Lecture Notes
Math 2400 - Calculus III
Spring 2024

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13.8 The Divergence Theorem

A region bounded between two functions of x and y

Question. What is a simple solid region? What are some examples?

We defined type 1, type 2, and type 3 regions in §12.7.

A simple solid region is a region that is simultaneously types 1, 2, and 3

e.g. regions bounded by ellipsoids or rectangular boxes.

Theorem. What is the divergence theorem?

Let E be a simple solid region and let S be the boundary surface of E, given with the positive (outward) orientation. Then

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV$$

In words, the flux of $\vec{F}$ across the boundary surface of E is equal to the triple integral of the divergence of $\vec{F}$ over E.

Rmk: Compare this to the vector form of Green's thm which said $\int_C \vec{F} \cdot \vec{n} \, ds = \iint_D \operatorname{div} \vec{F} \, dA$

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV$$

Proof.

1. Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$. Since $\operatorname{div}\mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z}$, what is $\iiint\limits_E \operatorname{div}\mathbf{F}\, dV$?

$$\iiint_E \operatorname{div}\vec{F}\, dV = \iiint_E \frac{\partial P}{\partial x}\, dV + \iiint_E \frac{\partial Q}{\partial y}\, dV + \iiint_E \frac{\partial R}{\partial z}\, dV$$

2. If $\mathbf{n}$ is the unit outward normal of S, what is $\iint\limits_S \mathbf{F}\cdot d\mathbf{S}$?

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_S (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k})\cdot\mathbf{n}\, dS$$

$$= \iint_S P\,\vec{i}\cdot\vec{n}\, dS + \iint_S Q\,\vec{j}\cdot\vec{n}\, dS + \iint_S R\,\vec{k}\cdot\vec{n}\, dS$$

3. What do we have to do to prove the Divergence Theorem?

Show that

$$\iint_S P\,\vec{i}\cdot\vec{n}\, dS = \iiint_E \frac{\partial P}{\partial x}\, dV \qquad \text{EQ1}$$

$$\iint_S Q\,\vec{j}\cdot\vec{n}\, dS = \iiint_E \frac{\partial Q}{\partial y}\, dV \qquad \text{EQ2}$$

$$\iint_S R\,\vec{k}\cdot\vec{n}\, dS = \iiint_E \frac{\partial R}{\partial z}\, dV \qquad \text{EQ3}$$

4. To prove the last equation, we will use the fact that E is a type 1 region:

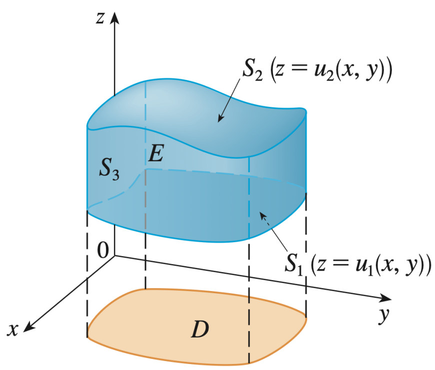
$$E = \{(x,y,z) \mid (x,y)\in D, u_1(x,y) \le z \le u_2(x,y)\}$$

where D is the projection of E onto the xy-plane. Using this, what is $\iiint\limits_E \dfrac{\partial R}{\partial z}\, dV$?

$$\iiint_E \frac{\partial R}{\partial z}\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} \frac{\partial R}{\partial z}(x,y,z)\, dz \right] dA$$

$$= \iint_D R(x,y,u_2(x,y)) - R(x,y,u_1(x,y))\, dA$$

5. What is the boundary of E?



The boundary surface S consists of three pieces: S_1, S_2, and S_3.

S_3 may or may not appear (e.g. no S_3 if E is a sphere)

6. How can we rewrite $\iint_S R\mathbf{k} \cdot \mathbf{n}\, dS$?

On S_3, $\vec{k} \cdot \vec{n} = 0$ since $\vec{k}$ is vertical and $\vec{n}$ is horizontal

$$\iint_S R\vec{k} \cdot \vec{n}\, dS = \iint_{S_1} R\vec{k} \cdot \vec{n}\, dS + \iint_{S_2} R\vec{k} \cdot \vec{n}\, dS + \cancel{\iint_{S_3} R\vec{k} \cdot \vec{n}\, dS}^{\,0}$$

(It is also this if there is no S_3)

7. What are $\iint_{S_1} R\mathbf{k} \cdot \mathbf{n}\, dS$ and $\iint_{S_2} R\mathbf{k} \cdot \mathbf{n}\, dS$?

On S_2, $\vec{n}$ points up. So $\iint_{S_2} R\vec{k} \cdot \vec{n}\, dS = \iint_D R(x, y, u_2(x,y))\, dA$

On S_1, $\vec{n}$ points down. So $\iint_{S_1} R\vec{k} \cdot \vec{n}\, dS = -\iint_D R(x, y, u_1(x,y))\, dA$

EQ3

8. Use the above to conclude that the Divergence Theorem holds.

$$\iint_S R\vec{k} \cdot \vec{n}\, dS = \iint_D R(x, y, u_2(x,y))\, dA - \iint_D R(x, y, u_1(x,y))\, dA = \iiint_E \frac{\partial R}{\partial z}\, dV$$

EQ1 and EQ2 are proved the same way, using the fact that E is type 2 and type 3. □

Example. Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x) = 1$$

The unit sphere is the boundary of the unit ball B given by $x^2+y^2+z^2 \leq 1$. By the divergence theorem,

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_B \operatorname{div} \vec{F}\, dV = \iiint_B 1\, dV = \operatorname{Vol}(B)$$

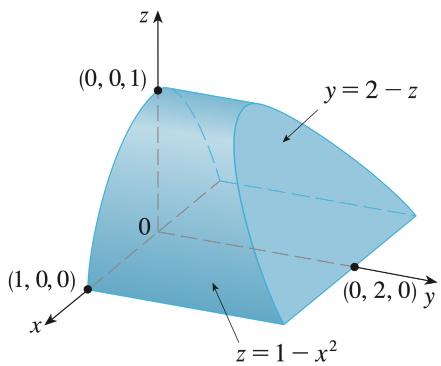
$$= \frac{4}{3}\pi (1)^3$$

$$= \frac{4\pi}{3}$$

Example. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = xy\,\mathbf{i} + (y^2 + e^{xz^2})\,\mathbf{j} + \sin(xy)\,\mathbf{k}$$

and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$.



We will use the divergence theorem instead of computing the four surface integrals directly.

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}\left(y^2 + e^{xz^2}\right) + \frac{\partial}{\partial z}(\sin xy) = y + 2y = 3y$$

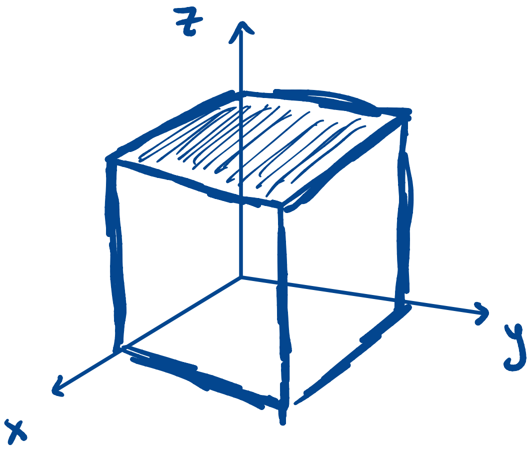
By the divergence theorem (need to put this on the exam),

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV = \iiint_E 3y\, dV$$

$$= \int_{-1}^{1} \int_0^{1-x^2} \int_0^{2-z} 3y\, dy\, dz\, dx$$

$$= \boxed{\frac{184}{35}}$$

Example. Let S be the unit cube with an open top. Find the flux of the vector field $\mathbf{F} = \langle z, y, x\rangle$ over S.



Let $\bar{S}$ be the union of S and the top of the cube

$$\iint_{\bar{S}} \vec{F}\cdot d\vec{S} = \iint_{S} \vec{F}\cdot d\vec{S} + \iint_{\text{lid}} \vec{F}\cdot d\vec{S}$$

$$\iint_{\bar{S}} \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV = \iiint_E 1\, dV = \operatorname{Vol}(E) = 1$$

Lid: $\vec{r}(x,y) = \langle x, y, 1\rangle \qquad 0\le x\le 1,\ 0\le y\le 1$

$\vec{r}_x = \langle 1, 0, 0\rangle$

$\vec{r}_y = \langle 0, 1, 0\rangle$

$\vec{r}_x \times \vec{r}_y = \langle 0, 0, 1\rangle$ → This agrees with the outward orientation ✓

$$\iint_{\text{lid}} \vec{F}\cdot d\vec{S} = \iint_D \langle z, y, x\rangle \cdot \langle 0,0,1\rangle\, dA = \iint_D x\, dA$$

$$\int_0^1\int_0^1 x\, dx\, dy = \int_0^1 x\, dx \cdot \int_0^1 dy = \frac{1}{2}$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{\bar{S}} \vec{F}\cdot d\vec{S} - \iint_{\text{lid}} \vec{F}\cdot d\vec{S} = 1 - \frac{1}{2} = \frac{1}{2}$$

2. [–/3 Points] DETAILS **SCALCCC4 13.8.017.**

z^2 y^2 x^2

Use the Divergence Theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = z^2x\mathbf{i} + (y^3/3+\tan(z))\mathbf{j} + (x^2z+y^2)\mathbf{k}$ and S is the top half of the sphere $x^2 + y^2 + z^2 = 1$. (Hint: Note that S is not a closed surface. First compute integrals over S_1 and S_2, where S_1 is the disk $x^2 + y^2 \le 1$, oriented downward, and S_2 is = $S_1 \cup S$.)

$$\iint_S \vec{F}\cdot d\vec{S} + \iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{S_2} \vec{F}\cdot d\vec{S}$$

S_2 :

S_1 is given by $\vec{r}(x,y) = \langle x, y, 0\rangle$, $x^2+y^2 \le 1$

$$\vec{F}(\vec{r}(x,y)) = \langle 0, y^3/3, y^2 \rangle$$

$$\vec{r}_x \times \vec{r}_y = \langle 1,0,0\rangle \times \langle 0,1,0\rangle = \langle 0,0,1\rangle$$

oriented outward $\rightsquigarrow$ $\vec{n} = -\vec{k}$

$$\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_D \langle 0, y^3/3, y^2\rangle \cdot \langle 0,0,-1\rangle \, dA = \iint_D -y^2 \, dA$$

$$= \int_0^{2\pi}\int_0^1 -r^2\sin^2\theta \cdot r \, dr \, d\theta$$

$$= -\int_0^{2\pi} \sin^2\theta \, d\theta \cdot \int_0^1 r^3 \, dr = -\frac{\pi}{4}$$

$\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$$\iint_{S_2} \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F} \, dV = \iiint_E x^2+y^2+z^2 \, dV$$

$\operatorname{div}\vec{F} = x^2+y^2+z^2$

$$= \int_0^{\pi/2}\int_0^{2\pi}\int_0^1 \rho^2 \cdot \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \int_0^{\pi/2} \sin\varphi \, d\varphi \cdot \int_0^{2\pi} d\theta \cdot \int_0^1 \rho^4 \, d\rho = 1 \cdot 2\pi \cdot \frac{1}{5} = \frac{2\pi}{5}$$

$$\iint_S \vec{F}\cdot d\vec{S} = \frac{2\pi}{5} - \left(-\frac{\pi}{4}\right) = \frac{8\pi}{20} + \frac{5\pi}{20} = \boxed{\frac{13\pi}{20}}$$