Lecture Notes
Math 2400 - Calculus III
Spring 2024

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### 9.2 Vectors

**Definition.** What is a vector?

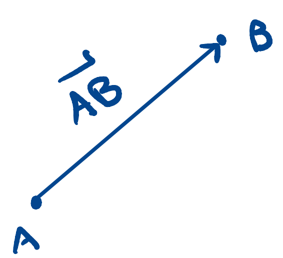
- A vector is a quantity that has both a <u>magnitude</u> and a <u>direction</u>.

- Examples: velocity or force

**Question.** How to represent vectors?

- We represent vectors by arrows
- The length of the arrow is the magnitude of the vector
- The arrow points in the direction of the vector
- We denote a vector by boldface font or by an arrow: $\mathbf{v}$ or $\vec{v}$

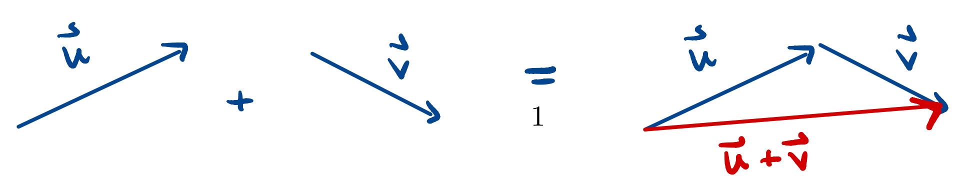
**Definition.** Suppose a particle moves along a line segment from point $A$ to point $B$. What is the displacement vector from $A$ to $B$?



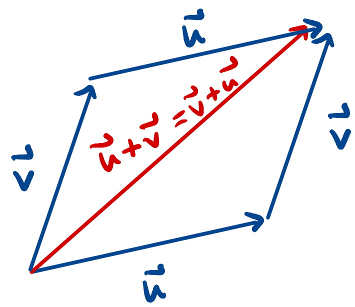

The displacement vector $\vec{AB}$ has initial point $A$ (tail) and terminal point $B$ (tip)

**Definition.** How do we add vectors?

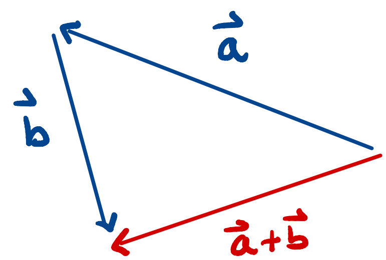
To add vectors $\vec{u}$ and $\vec{v}$, we move $\vec{v}$ so that it starts at the end of $\vec{u}$. $\vec{u}+\vec{v}$ is the vector that starts at the beginning of $\vec{u}$ and ends at the end of $\vec{v}$.



**Question.** Given vectors $\vec{u}$ and $\vec{v}$, why is $\vec{u}+\vec{v}=\vec{v}+\vec{u}$?

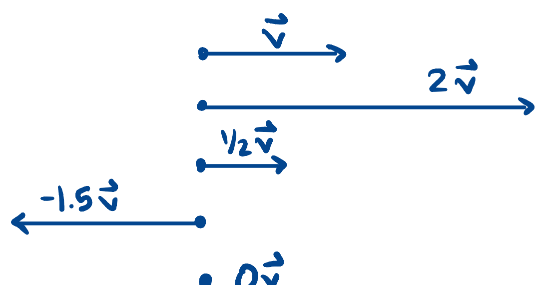



**Example.** Draw the sum of the vectors $\vec{a}$ and $\vec{b}$ shown below.

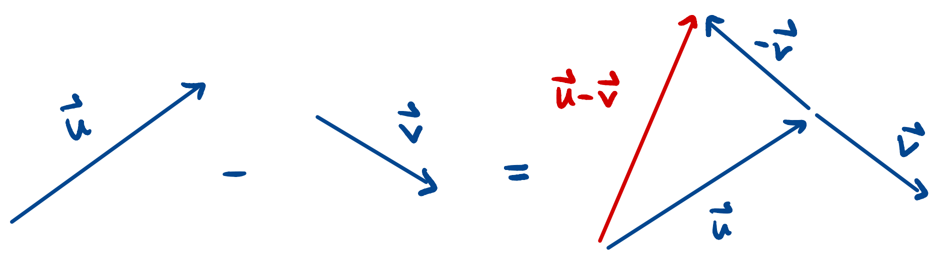

**Definition.** How do we scale vectors?

If $c$ is a real number, then $c\vec{v}$ is the vector whose length is $|c|$ times the length of $\vec{v}$ in the direction of $\vec{v}$ if $c>0$ and in the opposite direction if $c<0$.

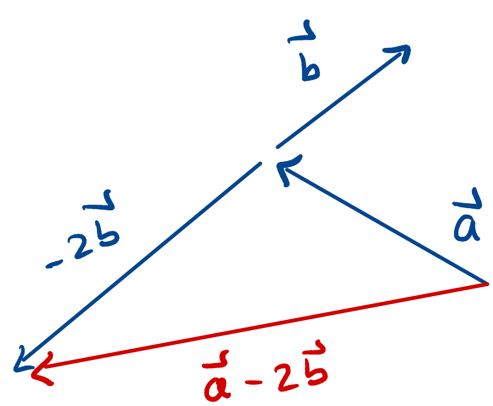


**Definition.** How do we subtract vectors?

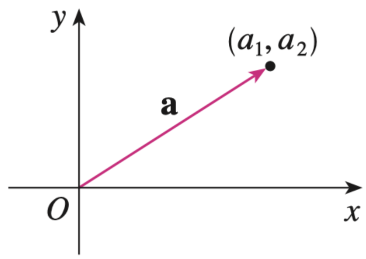
- Think about $\vec{u} - \vec{v}$ as $\vec{u} + (-\vec{v})$



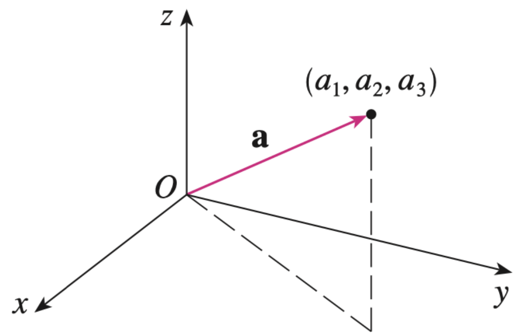

**Example.** If $\vec{a}$ and $\vec{b}$ are the vectors shown below, draw $\vec{a} - 2\vec{b}$.

**Question.** How to represent vectors using coordinates?




$\mathbf{a} = \langle a_1, a_2 \rangle$

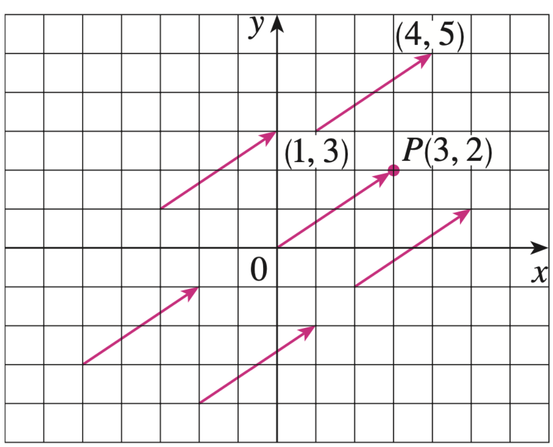



$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

- Place the initial point of the vector at the origin
- In 3D, the terminal point is at some point $(a_1, a_2, a_3)$
- We write $\vec{a} = \langle a_1, a_2, a_3 \rangle$

Components of $\vec{a}$

**Question.** Explain why the vectors below are all equivalent.

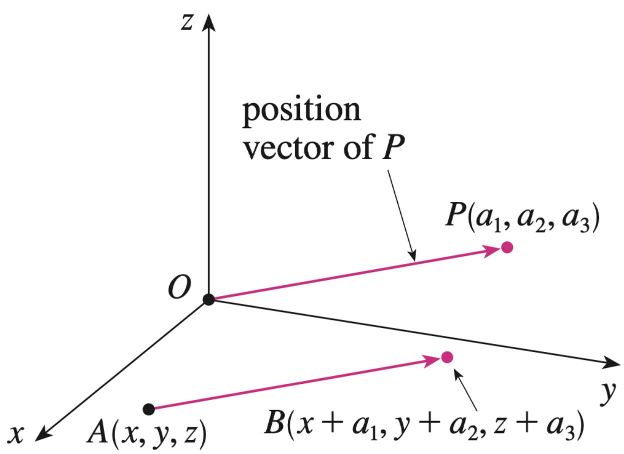



- Vectors are <u>equivalent</u> if they have the same magnitude and direction
- All of these vectors are equivalent to $\langle 3, 2 \rangle$

**Definition.** What is a position vector?

The vector $\vec{OP}$ starting at the origin and ending at a point $P$ is called the position vector of the point $P(a_1, a_2, a_3)$.

**Definition.** Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, what is the vector with representation $\vec{AB}$?




If we start at A and end at B then

$$\vec{AB} = \langle \underbrace{x_2 - x_1}_{a_1}, \underbrace{y_2 - y_1}_{a_2}, \underbrace{z_2 - z_1}_{a_3} \rangle$$

These vectors are two different geometric representations of $\langle a_1, a_2, a_3 \rangle$

**Example.** Find the vector represented by the directed line segment with initial point $A(2, -3, 4)$ and terminal point $B(-2, 1, 1)$.

$$\vec{AB} = \langle -2-2, 1-(-3), 1-4 \rangle = \langle -4, 4, -3 \rangle$$

**Definition.** How to calculate the magnitude of a vector $\vec{v}$?

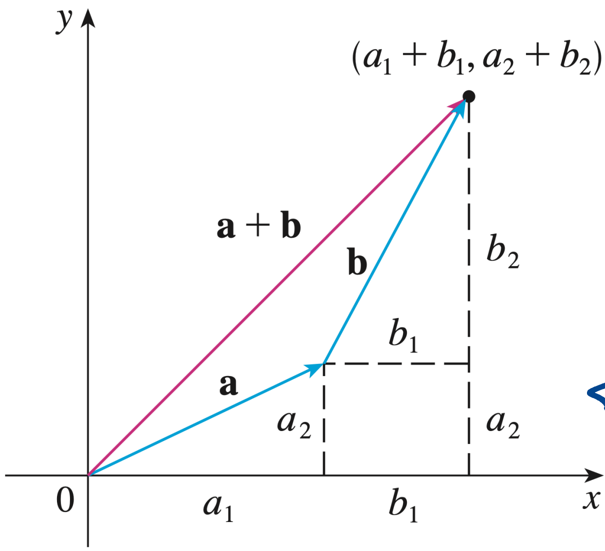
The magnitude (length) of $\vec{v}$ is denoted by $|\vec{v}|$

In 2D: $|\vec{v}| = |\langle a_1, a_2 \rangle| = \sqrt{a_1^2 + a_2^2}$

In 3D: $|\vec{v}| = |\langle a_1, a_2, a_3 \rangle| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Pythagorean Thm

**Question.** How do we add vectors algebraically?




To add vectors

$\vec{a} = \langle a_1, a_2 \rangle$

$\vec{b} = \langle b_1, b_2 \rangle$

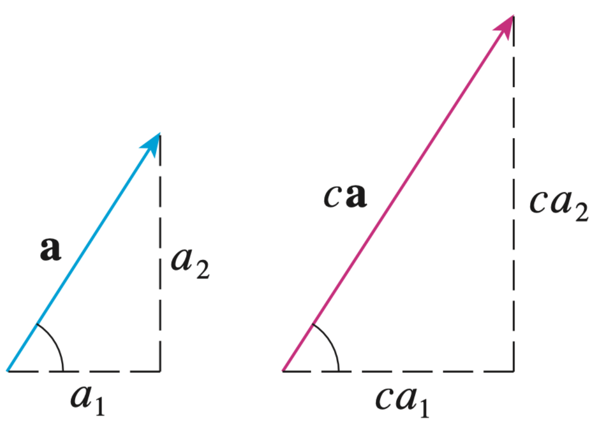
we just add the components

$\langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle$

. Similarly, to subtract vectors, we subtract the components

$\langle a_1, a_2 \rangle - \langle b_1, b_2 \rangle = \langle a_1 - b_1, a_2 - b_2 \rangle$

**Question.** How do we scale vectors algebraically?




To multiply a vector by a scalar, we multiply each component by the scalar

$c \langle a_1, a_2 \rangle = \langle ca_1, ca_2 \rangle$

**Example.** If $\vec{a} = \langle 4, 0, 3 \rangle$ and $\vec{b} = \langle -2, 1, 5 \rangle$, find $|\vec{a}|$ and the vectors $\vec{a}+\vec{b}$, $\vec{a}-\vec{b}$, $3\vec{b}$ and $2\vec{a}+5\vec{b}$.

$$|\vec{a}| = \sqrt{4^2+0^2+3^2} = \sqrt{25} = 5$$

$$\vec{a}+\vec{b} = \langle 4+(-2), 0+1, 3+5 \rangle$$
$$= \langle 2, 1, 8 \rangle$$

$$\vec{a}-\vec{b} = \langle 4-(-2), 0-1, 3-5 \rangle$$
$$= \langle 6, -1, -2 \rangle$$

$$3\vec{b} = \langle 3\cdot(-2), 3\cdot 1, 3\cdot 5 \rangle$$
$$= \langle -6, 3, 15 \rangle$$

$$2\vec{a}+5\vec{b} = \langle 2\cdot 4, 2\cdot 0, 2\cdot 3 \rangle + \langle 5\cdot(-2), 5\cdot 1, 5\cdot 5 \rangle$$
$$= \langle 8, 0, 6 \rangle + \langle -10, 5, 25 \rangle$$
$$= \langle -2, 5, 31 \rangle$$

**Definition.** What is the set $V_n$?

- $V_n$ is the set of all n-dimensional vectors
- an n-dimensional vector is an ordered tuple

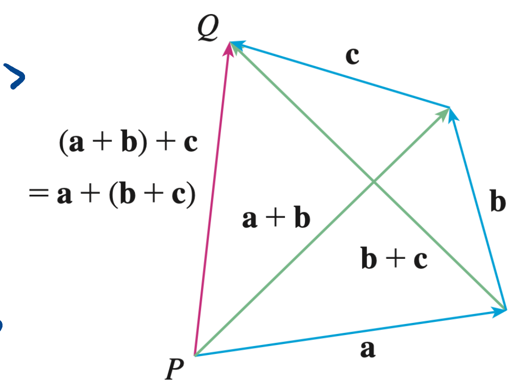
$$\vec{a} = \langle a_1, a_2, \dots, a_n \rangle$$

where $a_1, a_2, \dots, a_n$ are real numbers

**Theorem.** Let $\vec{a}, \vec{b}$, and $\vec{c}$ are vectors in $V_n$ and $c$ and $d$ are scalars. Prove the following properties of these vectors.

1. $\vec{a}+\vec{b}=\vec{b}+\vec{a}$
2. $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$
3. $\vec{a}+\vec{0}=\vec{a}$
4. $\vec{a}+(-\vec{a})=\vec{0}$
5. $c(\vec{a}+\vec{b})=c\vec{a}+c\vec{b}$
6. $(c+d)\vec{a}=c\vec{a}+d\vec{a}$
7. $(cd)\vec{a}=c(d\vec{a})$
8. $1\cdot\vec{a}=\vec{a}$

*Proof.*




① $\vec{a}+\vec{b} = \langle a_1, a_2, \ldots, a_n\rangle + \langle b_1, b_2, \ldots, b_n\rangle$

$= \langle a_1+b_1, a_2+b_2, \ldots, a_n+b_n\rangle$

$= \langle b_1+a_1, b_2+a_2, \ldots, b_n+a_n\rangle$

$= \langle b_1, b_2, \ldots, b_n\rangle + \langle a_1, a_2, \ldots, a_n\rangle$

$= \vec{b}+\vec{a}$

② Given vectors $\vec{a}, \vec{b}, \vec{c}$, construct $\vec{a}+\vec{b}$ and $\vec{b}+\vec{c}$

$\vec{PQ} = (\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$

(See figure)

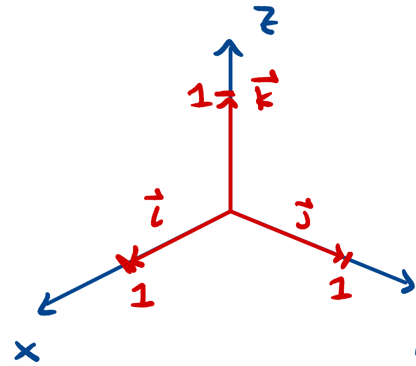
$\square$

**Definition.** What are the vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$?

$\vec{i} = \langle 1, 0, 0 \rangle$

$\vec{j} = \langle 0, 1, 0 \rangle$

$\vec{k} = \langle 0, 0, 1 \rangle$




**Example.** Show that any vector in $V_3$ can be expressed in terms of the vectors $\vec{i}$, $\vec{j}$, $\vec{k}$.

$$\vec{a} = \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle$$
$$= a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle$$
$$= a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$

($\vec{i}, \vec{j}, \vec{k}$ is called a basis of $\mathbb{R}^3$ for this reason)

**Example.** If $\vec{a} = \vec{i} + 2\vec{j} - 3\vec{k}$ and $\vec{b} = 4\vec{i} + 7\vec{k}$, express the vector $2\vec{a} + 3\vec{b}$ in terms of $\vec{i}, \vec{j}$, and $\vec{k}$.

$$2\vec{a} + 3\vec{b} = 2(\vec{i} + 2\vec{j} - 3\vec{k}) + 3(4\vec{i} + 7\vec{k})$$
$$= 2\vec{i} + 4\vec{j} - 6\vec{k} + 12\vec{i} + 21\vec{k}$$
$$= 14\vec{i} + 4\vec{j} + 15\vec{k}$$

$\langle 14, 4, 15 \rangle$

**Definition.** What is a unit vector? If $\vec{a} \neq 0$, find a unit vector that has the same direction as $\vec{a}$.

Unit vector: a vector of length 1

$\vec{u} = \dfrac{\vec{a}}{|\vec{a}|}$ is a vector with magnitude 1 in the direction of $\vec{a}$

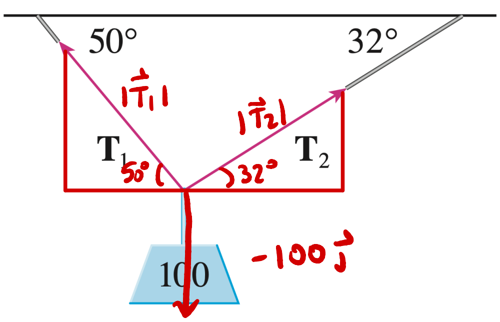
(check: since $|c\vec{a}| = |c||\vec{a}|$, we have $|\vec{u}| = \left|\frac{1}{|\vec{a}|}\right| \cdot |\vec{a}| = \frac{1}{|\vec{a}|} \cdot |\vec{a}| = 1$)

**Example.** Find the unit vector in the direction of the vector $2\vec{i} - \vec{j} - 2\vec{k}$.

$$|2\vec{i} - \vec{j} - 2\vec{k}| = \sqrt{(2)^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$$

We obtain $\frac{1}{3}(2\vec{i} - \vec{j} - 2\vec{k}) = \frac{2}{3}\vec{i} - \frac{1}{3}\vec{j} - \frac{2}{3}\vec{k} \leftrightarrow \langle \frac{2}{3}, \frac{-1}{3}, \frac{-2}{3} \rangle$

**Example.** A 100-lb weight hangs from two wires as shown below. Find the tensions (forces) $\vec{T}_1$ and $\vec{T}_2$ in both wires and the magnitudes of the tensions.




① Find the components of $\vec{T}_1$ and $\vec{T}_2$

$$\vec{T}_1 = -|\vec{T}_1|\cos(50°)\vec{i} + |\vec{T}_1|\sin(50°)\vec{j}$$
$$\vec{T}_2 = |\vec{T}_2|\cos(32°)\vec{i} + |\vec{T}_2|\sin(32°)\vec{j}$$

② $\vec{T}_1 + \vec{T}_2$ counterbalances 100 lbs $\Rightarrow \vec{T}_1 + \vec{T}_2 = 100\vec{j}$

③ Hence

$$-|\vec{T}_1|\cos(50°) + |\vec{T}_2|\cos(32°) = 0 \quad \leftarrow \text{sum of } \vec{i}\text{-components}$$
$$|\vec{T}_1|\sin(50°) + |\vec{T}_2|\sin(32°) = 100 \quad \leftarrow \text{sum of } \vec{j}\text{-components}$$

④ By EQ1, $|\vec{T}_2| = |\vec{T}_1| \cdot \dfrac{\cos(50°)}{\cos(32°)}$

Hence by EQ 2,

$$|\vec{T}_1| \sin(50^\circ) + |\vec{T}_1| \frac{\cos(50^\circ)}{\cos(32^\circ)} \sin(32^\circ) = 100$$

$$\Rightarrow |\vec{T}_1| \approx 85.64 \text{ lbs}$$

$$\Rightarrow |\vec{T}_2| \approx 64.91 \text{ lbs}$$

5 Plugging these into 1 we get the vector components

$$\vec{T}_1 = -55.05\vec{i} + 65.60\vec{j}$$

$$\vec{T}_2 = 55.05\vec{i} + 34.40\vec{j}$$