Lecture Notes
Math 2400 - Calculus III
Spring 2024

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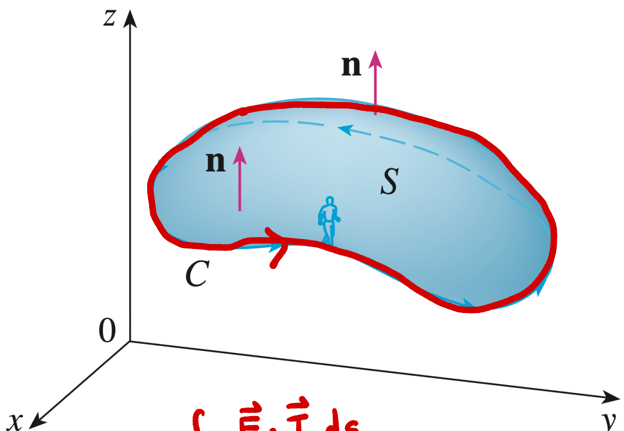
13.7 Stokes' Theorem

Theorem (Stokes' Theorem)**.** Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. How can we compute $\int_C \mathbf{F} \cdot d\mathbf{r}$?



Note: The surface S has a boundary curve C. $\vec{n}$ induces an orientation on C by the right hand rule.

$\int_C \vec{F} \cdot \vec{T}\, ds$

$\iint_S \text{curl}\, \vec{F} \cdot \vec{n}\, dS$

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\, \vec{F} \cdot d\vec{S}}$$

- In words, the line integral around the boundary curve of S of the tangential component of $\vec{F}$ is equal to the surface integral of the normal component of the curl of $\vec{F}$.

- Other notation: $\iint_S \text{curl}\, \vec{F} \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$

$\int_S d\omega = \int_{\partial S} \omega$

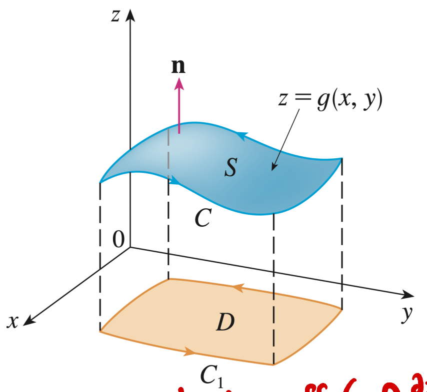
- ω is a differential form
- S is a manifold

- If S is flat and lies in the xy-plane, Stokes' Thm says

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\, \vec{F} \cdot d\vec{S} = \iint_S \text{curl}\, \vec{F} \cdot \vec{k}\, dA$$

This is Green's Thm!

Proof. → Special case of Stokes' Thm



- Suppose that the equation of S is $z = g(x, y)$, for $(x, y) \in D$. Assume that g has continuous second-order partial derivatives
- The boundary curve C_1 of D corresponds to C.
- Orienting S upward, the positive orientation of C corresponds to the positive orientation of C_1.
- We have $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, where the partial derivatives of P, Q, and R are continuous.

- Since S is a graph of a function, → In §13.6 we showed $\iint_S \vec{F}\cdot d\vec{S} = \iint_D \left(-P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R\right) dA$

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_D \left[-\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\frac{\partial z}{\partial x} - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\frac{\partial z}{\partial y} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right] dA$$

- If a parametric representation of C_1 is given by

$$x = x(t) \qquad y = y(t) \qquad a \le t \le b$$

then a parametric representation of C is

$$x = x(t) \qquad y = y(t) \qquad z = g(x(t), y(t)) \qquad a \le t \le b$$

- Hence,

$\vec{F}\cdot d\vec{r}$ is $\langle P, Q, R\rangle \cdot \vec{r}\,'(t)\,dt$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}\right) dt$$

use the chain rule to compute $\frac{dz}{dt}$ →
$$= \int_a^b \left[P\frac{dx}{dt} + Q\frac{dy}{dt} + R\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right)\right] dt$$

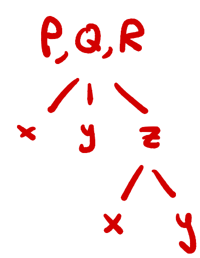
Collect $\frac{dx}{dt}$ and $\frac{dy}{dt}$ terms →
$$= \int_a^b \left[\left(P + R\frac{\partial z}{\partial x}\right)\frac{dx}{dt} + \left(Q + R\frac{\partial z}{\partial y}\right)\frac{dy}{dt}\right] dt$$

definition of line integral w.r.t. x and y →
$$= \int_{C_1} \left(P + R\frac{\partial z}{\partial x}\right) dx + \left(Q + R\frac{\partial z}{\partial y}\right) dy$$

Green's Thm →
$$= \iint_D \left[\frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial z}{\partial x}\right)\right] dA$$

need product rule

- By the chain rule, we obtain

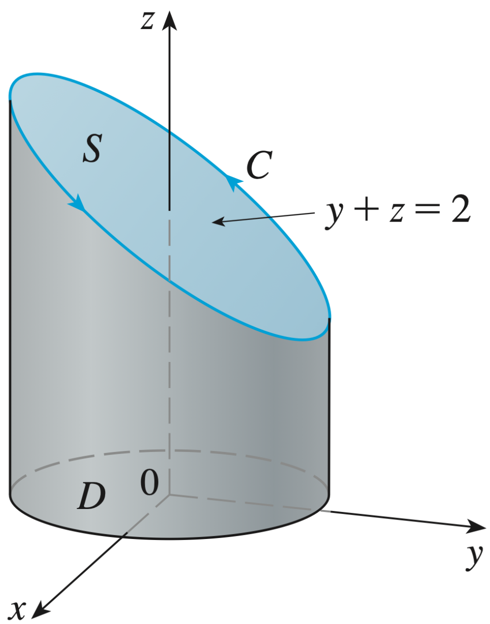


$$= \iint_D \left[\left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial z}{\partial y} + \cancel{\frac{\partial R}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}} + \cancel{R\frac{\partial^2 z}{\partial x\partial y}}\right) - \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial y}\frac{\partial z}{\partial x} + \cancel{\frac{\partial R}{\partial z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}} + \cancel{R\frac{\partial^2 z}{\partial y\partial x}}\right)\right] dA$$

See bullet point 5 →
$$= \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$$

$\square$

Example. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = -y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$ and C is the curve of intersection of the plane $y+z=2$ and the cylinder $x^2+y^2=1$. (Orient C to be counterclockwise when viewed from above.)



We could evalutate $\int_C \vec{F}\cdot d\vec{r}$ directly, but let's use Stokes' Thm

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$$

$$= \iint_D \text{curl}\,\vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\,dA$$

There are many surfaces with boundary C. The most convenient is the elliptical region in the plane $y+z=2$

A parametrization for S is $\vec{r}(x,y) = \langle x, y, 2-y\rangle$

$$\text{curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^2 & x & z^2 \end{vmatrix} = (1+2y)\,\vec{k}$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \langle 0, 1, 1\rangle$$

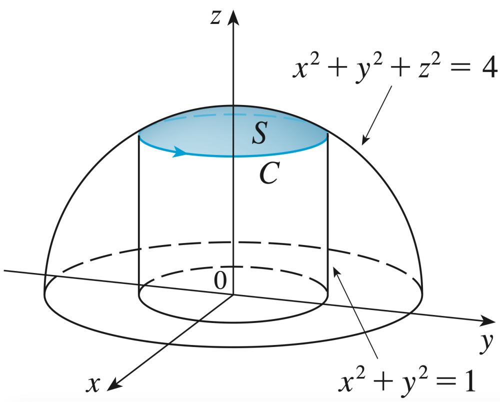
Correct orientation ✓
If C were oriented clockwise, we would multiply this by -1.

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_D 1+2y\,dA$$

$$= \int_0^{2\pi}\int_0^1 (1+2r\sin\theta)\,r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2}{2} + 2\frac{r^3}{3}\sin\theta\right]_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} + \frac{2}{3}\sin\theta\,d\theta = \pi$$

Example. Use Stokes' Theorem to compute $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yz\,\mathbf{j} + xy\,\mathbf{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane.



What is C? We solve the system

$$x^2 + y^2 + z^2 = 4$$
$$x^2 + y^2 = 1$$

$$z^2 = 3$$
$$z = \sqrt{3}$$

Conclude: C is the circle with $x^2+y^2=1$ and $z=\sqrt{3}$

A parametrization for C is $\vec{r}(t) = \langle \cos t, \sin t, \sqrt{3} \rangle \quad 0 \le t \le 2\pi$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 0 \rangle$$

$$\vec{F}(\vec{r}(t)) = \langle \sqrt{3}\cos t, \sqrt{3}\sin t, \cos t \sin t \rangle$$

By Stokes' Thm,

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$\int_0^{2\pi} -\sqrt{3}\cos t \sin t + \sqrt{3} \sin t \cos t \; dt$$

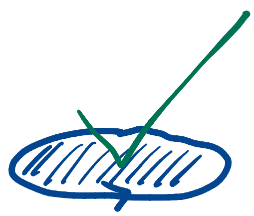
$$\int_0^{2\pi} 0 \, dt = 0$$

Question. What can we say if S_1 and S_2 are oriented surfaces with the same oriented boundary curve C?

If two oriented surfaces have the same boundary

$$\iint_{S_1} \text{curl}\,\vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} = \iint_{S_2} \text{curl}\,\vec{F} \cdot d\vec{S}$$

This is useful when it is difficult to integrate over one surface, but easy to integrate over another.



Theorem. Use Stokes' Theorem to prove that if $\text{curl}\,\mathbf{F} = 0$ on all of $\mathbb{R}^3$, then $\mathbf{F}$ is conservative.

- To show $\vec{F}$ is conservative, we will show that the line integral around any closed loop is 0.

- Given a simple closed curve C, we can find an orientable surface S whose boundary is C

- $$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \iint_S 0 \cdot d\vec{S} = 0$$