Lecture Notes
Math 2400 - Calculus III
Spring 2024

Name: ____________________________

13.8 The Divergence Theorem

Question. What is a simple solid region? What are some examples?

Theorem. What is the divergence theorem?

Proof.

1. Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$. Since $\operatorname{div}\mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z}$, what is $\iiint_E \operatorname{div}\mathbf{F}\,dV$?

2. If $\mathbf{n}$ is the unit outward normal of S, what is $\iint_S \mathbf{F}\cdot d\mathbf{S}$?

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_S (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k})\cdot\mathbf{n}\,dS$$

$$=$$

3. What do we have to do to prove the Divergence Theorem?

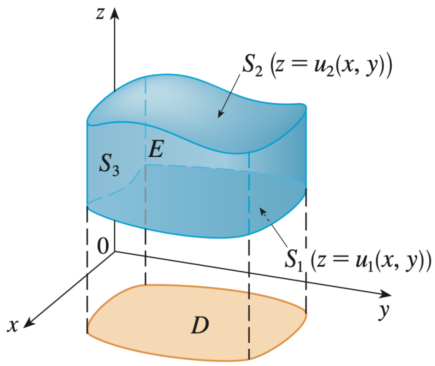
4. To prove the last equation, we will use the fact that E is a type 1 region:

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}$$

where D is the projection of E onto the xy-plane. Using this, what is $\iiint_E \dfrac{\partial R}{\partial z}\,dV$?

$$\iiint_E \frac{\partial R}{\partial z}\,dV =$$

5. What is the boundary of E?



6. How can we rewrite $\iint_S R\mathbf{k} \cdot \mathbf{n}\, dS$?

7. What are $\iint_{S_1} R\mathbf{k} \cdot \mathbf{n}\, dS$ and $\iint_{S_2} R\mathbf{k} \cdot \mathbf{n}\, dS$?

8. Use the above to conclude that the Divergence Theorem holds.

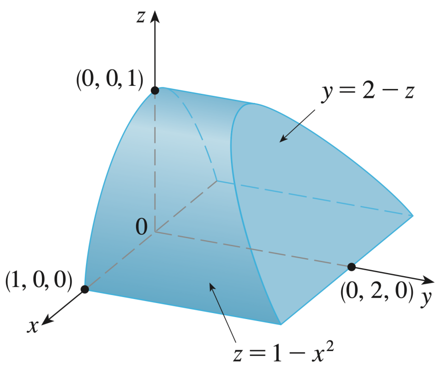
$\square$

Example. Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

Example. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = xy\,\mathbf{i} + (y^2 + e^{xz^2})\,\mathbf{j} + \sin(xy)\,\mathbf{k}$$

and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$.



Example. Let S be the unit cube with an open top. Find the flux of the vector field $\mathbf{F} = \langle z, y, x \rangle$ over S.