Lecture Notes
Math 2400 - Calculus III
Spring 2024

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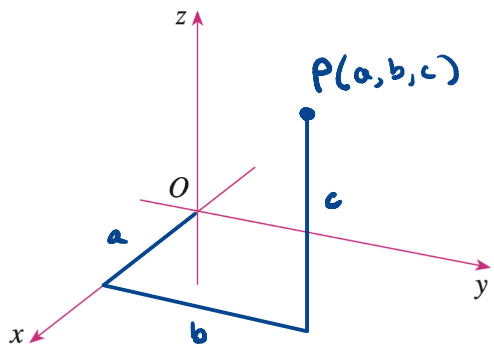
### 9.1 Three-Dimensional Coordinate Systems

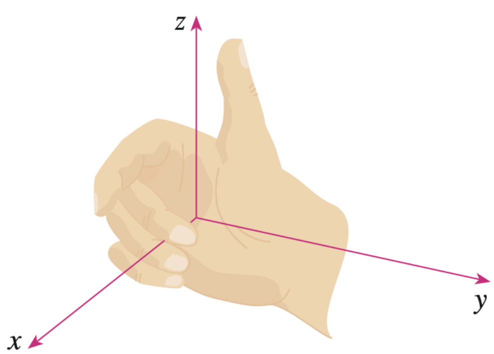
**Question.** How do we represent points in space?




A point in space is a triple $(a, b, c)$ of real numbers

x-coord y-coord z-coord

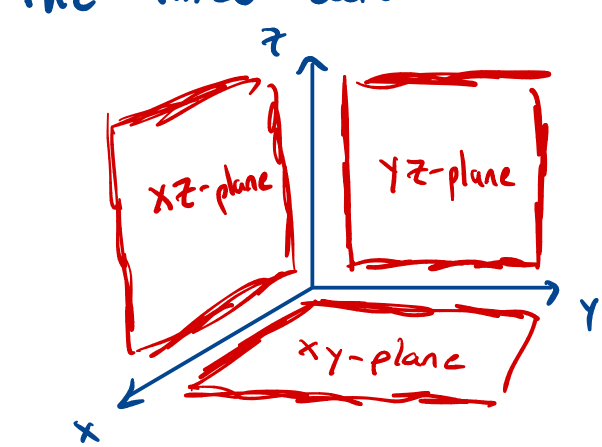
**Definition.** What is the right-hand rule?



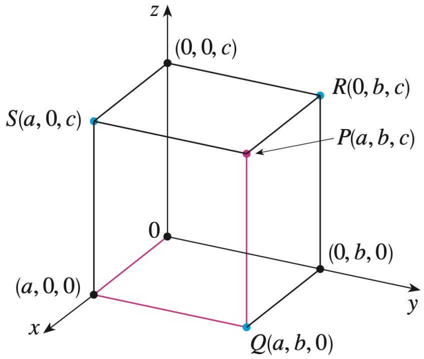

- The right-hand rule determines the direction of the positive z-axis
- Curl your right hand counterclockwise from pos. x-axis to pos. y-axis
- Your thumb points in the direction of the pos. z-axis

**Example.** What are the three coordinate planes?

The three coordinate axes determine three coordinate planes



**Question.** Explain how a point $P(a, b, c)$ can be projected to each of the coordinate planes.




- Any point $P(a,b,c)$ determines a rectangular box
- $Q(a,b,0)$ = Projection to $xy$-plane
- $S(a,0,c)$ = Projection to $xz$-plane
- $R(0,b,c)$ = Projection to $yz$-plane

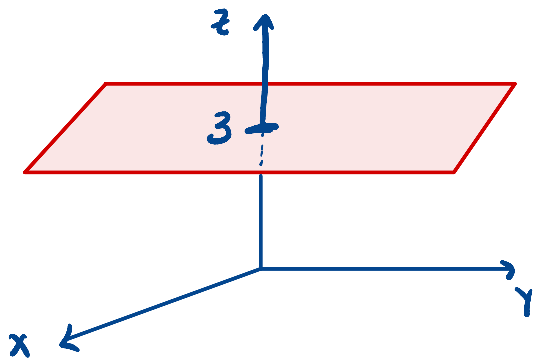
**Definition.** What is $\mathbb{R}^3$?

The set of all triples

$\mathbb{R}^3$ is the Cartesian Product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}$

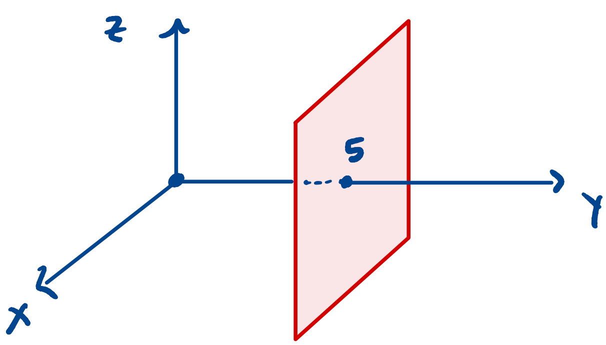
1 : 1 correspondence $\{\text{points in space}\} \longleftrightarrow$ ordered triples $(a,b,c) \in \mathbb{R}^3$

**Example.** What surface in $\mathbb{R}^3$ is represented by the equation $z = 3$?




This is all points $(x,y,z)$ in $\mathbb{R}^3$ with $z$-coordinate 3.

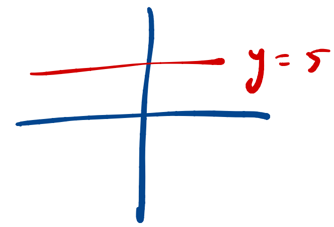
**Example.** What surface in $\mathbb{R}^3$ is represented by the equation $y = 5$?




This is all points $(x,y,z)$ in $\mathbb{R}3$ with $y$-coordinate 5

**Remark.** Why do we have to be careful about the context of our equations?

They could represent different surfaces depending on the ambient space. In $\mathbb{R}^2$, $y=5$ is a line



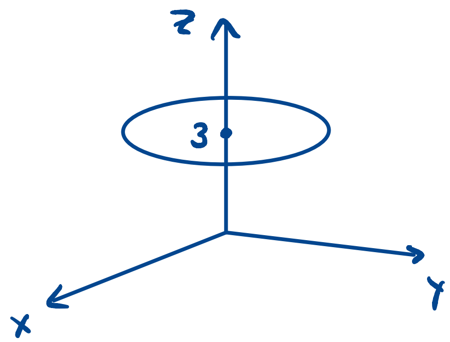

**Example.**

(a) Which points $(x, y, z)$ satisfy the equations $x^2 + y^2 = 1$ and $z = 3$?

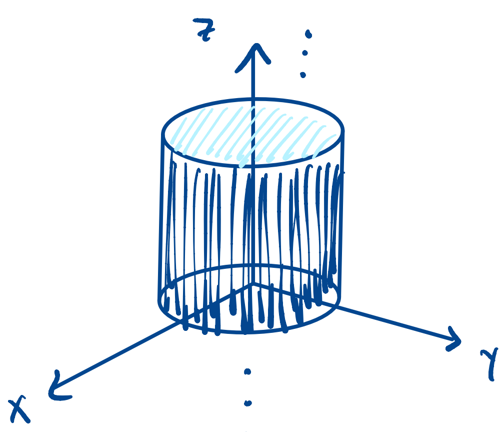
(b) What does the equation $x^2 + y^2 = 1$ represent as a surface in $\mathbb{R}^3$?

(a) Since $z=3$, the points lie in the plane $z=3$

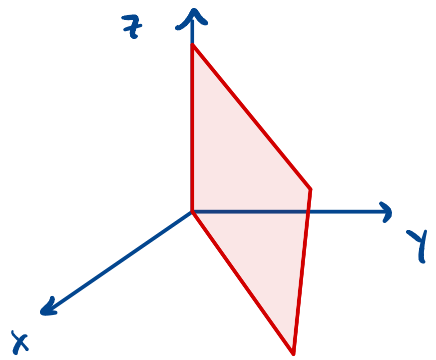
Since $x^2+y^2=1$, the points lie on a circle of radius 1 centered around the $z$-axis




(b) Without the restriction on $z$-coordinate, we get an infinite cylinder

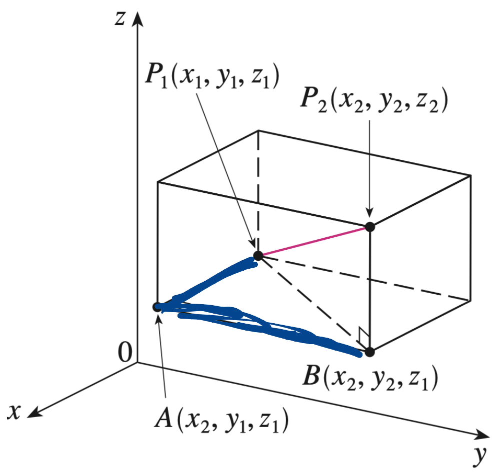


**Example.** Describe and sketch the surface in $\mathbb{R}^3$ represented by the equation $y = x$.




- Start by graphing $y=x$ in the $xy$-plane
- Extend this line to all $z$-coordinates

**Theorem** (Distance Formula in Three Dimensions)**.** Find a formula that represents the distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$.




- $P_1$ and $P_2$ determine a box
- We know:

$$|P_1A| = |x_2 - x_1|$$
$$|AB| = |y_2 - y_1|$$
$$|BP_2| = |z_2 - z_1|$$

- Hence

$$\begin{aligned} |P_1P_2|^2 &= |P_1B|^2 + |BP_2|^2 \\ &= |P_1A|^2 + |AB|^2 + |BP_2|^2 \\ &= |x_2-x_1|^2 + |y_2-y_1|^2 + |z_2-z_1|^2 \\ &= (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 \end{aligned}$$

- Therefore, $|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

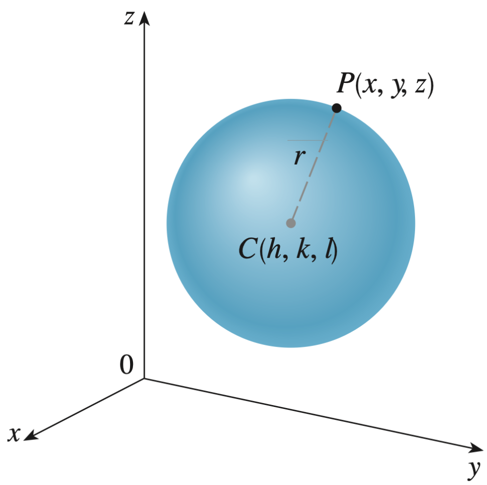
**Example.** Find the distance from the point $P(2, -1, 7)$ to the point $Q(1, -3, 5)$.

$$|PQ| = \sqrt{(1-2)^2 + (-3-(-1))^2 + (5-7)^2}$$

$$= \sqrt{(-1)^2 + (-2)^2 + (-2)^2}$$

$$= \sqrt{1+4+4} = \sqrt{9} = 3$$

**Example.** Find an equation of a sphere with radius $r$ and center $C(h, k, l)$.




- The sphere is all points $(x, y, z)$ that are $r$ units away from $(h, k, l)$

$\Rightarrow$ All points $P(x, y, z)$ with $|PC| = r$

$$\Rightarrow |PC|^2 = r^2$$

$$\Rightarrow \boxed{(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2}$$

Distance Formula

Note: if the sphere is centered at the origin, then $(h, k, l) = (0, 0, 0)$ and $\boxed{x^2 + y^2 + z^2 = r^2}$

**Example.** Show that $x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$ is the equation of a sphere, and find its center and radius.

· Completing the squares,

$$x^2 + 4x + 4 + y^2 - 6y + 9 + z^2 + 2z + 1 + 6 = 14$$

$$(x+2)^2 + (y-3)^2 + (z+1)^2 = 8$$

Center: $(-2, 3, -1)$

Radius: $\sqrt{8}$

**Example.** What region in $\mathbb{R}^3$ is represented by the following inequalities?

$$1 \leq x^2 + y^2 + z^2 \leq 4 \qquad z \leq 0$$

$x^2+y^2+z^2 = 1$ is a sphere of radius 1

$x^2+y^2+z^2 = 4$ is a sphere of radius 2

$1 \leq x^2+y^2+z^2 \leq 4$ is the set of points bounded by these spheres

The inequality $z \leq 0$ requires that the points lie on or below the xy-plane