Lecture Notes
Math 2400 - Calculus III
Spring 2024

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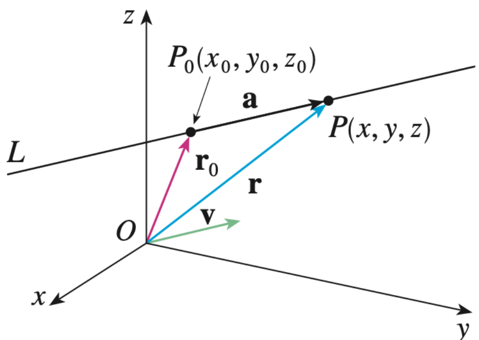
9.5 Equations of Lines and Planes

Question. How do we describe a line in three-dimensional space?



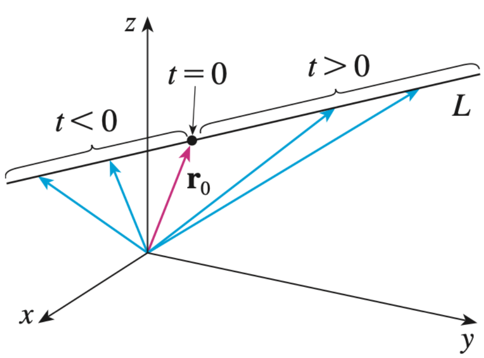
- Let P_0 be any point on L with position vector $\vec{r}_0$.
- Let P be any other point on L with position vector $\vec{r}$.

- Key observation: if $\vec{v}$ is a vector parallel to L then

Vector equation of L → $\boxed{\vec{r} = \vec{r}_0 + t\vec{v}}$ for some value t.

(as t varies, $\vec{r}_0 + t\vec{v}$ traces out the whole line)

Question. How can we describe a line in three-dimensional space parametrically?



In the above scenario, let

$$\vec{v} = \langle a, b, c \rangle$$
$$\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$$
$$\vec{r} = \langle x, y, z \rangle$$

The equation $\vec{r} = \vec{r}_0 + t\vec{v}$ becomes

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

parametric equations of L

$$\Rightarrow \boxed{x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct}$$

Example. Find a vector equation and parametric equations for the line that passes through the point (5,1,3) and is parallel to the vector $\vec{i} + 4\vec{j} - 2\vec{k}$.

- We need an equation $\vec{r} = \vec{r}_0 + t\vec{v}$
- $\vec{r}_0 = \langle 5, 1, 3 \rangle$ and $\vec{v} = \langle 1, 4, -2 \rangle$
- Vector equation: $\vec{r} = \langle 5,1,3 \rangle + t\langle 1,4,-2 \rangle$

  $= \langle 5+t, 1+4t, 3-2t \rangle$

- Parametric equations: $x = 5+t \quad y = 1+4t \quad z = 3-2t$

Question. Are the vector equation and parametric equations of a line unique?

No! We have many choices for $\vec{r}_0$ and also many choices for the parallel vector $\vec{v}$.

Definition. What are the symmetric equations a line L?

From the parametric equations,

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

We can solve for t in each case to obtain

$$\boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}}$$

Symmetric equations of L

(if, say, $a=0$ then it would be $x = x_0 \quad \frac{y-y_0}{b} = \frac{z-z_0}{c}$)

Example.

(a) Find parametric equations and symmetric equations of the line that passes through the points $A(2,4,-3)$ and $B(3,-1,1)$.

(b) At what point does this line intersect the xy-plane?

(a) · The parallel vector is $\vec{v} = \overrightarrow{AB} = \langle 1,-5,4\rangle$

· Since A is on the line, take $\vec{r}_0 = \langle 2,4,-3\rangle$

· Vector equation: $\vec{r} = \vec{r}_0 + t\vec{v}$

$$\langle x,y,z\rangle = \langle 2,4,-3\rangle + t\langle 1,-5,4\rangle$$

· Parametric Equations: $x = 2+t \quad y = 4-5t \quad z = -3+4t$

· Symmetric Equations: $x-2 = \dfrac{y-4}{-5} = \dfrac{z+3}{4}$

(b) L intersects the xy-plane when $z=0$. Hence $x-2 = \dfrac{y-4}{-5} = \dfrac{3}{4}$

Solving gives $x = \frac{11}{4}$ and $y = \frac{1}{4}$. $\boxed{\left(\frac{11}{4}, \frac{1}{4}, 0\right)}$

Example. Show that the lines L_1 and L_2 with parametric equations

$$\begin{array}{lll} x = 1+t & y = -2+3t & z = 4-t \\ x = 2s & y = 3+s & z = -3+4s \end{array}$$

are skew lines.

· We need to show these lines don't intersect and are not parallel

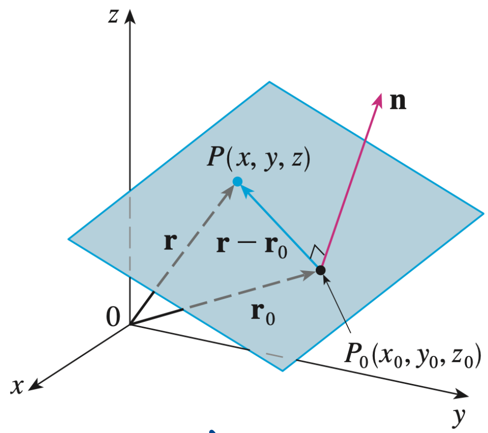
· The corresponding parallel vectors are $\langle 1,3,-1\rangle$ and $\langle 2,1,4\rangle$

· These vectors aren't parallel ✓

· To show no point of intersection, verify that the system

$$\left.\begin{aligned} 1+t &= 2s \\ -2+3t &= 3+s \\ 4-t &= -3+4s \end{aligned}\right\} \text{has no solution.}$$

Question. How to describe a plane in space?



- A plane is determined by a point P_0 and a vector $\vec{n}$ that is orthogonal to the plane. "normal vector"

- Our plane will be "all points P so that $\overrightarrow{P_0P}$ is perpendicular to $\vec{n}$"

- Since $\overrightarrow{P_0P}$ is the vector $\vec{r}-\vec{r}_0$, we want all $\vec{r}$ so that

$$\boxed{\vec{n}\cdot(\vec{r}-\vec{r}_0)=0}$$

Vector equation of the plane

Definition. How to describe a plane using a scalar equation?

If $\vec{n}=\langle a,b,c\rangle$ $\vec{r}=\langle x,y,z\rangle$ and $\vec{r}_0=\langle x_0,y_0,z_0\rangle$ the vector equation becomes

$$\boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0}$$

Scalar equation of a plane through the point (x_0,y_0,z_0) with normal vector $\langle a,b,c\rangle$

Example. Find an equation of the plane through the point $(2,4,-1)$ with normal vector $\vec{n}=\langle 2,3,4\rangle$.

From the above, we get

$$\boxed{2(x-2)+3(y-4)+4(z+1)=0}$$

Question. How to rewrite the equation of a plane using a linear equation?

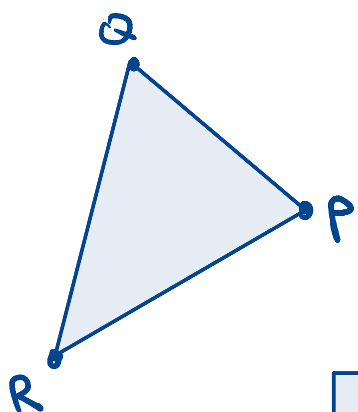
If we collect the terms in the scalar equation

$$\boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0}$$

we obtain an equation of the form

$$\boxed{ax+by+cz+d=0}$$ Linear equation

Example. Find an equation of the plane that passes through the points $P(1,3,2)$, $Q(3,-1,6)$ and $R(5,2,0)$.



- We need a point and a normal vector
- Point: $(1,3,2)$
- Normal vector: $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$

$$= \begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = \langle 12, 20, 14 \rangle$$

$$\boxed{12(x-1)+20(y-3)+14(z-2)=0}$$

Example. Find the point at which the line with parametric equations $x = 2+3t$, $y = -4t$, $z = 5+t$ intersects the plane $4x+5y-2z = 18$.

Solve for t so that

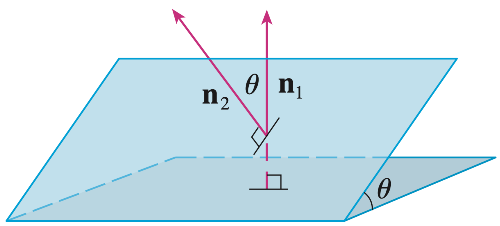
$$4(2+3t)+5(-4t)-2(5+t)=18$$

$$\iff -10t = 20$$

$$\iff t = -2$$

The point is $\boxed{(-4,8,3)}$

Definition. How can we determine if two planes are parallel? If two planes are not parallel, what is the angle between the two planes?

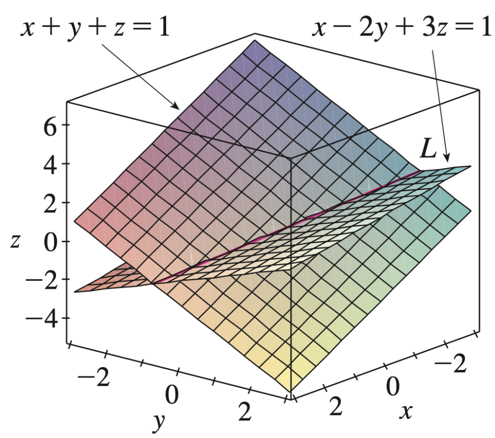


- Two planes are parallel if their normal vectors are parallel
- The angle between two planes is the acute angle between their normal vectors.

Example.

(a) Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.

(b) Find symmetric equations for the line of intersection L of these two planes.



(a) $\vec{n}_1 = \langle 1, 1, 1 \rangle$

$\vec{n}_2 = \langle 1, -2, 3 \rangle$

$$\theta = \cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right) = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right) \approx 72^\circ$$

(b) We need a point and a parallel vector

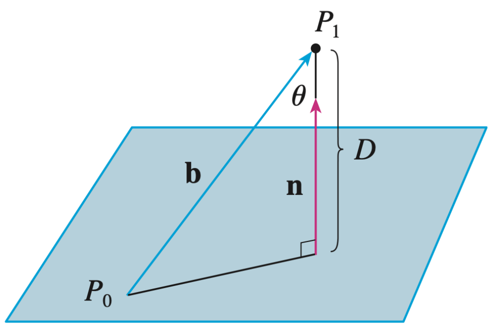
- Find where L intersects the xy-plane
- Setting $z=0$, $x+y=1$ and $x-2y=1 \Rightarrow x=1$ and $y=0$
- $P = (1, 0, 0)$

Since L is in both planes, L is perpendicular to both $\vec{n}_1$ and $\vec{n}_2$

$\Rightarrow$ a parallel vector is $\vec{v} = \vec{n}_1 \times \vec{n}_2 = \langle 5, -2, -3 \rangle$

We obtain: $\boxed{\dfrac{x-1}{5} = \dfrac{y}{-2} = \dfrac{z}{-3}}$

Example. Find a formula for the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.



- Let $P_0(x_0, y_0, z_0)$ be any point in the plane
- If $\vec{b} = \overrightarrow{P_0P_1}$, then

$$D = \left| \operatorname{comp}_{\vec{n}} \vec{b} \right| = \frac{|\vec{n} \cdot \vec{b}|}{|\vec{n}|}$$

absolute value of the scalar projection of $\vec{b}$ onto $\vec{n}$.

$$\boxed{D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}}$$

Example. Find the distance between the parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.

- Choose any point on one plane and calculate its distance to the other plane
- On plane 1, if $y = 0$ and $z = 0$, then $10x = 5$
- Conclude that $(\frac{1}{2}, 0, 0)$ is on the plane
- $D = \dfrac{|5(\frac{1}{2}) + 1(0) - 1(0) - 1|}{\sqrt{5^2 + 1 + (-1)^2}} = \dfrac{3/2}{3\sqrt{3}} = \dfrac{\sqrt{3}}{6}$

Example. We showed that the lines

$$\begin{array}{llll} L_1: & x = 1 + t & y = -2 + 3t & z = 4 - t \\ L_2: & x = 2s & y = 3 + s & z = -3 + 4s \end{array}$$

are skew. Find the distance between them.

- Strategy: L_1 and L_2 lie in parallel planes P_1 and P_2.

  We will find the distance between these planes.

- A vector parallel to L_1 is $\vec{v}_1 = \langle 1, 3, -1 \rangle$
- A vector parallel to L_2 is $\vec{v}_2 = \langle 2, 1, 4 \rangle$
- $\vec{n} = \vec{v}_1 \times \vec{v}_2 = \langle 13, -6, -5 \rangle$ is normal to both P_1 and P_2
- Set $s=0$. $(0, 3, -3)$ is on L_2 (and hence P_2)
- Equation for P_2: $13(x-0) - 6(y-3) - 5(z+3) = 0$

  $$\iff 13x - 6y - 5z + 3 = 0$$

- Set $t=0$. $(1,-2,4)$ is on L_1 (and hence P_1).
- Find distance from $(1,-2,4)$ to P_2
- $$D = \frac{|13(1) - 6(-2) - 5(4) + 3|}{\sqrt{13^2 + (-6)^2 + (-5)^2}} = \frac{8}{\sqrt{230}}$$