Lecture Notes
Math 2400 - Calculus III
Spring 2024

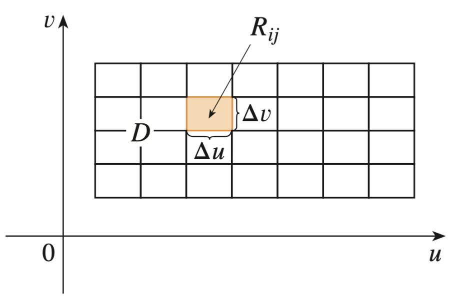
Name: ______________________________

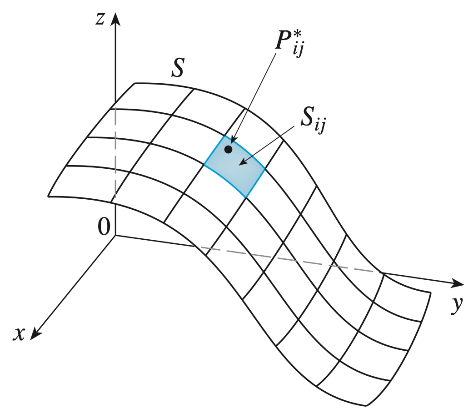
13.6 Surface Integrals

Definition. Let S be a parametric surface defined by a vector function

$$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k} \qquad (u,v) \in D$$

What is the surface integral of a function $f(x,y,z)$ over the surface S?



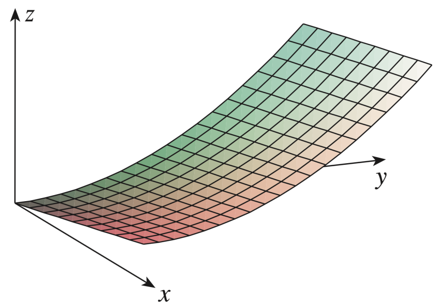


Example. Compute the surface integral $\iint_S x^2 \, dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

Question. What is an application of surface integrals?

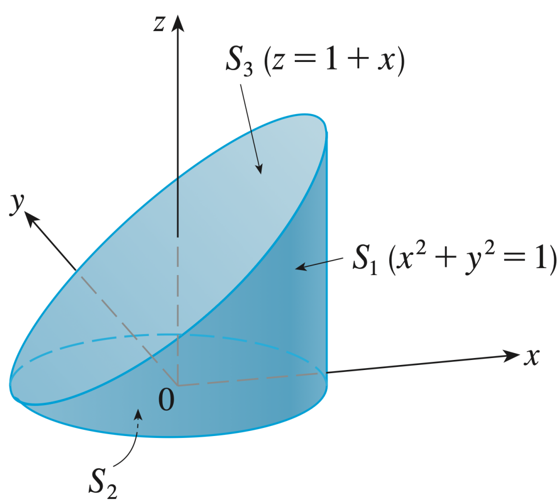
Definition. If S is the graph of a function $g(x, y)$, how can we compute the surface integral of a function $f(x, y, z)$ over S?

Example. Evaluate $\iint_S y\, dS$, where S is the surface $z = x + y^2, 0 \le x \le 1, 0 \le y \le 2$.

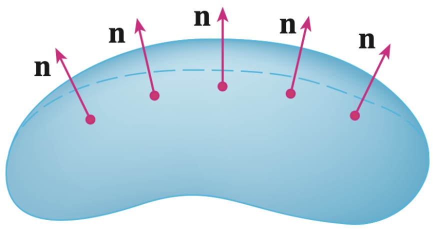


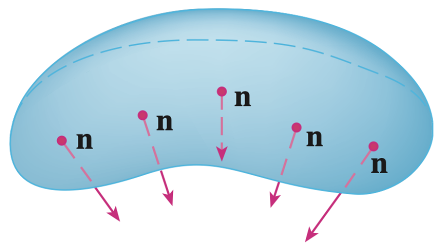
Question. How can we evaluate a surface integral over S if S is a finite union of smooth surfaces that intersect only along their boundaries?

Example. Evaluate $\iint_S z\,dS$, where S is the surface whose sides S_1 are given by the cylinder $x^2 + y^2 = 1$, whose bottom S_2 is the disk $x^2 + y^2 \leq 1$ in the plane $z = 0$, and whose top S_3 is the part of the plane $z = 1 + x$ that lies above S_2.

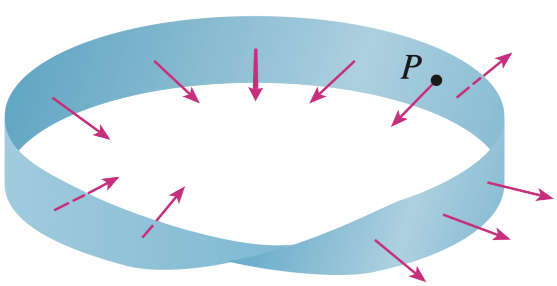


Definition. What is an orientable surface?





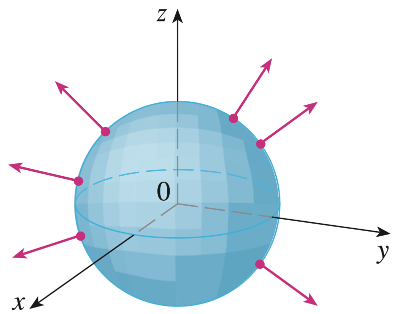
Example. Give an example of a surface that is not orientable.

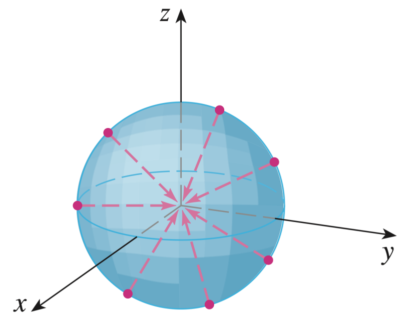


Definition. If S is the graph of a function $g(x, y)$, how can we give S a natural orientation?

Definition. If S is a smooth orientable surface given in parametric form, how can we give S a natural orientation?

Definition. For a closed surface, what is the positive orientation?



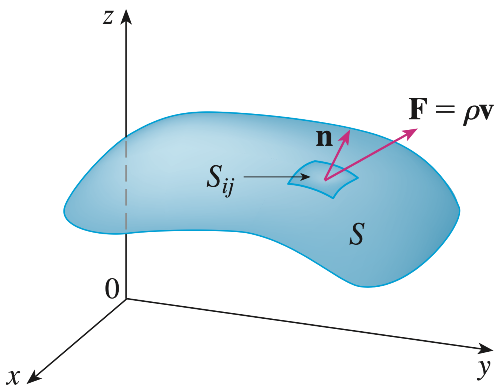


Example. A parametric representation for the sphere $x^2 + y^2 + z^2 = a^2$ is given by

$$\mathbf{r}(\phi, \theta) = \langle a \sin \phi \cos \theta, a \sin \phi \sin \theta, a \cos \phi \rangle$$

What is the orientation induced by $\mathbf{r}(\phi, \theta)$?

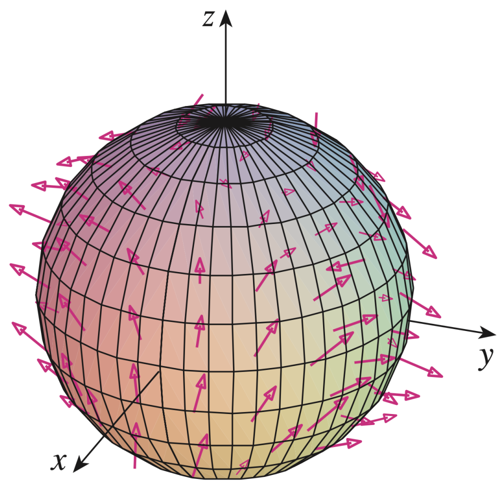
Definition. If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, what is the surface integral of $\mathbf{F}$ over S?



Definition. If S is given by a vector function $\mathbf{r}(u, v)$, what does the surface integral of $\mathbf{F}$ over S look like?

Definition. If S is the graph of a function $g(x, y)$, what does the surface integral of $\mathbf{F}$ over S look like?

Example. Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.



Example. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.