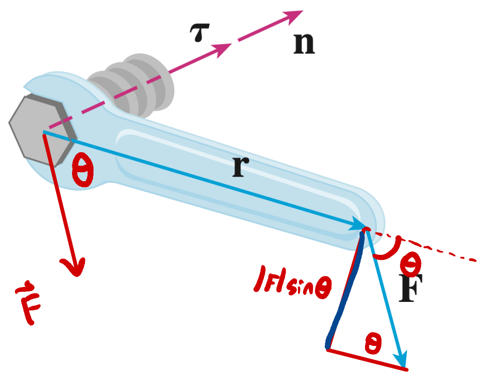
Lecture Notes
Math 2400 - Calculus III
Spring 2024

Name: Champ

9.4 The Cross Product

Definition. What is torque?



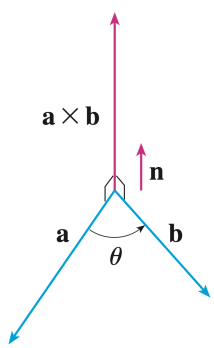
• Torque measures how much a force acting on an object causes that object to rotate

$$|\vec{\tau}| = |\vec{r}||\vec{F}|\sin\theta$$

$$\vec{\tau} = (|\vec{r}||\vec{F}|\sin\theta)\,\vec{n}$$

($\vec{n}$ is a unit vector orthogonal to $\vec{r}$ and $\vec{F}$, determined by the RHR)

Definition. If $\vec{a}$ and $\vec{b}$ are nonzero vectors in V_3, what is the cross product of $\vec{a}$ and $\vec{b}$?



$$\vec{a} \times \vec{b} = (|\vec{a}||\vec{b}|\sin\theta)\,\vec{n}$$

where $0 \leq \theta \leq \pi$

If you curl your right hand from $\vec{a}$ to $\vec{b}$ through θ, thumb points in the direction of the unit vector $\vec{n}$

(In particular, $\vec{\tau} = \vec{r} \times \vec{F}$)

Theorem. Show that two nonvectors $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} \times \vec{b} = 0$.

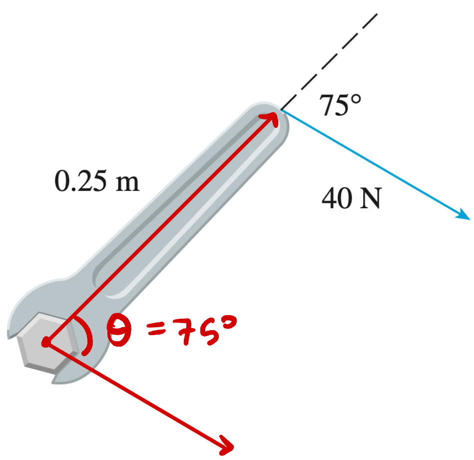
$\vec{a}$ and $\vec{b}$ are parallel $\iff$ the angle between them is 0 or π

$$\iff \sin\theta = 0$$

$$\iff |\vec{a}||\vec{b}| \sin\theta = 0$$

$$\iff \vec{a} \times \vec{b} = \vec{0}$$

Example. A bolt is tightened by applying a 40-N force to a 0.25-m wrench, as shown below. Find the magnitude of the torque about the center of the bolt.

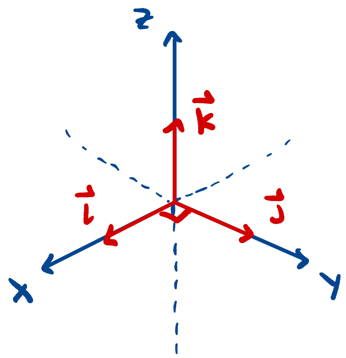


$$|\vec{\tau}| = |\vec{r}||\vec{F}| \sin(\theta)$$

$$= 0.25 \cdot (40) \sin(75^\circ)$$

$$\approx 9.66 \text{ N} \cdot \text{m}$$

Example. Find $\vec{i} \times \vec{j}$ and $\vec{j} \times \vec{i}$.



$$\vec{i} \times \vec{j} = \left(|\vec{i}||\vec{j}| \sin 90^\circ \right) \vec{k} = \vec{k}$$

$$\vec{j} \times \vec{i} = \left(|\vec{j}||\vec{i}| \sin 90^\circ \right) (-\vec{k}) = -\vec{k}$$

Example. Summarize the cross products of $\vec{i}$, $\vec{j}$, and $\vec{k}$.

$\times$	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	$\vec{0}$	$\vec{k}$	$-\vec{j}$
$\vec{j}$	$-\vec{k}$	$\vec{0}$	$\vec{i}$
$\vec{k}$	$\vec{j}$	$-\vec{i}$	$\vec{0}$

This means $\vec{j} \times \vec{k} = \vec{i}$

Example. Is $\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$?

No! For example, $\vec{i} \times \vec{j} \neq \vec{j} \times \vec{i}$

Example. Is $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$?

No! $(\vec{i} \times \vec{i}) \times \vec{j} = \vec{0} \times \vec{j} = \vec{0}$

$\vec{i} \times (\vec{i} \times \vec{j}) = \vec{i} \times \vec{k} = -\vec{j}$

Theorem. What are some properties of the cross product? If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors and c is a scalar, then

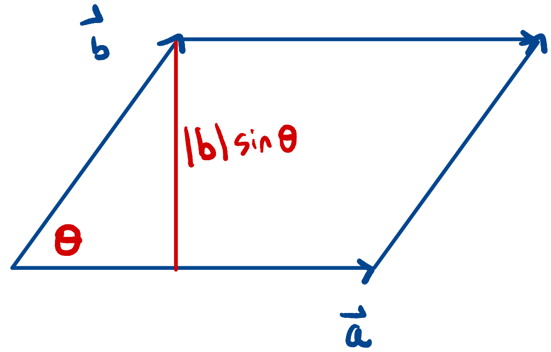
1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

1 Use RHR

2 Apply the def. of $\times$

3, 4 More difficult...

Question. Give a geometric interpretation of the length of the cross product.



$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$$

This is the area of the parallelogram.

Question. How to calculate $\vec{a} \times \vec{b}$ using components?

$$\vec{a} \times \vec{b} = (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \times (b_1\vec{i} + b_2\vec{j} + b_3\vec{k})$$

Using the distributive property of the cross product, we get

$$= (a_1b_1)\,\vec{i} \times \vec{i} + (a_1b_2)\,\vec{i} \times \vec{j} + (a_1b_3)\,\vec{i} \times \vec{k}$$
$$+ \cdots + (a_3b_3)\,\vec{k} \times \vec{k}$$

$$= (a_2b_3 - a_3b_2)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}$$

Read about "determinant of a 3x3 matrix"

Remark. How can we use determinants to simplify the cross product calculation?

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= (a_2 b_3 - a_3 b_2)\vec{i} - (a_1 b_3 - a_3 b_1)\vec{j} + (a_1 b_2 - a_2 b_1)\vec{k}$$

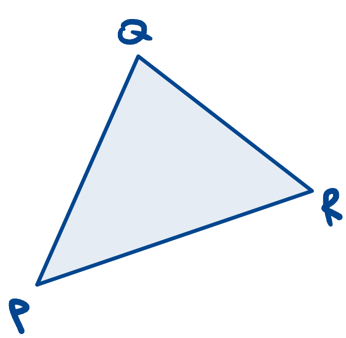
Example. Use determinants to calculate the cross product of $\vec{a} = \langle 1, 3, 4 \rangle$ and $\vec{b} = \langle 2, 7, -5 \rangle$.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = \vec{i} \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix}$$

$$= (-15 - 28)\vec{i} - (-5 - 8)\vec{j} + (7 - 6)\vec{k}$$

$$= -43\vec{i} + 13\vec{j} + \vec{k}$$

Example. Find a vector perpendicular to the plane that passes through the points $P(1, 4, 6)$, $Q(-2, 5, -1)$, $R(1, -1, 1)$.



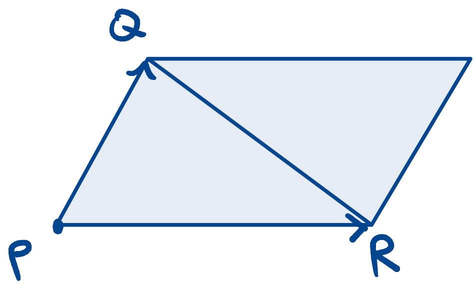
We can compute $\vec{PQ} \times \vec{PR}$

$\vec{PQ} = \langle -3, 1, -7 \rangle$

$\vec{PR} = \langle 0, -5, -5 \rangle$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} = \langle -40, -15, 15 \rangle$$

Example. Find the area of the triangle with vertices $P(1,4,6)$, $Q(-2,5,-1)$, and $R(1,-1,1)$.



Area of parallelogram is

see above

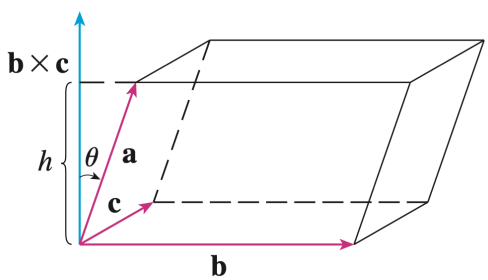
$$|\vec{PQ} \times \vec{PR}| = |\langle -40, -15, 15 \rangle|$$

$$= \sqrt{(-40)^2 + (-15)^2 + (15)^2}$$

$$= \sqrt{2050}$$

Area of ΔPQR is $\frac{1}{2}\sqrt{2050}$

Definition. What is the scalar triple product of the vectors $\vec{a}, \vec{b}$, and $\vec{c}$?



- Scalar triple product

  $\vec{a} \cdot (\vec{b} \times \vec{c})$

- The magnitude of this is the volume of the parallelepiped determined by $\vec{a}, \vec{b}, \vec{c}$

$$V = B \cdot h \quad = |\vec{b} \times \vec{c}| \cdot |\vec{a}| \, |\cos\theta|$$

need $|\cos\theta|$ in case $\theta > \pi/2$

$$= |\vec{a} \cdot (\vec{b} \times \vec{c})|$$

Theorem. Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$, and $\vec{c} = \langle c_1, c_2, c_3 \rangle$. Show that

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot \left[\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \vec{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \vec{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \vec{k} \right]$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Example. Use the scalar triple product to show that the vectors $\vec{a} = \langle 1, 4, -7 \rangle$, $\vec{b} = \langle 2, -1, 4 \rangle$, and $\vec{c} = \langle 0, -9, 18 \rangle$ lie in the same plane.

We just need to show $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$ → Volume of parallelepiped is 0

$$\begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = 1(-18+36) - 4(36-0) - 7(-18-0)$$

$$= 18 - 144 + 126$$

$$= 0$$