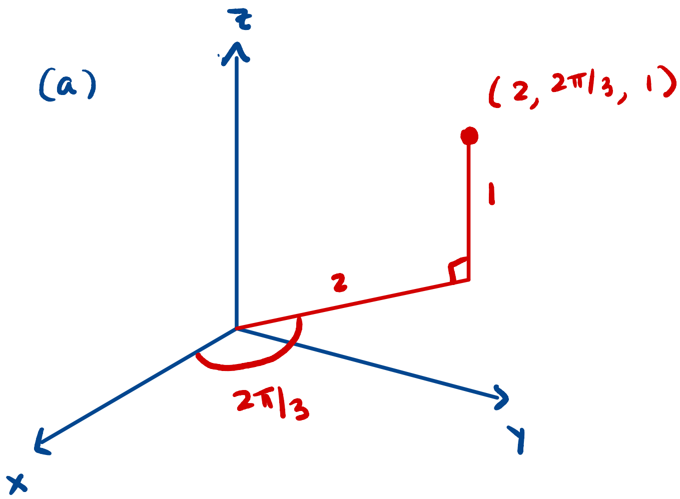
$$r^2 = x^2 + y^2$$
$$\tan\theta = \frac{y}{x}$$
$$z = z$$

Example.

(a) Plot the point with cylindrical coordinates $(\underset{r}{2}, \underset{\theta}{2\pi/3}, \underset{z}{1})$ and find its rectangular coordinates.

(b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

(a)



To convert to rectangular coordinates

$$x = 2\cos(2\pi/3) = -1$$

$$y = 2\sin(2\pi/3) = \sqrt{3}$$

$$z = 1$$

$$\boxed{(-1, \sqrt{3}, 1)}$$

(b) Given $(3, -3, -7)$

$$r = \sqrt{(3)^2 + (-3)^2} = 3\sqrt{2}$$

$$\tan\theta = \frac{-3}{3} = -1 \Rightarrow \theta = \frac{7\pi}{4} + 2\pi n$$

can choose any n. Let's pick $n = 0$

$$\cancel{\theta = \frac{3\pi}{4} + 2\pi n}$$

our point is in the 4^{th} quadrant

$$z = -7$$

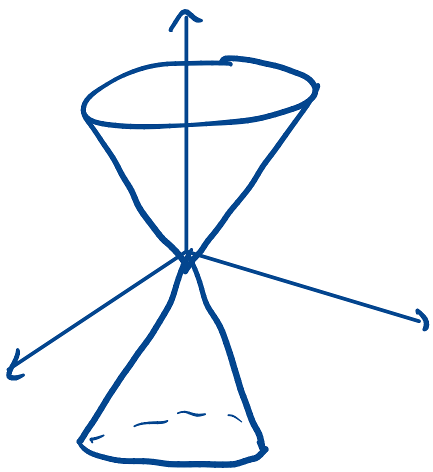
$$\boxed{(3\sqrt{2}, 7\pi/4, -7)}$$

Remark. When are cylindrical coordinates useful? What is the equation of a cylinder in cylindrical coordinates?

- Useful in problems that involve symmetry about an axis
- Cylinder in rectangular coords: $x^2 + y^2 = c^2$
- Cylinder in cylindrical coords: $r = c$

Example. Describe the surface whose equation in cylindrical coordinates is $z = r$

- Consider the horizontal trace $z = k$
- It is a circle of radius k since $r = z = k$ and θ can be any angle $0 \le \theta \le 2\pi$
- We obtain a cone (note: r can be negative)



- Can also see this using rectangular coords:

$$z^2 = r^2 = x^2 + y^2$$

$$z^2 = x^2 + y^2$$

cone. See §9.6

Example. Find an equation in cylindrical coordinates for the ellipsoid $4x^2 + 4y^2 + z^2 = 1$.

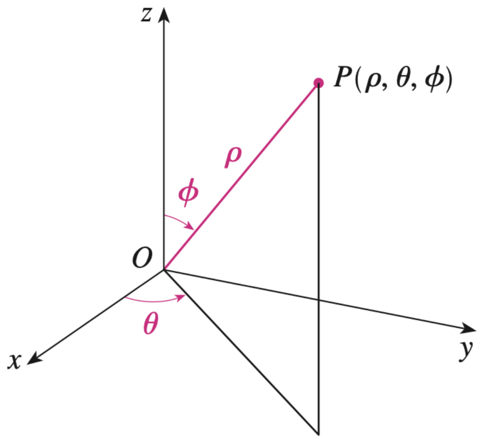
Since $r^2 = x^2 + y^2$,

$$z^2 = 1 - 4x^2 - 4y^2$$

$$z^2 = 1 - 4(x^2 + y^2)$$

$$z^2 = 1 - 4r^2$$

Definition. What is the spherical coordinate system?

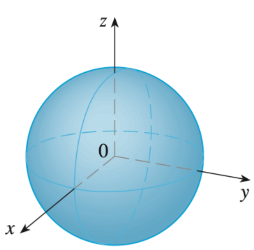


- A point P in space is (ρ, θ, ϕ)
- ρ: distance from origin
- θ: same as cylindrical coords
- ϕ: the angle from positive z-axis to $\overrightarrow{OP}$
- $\rho \geq 0$ and $0 \leq \phi \leq \pi$

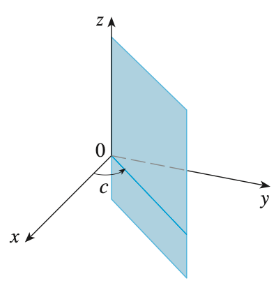
Remark. When are spherical coordinates useful? What is the equation of a sphere in spherical coordinates?

- Useful in problems involving symmetry around a point
- Sphere in rectangular coords: $x^2 + y^2 + z^2 = c^2$
- Sphere in spherical coords: $\rho = c$

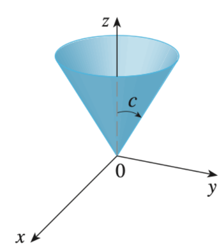
Example. Find an equation for each of the following surfaces using spherical coordinates.



$\rho = c$

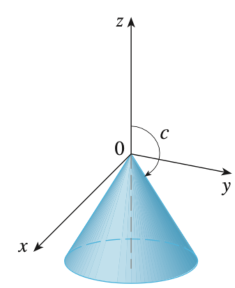


$\theta = c$



$0 < c < \pi/2$

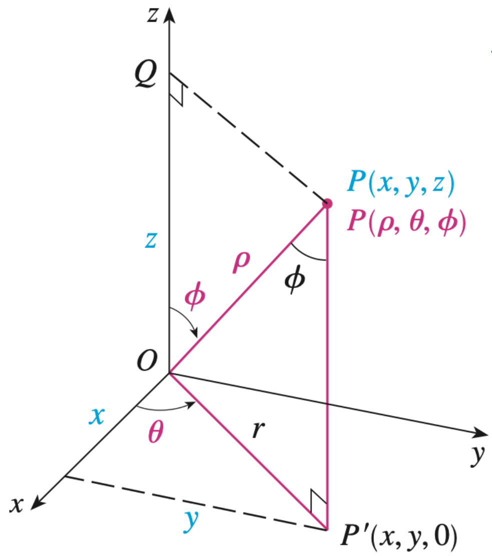
$\phi = c$



$\pi/2 < c < \pi$

$\phi = c$

Question. What is the relationship between rectangular and spherical coordinates?



- From ΔOPQ, know $\Delta OPP'$
- From $\Delta OPP'$, know

$$z = \rho\cos\phi$$
$$r = \rho\sin\phi$$

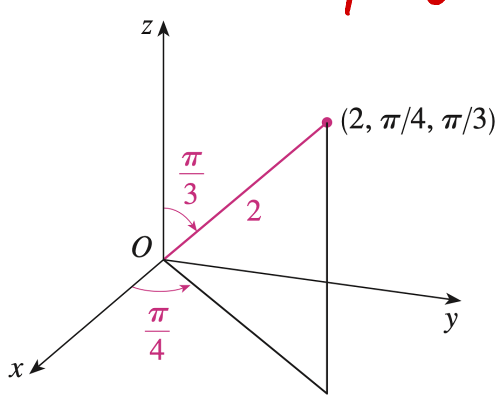
- since $x = r\cos\theta$ and $y = r\sin\theta$

$$z = \rho\cos\phi$$
$$x = \underbrace{\boxed{\rho\sin\phi}}_{r}\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$

- Also, $x^2 + y^2 + z^2 = \rho^2$

Example. The point $(2, \pi/4, \pi/3)$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

$\rho \quad \theta \quad \phi$



$$x = \rho\sin\phi\cos\theta$$
$$= 2\sin(\pi/3)\cos(\pi/4) = \sqrt{3/2}$$

$$y = \rho\sin\phi\sin\theta$$
$$= 2\sin(\pi/3)\sin(\pi/4) = \sqrt{3/2}$$

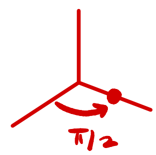
$$z = \rho\cos\phi = 2\cos(\pi/3) = 1$$

$$(2, \pi/4, \pi/3) \longleftrightarrow \left(\sqrt{3/2}, \sqrt{3/2}, 1\right)$$

Example. The point $(0, 2\sqrt{3}, -2)$ is given in rectangular coordinates. Find spherical coordinates for this point.

- We need (ρ, θ, ϕ)
- $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0 + 12 + 4} = 4$
- From $z = \rho \cos\phi$, $\quad \cos\phi = \frac{z}{\rho} = \frac{-2}{4} = -\frac{1}{2} \Rightarrow \phi = \frac{2\pi}{3}$
- From $x = \rho \sin\phi \cos\theta$, $\quad \cos\theta = \frac{x}{\rho \sin\phi} = 0 \Rightarrow \theta = \pi/2$

$\cancel{\theta = 3\pi/2}$

(θ is not $\frac{3\pi}{2}$ since our point is here: )

- Answer: $(4, \pi/2, 2\pi/3)$

Example. Find an equation in spherical coordinates for the hyperboloid of two sheets with equation $x^2 - y^2 - z^2 = 1$.

$$(\rho \sin\phi \cos\theta)^2 - (\rho \sin\phi \sin\theta)^2 - (\rho \cos\phi)^2 = 1$$

$$\rho^2 \left[\sin^2\phi \cos^2\theta - \sin^2\phi \sin^2\theta - \cos^2\phi \right] = 1$$

$$\rho^2 \left[\sin^2\phi \left(\cos^2\theta - \sin^2\theta\right) - \cos^2\phi \right] = 1$$

Example. Find a rectangular equation for the surface whose spherical equation is $\rho = \sin\theta \sin\phi$.

$$\rho = \sin\theta \sin\phi$$

$$\Leftrightarrow \quad \rho^2 = \rho \sin\theta \sin\phi$$

$$\Leftrightarrow \quad x^2 + y^2 + z^2 = y$$

$$\Leftrightarrow \quad x^2 + y^2 - y + \frac{1}{4} + z^2 = \frac{1}{4}$$

$$\Leftrightarrow \quad x^2 + \left(y - \frac{1}{2}\right)^2 + z^2 = \frac{1}{4}$$

This is a sphere with center $(0, \frac{1}{2}, 0)$ and radius $\frac{1}{2}$.