Lecture Notes
Math 2400 - Calculus III
Spring 2024

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9.6 Functions and Surfaces

$f(x,y)$

Definition. What is a function of two variables?

A rule that assigns to each ordered pair (x,y) in a set D a unique real number $f(x,y)$

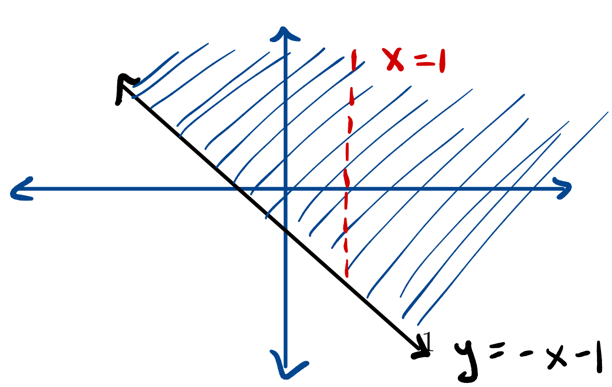
Example. If $f(x,y) = 4x^2 + y^2$, what are the domain and range of $f(x,y)$?

- $f(x,y)$ is defined for all possible x and y
- Domain: the whole xy-plane
- Range: since $x^2 \geq 0$ and $y^2 \geq 0$ the range is $[0, \infty)$

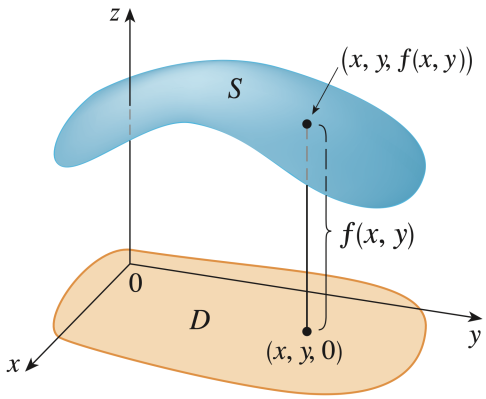
Example. If $f(x,y) = \dfrac{\sqrt{x+y+1}}{x-1}$, evaluate $f(3,2)$ and find and sketch the domain.

- We need $x+y+1 \geq 0$ and $x - 1 \neq 0$

$\Rightarrow D = \{ (x,y) \mid y \geq -x-1 \text{ and } x \neq 1 \}$



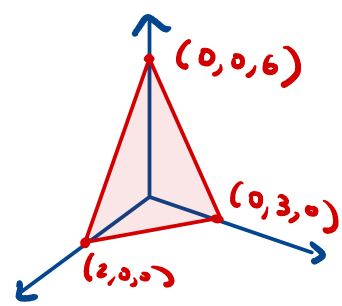
Definition. If f is a function of two variables with domain D, what is the graph of f?



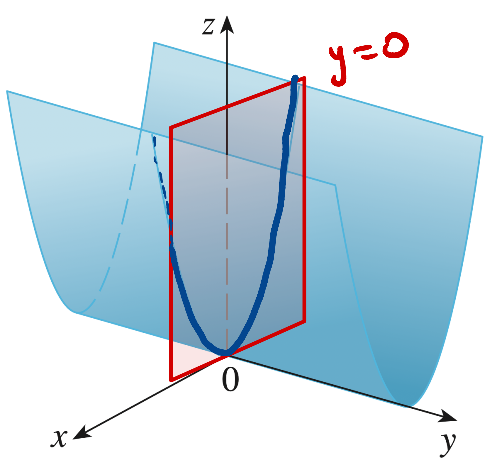
The graph of $f(x,y)$ is the set of all points (x,y,z) such that $z = f(x,y)$

Example. Sketch the graph of the function $f(x, y) = 6 - 3x - 2y$.

- The graph has equation $z = 6 - 3x - 2y$
- This is a plane. Intercepts?
  - If $y = z = 0$, $x = 2$
  - If $x = z = 0$, $y = 3$
  - If $x = y = 0$, $z = 6$



Example. Sketch the graph of the function $f(x, y) = x^2$.



- The graph has equation $z = x^2$
- This equation doesn't depend on y
- Each plane $y = k$ has a copy of the $z = x^2$ parabola

traces

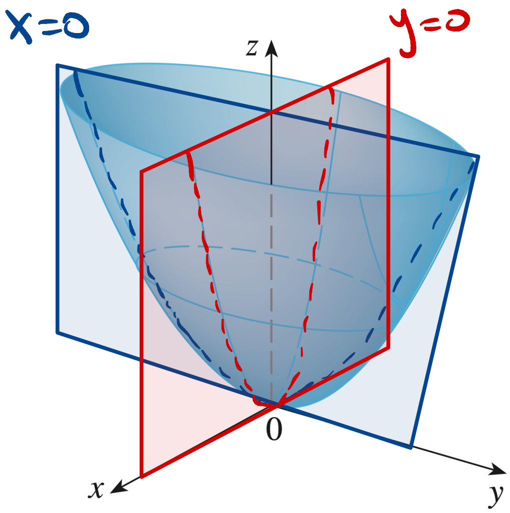
Remark. How can we use cross-sections to help us sketch graphs of functions of two variables?

- It helps to keep one of the variables fixed to graph "slices" of the function
- Set $x=k$ or $y=k$ to see vertical slices
- Set $z=k$ to see horizontal slices

(k is some number)

$z = 4x^2+y^2$

Example. Use traces to sketch the graph of the function $f(x, y) = 4x^2 + y^2$.



- If $x=0$, $z=y^2$
- If $x=k$, $z=y^2+4k^2$

$\Rightarrow$ We get a family of parabolas as we move the $x=0$ plane along the x-axis

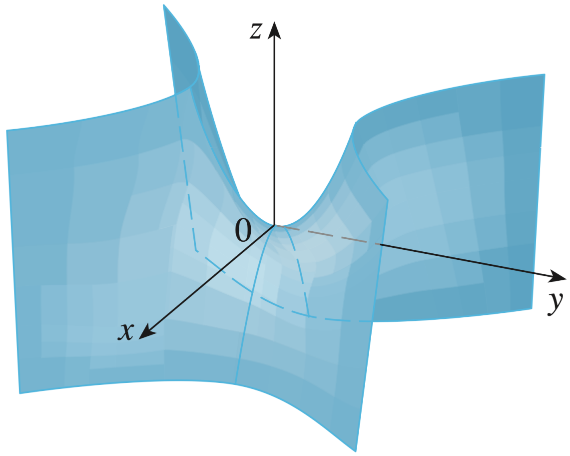
- If $y=k$, $z=4x^2+k^2$

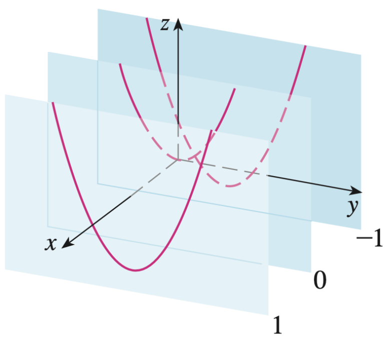
$\Rightarrow$ we get a family of parabolas as we move the $y=0$ plane along the y-axis

- If $z=k$, then $4x^2+y^2=k$

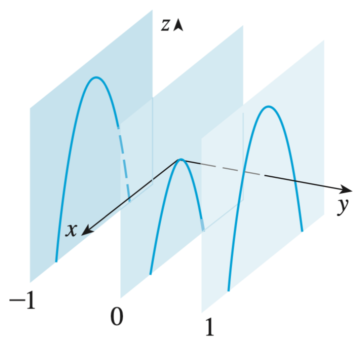
$\Rightarrow$ We get a family of ellipses as we move the $z=0$ plane through the z-axis

Example. Sketch the graph of $f(x, y) = y^2 - x^2$. $\to z = y^2 - x^2$

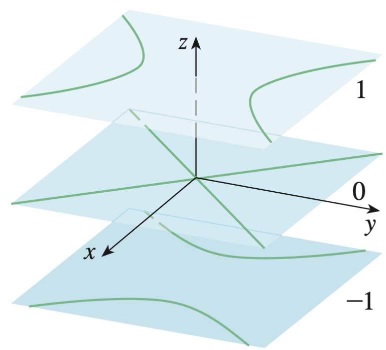




Traces in $x = k$



Traces in $y = k$



Traces in $z = k$

$z = y^2 - k^2$

(upward parabolas)

$z = -x^2 + k^2$

(downward parabolas)

$y^2 - x^2 = k$

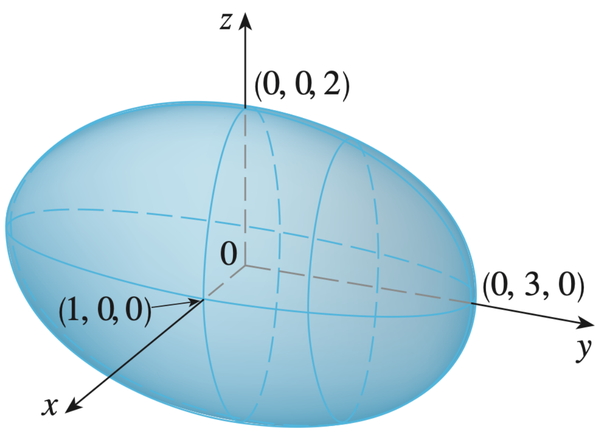
(hyperbolas)

Definition. What is a quadric surface?

The graph of a degree 2 equation in the variables x, y, z

The highest power of each variable is ≤ 2

Example. Sketch the quadric surface with equation $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$.

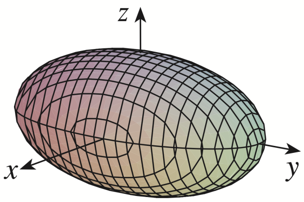
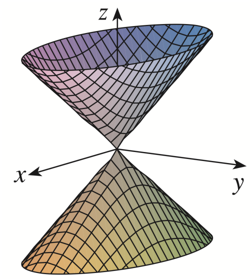
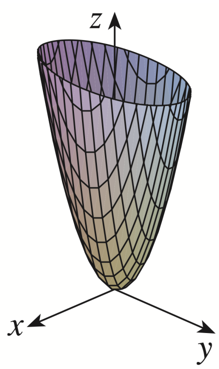
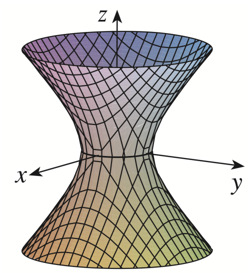
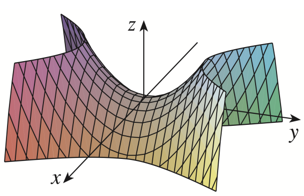
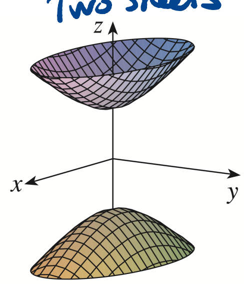


Question. Is the ellipsoid $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$ the graph of a function?

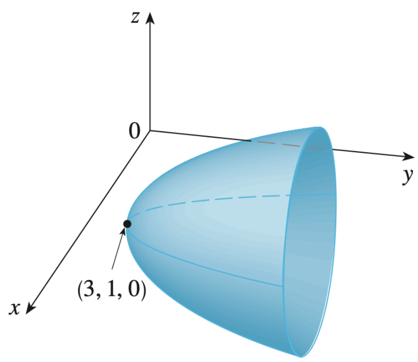
These surfaces are symmetric about the z-axis

If symmetric about a different axis, equation changes accordingly

Example. Below is a table of the six basic types of quadric surfaces in standard form.

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	Cone 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$
Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Variable raised to 1st power indicates the axis of symmetry	Hyperboloid of One Sheet 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Variable with negative coefficient indicates axis of symmetry.
Hyperbolic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ This is the case $c<0$.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ two minus signs indicates two sheets

Example. Classify the quadric surface $x^2 + 2z^2 - 6x - y + 10 = 0$.



Complete the square:

$$x^2 - 6x + 9 + 2z^2 - y + 10 = 9$$

$$(x-3)^2 + 2z^2 = y - 1$$

$\Rightarrow$ Elliptic paraboloid

$\Rightarrow$ The axis of symmetry is parallel to the y-axis

$\Rightarrow$ Vertex has been shifted from $(0,0,0)$ to $(3,1,0)$