Lecture Notes
Math 2400 - Calculus III
Spring 2024

Name: Champ

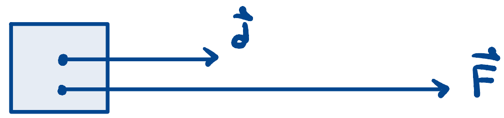
9.3 The Dot Product

measured in Joules

Definition. What is the work done by a force F in moving an object through a distance d?

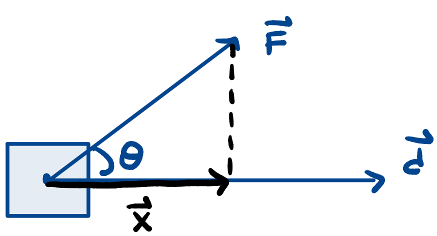
The work W is the product of "length of the path" and the "component of the force acting along the path"

e.g. Push a cart 5m with a force of 10 N



$$W = |\vec{d}| \cdot |\vec{F}| = 5 \cdot 10 = 50 \text{ J}$$

In general, the force does not go exactly along the displacement vector.



- The length of the path is $|\vec{d}|$
- The component of the force along the path is
$$|\vec{x}| = |\vec{F}| \cos\theta$$

$$\Rightarrow W = |\vec{d}| \cdot |\vec{F}| \cos\theta$$

(Work is somehow capturing how much of $\vec{F}$ is in the direction of $\vec{d}$)

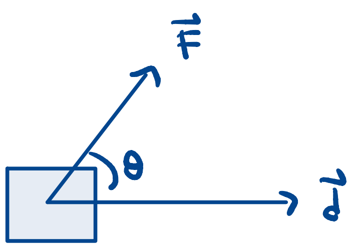
Definition. What is the dot product of two nonzero vectors $\vec{a}$ and $\vec{b}$?

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$$

where $0 \le \theta \le \pi$ is the angle between $\vec{a}$ and $\vec{b}$

⚠ The dot product of two vectors is a number

Example. How can we reinterpret work in terms of the dot product?

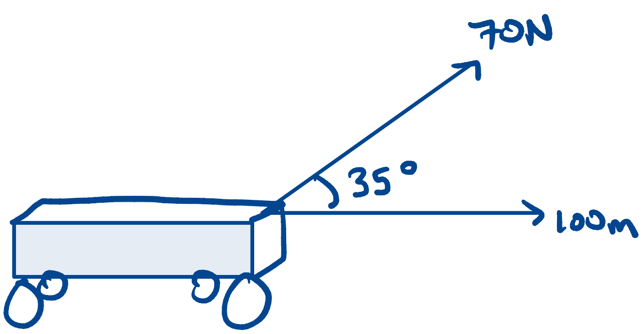


$$W = |\vec{d}|\,|\vec{F}| \cos\theta$$

$$= \vec{F} \cdot \vec{d}$$

(or $\vec{d} \cdot \vec{F}$)

Example. A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of 35° above the horizontal. Find the work done by the force.



$$W = \vec{F} \cdot \vec{d}$$

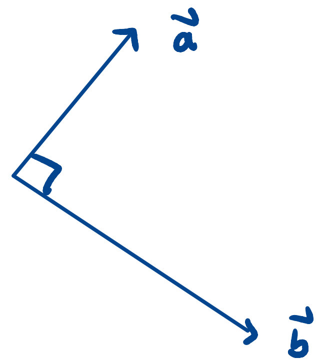
$$= |\vec{F}|\,|\vec{d}| \cos\theta$$

$$= 70 \cdot 100 \cos(35°)$$

$$= 5734 \text{ J}$$

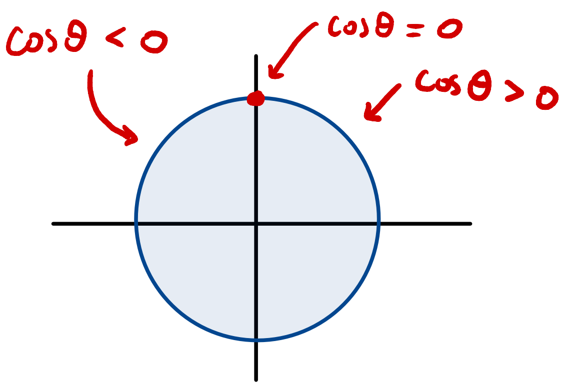
Definition. What does it mean for two vectors to be orthogonal? perpendicular

Vectors $\vec{a}$ and $\vec{b}$ are orthogonal if the angle between them is $\pi/2$



Theorem. Prove that two vectors $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\vec{a} \cdot \vec{b} = 0$.

- Let $\vec{a}$ and $\vec{b}$ be nonzero vectors
- By definition, $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$
- Since $|\vec{a}|$ and $|\vec{b}|$ are positive numbers, $\vec{a} \cdot \vec{b}$ has the same sign as $\cos\theta$



Unit Circle

- Hence if $\vec{a} \cdot \vec{b} = 0$, we must have $\cos\theta = 0$
- $\cos\theta = 0 \Rightarrow \theta = \pi/2$

Definition. What is the dot product in terms of components?

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$

Then $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

(This can be proved from $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$
using the Law of Cosines)

Example. Compute the following dot products:

(a) $\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle$

(b) $\langle -1, 7, 4 \rangle \cdot \langle 6, 2, -\frac{1}{2} \rangle$

(c) $(\vec{i} + 2\vec{j} - 3\vec{k}) \cdot (2\vec{j} - \vec{k})$

(a) $2 \cdot 3 + 4 \cdot (-1) = 6 - 4 = 2$

(b) $(-1) \cdot 6 + 7 \cdot 2 + 4 \cdot (-\frac{1}{2})$

$= -6 + 14 - 2$

$= 6$

(c) $\langle 1, 2, -3 \rangle \cdot \langle 0, 2, -1 \rangle$

$= 1 \cdot 0 + 2 \cdot 2 + (-3) \cdot (-1)$

$= 0 + 4 + 3$

$= 7$

Example. Show that $2\vec{\imath} + 2\vec{\jmath} - \vec{k}$ is perpendicular to $5\vec{\imath} - 4\vec{\jmath} + 2\vec{k}$.

To show two vectors are perpendicular, we just need to show that the dot product is 0

$$(2\vec{\imath} + 2\vec{\jmath} - \vec{k}) \cdot (5\vec{\imath} - 4\vec{\jmath} + 2\vec{k}) = 2\cdot 5 + 2\cdot(-4) + (-1)\cdot 2$$
$$= 10 - 8 - 2$$
$$= 0 \quad \checkmark$$

Example. Find the angle between the vectors $\vec{a} = \langle 2, 2, -1\rangle$ and $\vec{b} = \langle 5, -3, 2\rangle$.

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$$

$$\vec{a} \cdot \vec{b} = 2\cdot 5 + 2\cdot(-3) + (-1)\cdot 2 = 10 - 6 - 2 = 2$$

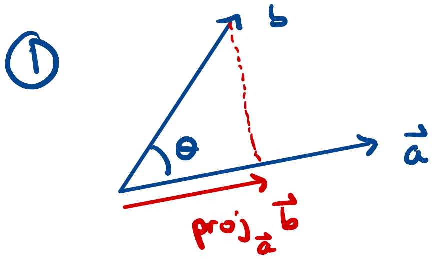
$$|\vec{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{9} = 3$$

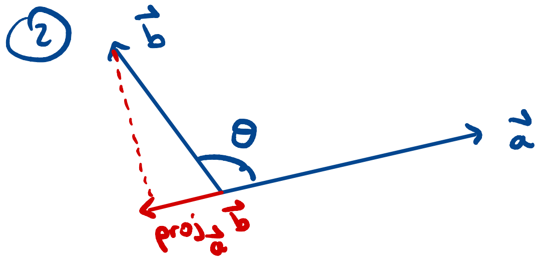
$$|\vec{b}| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{38}$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{2}{3\sqrt{38}}$$

$$\theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right) \approx 1.46 \text{ rad} \quad (\text{or } 84^\circ)$$

Definition. What is the scalar projection of $\vec{b}$ onto $\vec{a}$?





The scalar projection is the length of $proj_{\vec{a}}\vec{b}$

$$\Rightarrow comp_{\vec{a}}\vec{b} = |\vec{b}|\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

(Think: the "shadow" of $\vec{b}$ on $\vec{a}$)

Definition. What is the vector projection of $\vec{b}$ onto $\vec{a}$?

To find the vector projection, multiply the scalar projection by a unit vector in the direction of $\vec{a}$

$$proj_{\vec{a}}\vec{b} = \underbrace{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}_{\text{Scalar projection}}\underbrace{\left(\frac{\vec{a}}{|\vec{a}|}\right)}_{\text{unit vector in the direction of }\vec{a}} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}$$

Example. Find the scalar projection and vector projection of $\vec{b} = \langle 1, 1, 2\rangle$ onto $\vec{a} = \langle -2, 3, 1\rangle$.

First, $|\vec{a}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$

$$\vec{a}\cdot\vec{b} = (-2)\cdot 1 + 3\cdot 1 + 1\cdot 2 = -2+3+2 = 3$$

$$comp_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{3}{\sqrt{14}}$$

$$proj_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\cdot\frac{\vec{a}}{|\vec{a}|} = \frac{3}{\sqrt{14}}\cdot\frac{\vec{a}}{\sqrt{14}} = \frac{3}{14}\vec{a} = \left\langle \frac{-6}{14}, \frac{9}{14}, \frac{3}{14}\right\rangle$$