Lecture Notes
Math 2400 - Calculus III
Spring 2024

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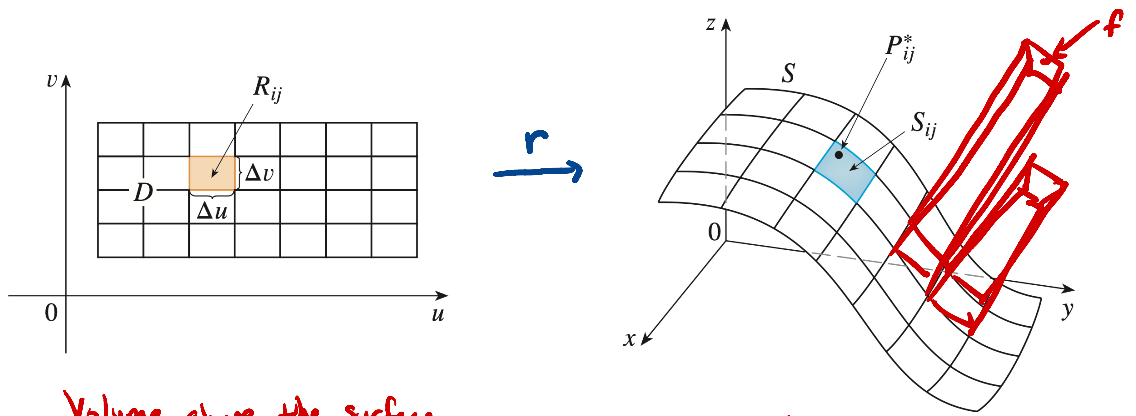
13.6 Surface Integrals

Definition. Let S be a parametric surface defined by a vector function

$$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k} \qquad (u,v) \in D$$

What is the surface integral of a function $f(x,y,z)$ over the surface S?



Volume above the surface

height of column

base of column

$$\iint_S f(x,y,z)\,dS = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\,\Delta S_{ij}$$

$$= \iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA$$

$\Delta S_{ij} \approx |\vec{r}_u \times \vec{r}_v|\,\Delta u\,\Delta v$ and $dS = |\vec{r}_u \times \vec{r}_v|\,dA$

Example. Compute the surface integral $\iint_S x^2\, dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

- $\vec{r}(\phi, \theta) = \langle \sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi \rangle$

  for $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$

- In the §12.6 notes, we computed that $|\vec{r}_\phi \times \vec{r}_\theta| = \sin\phi$

- $$\iint_S x^2\, dS = \iint_D (\sin\phi\cos\theta)^2\, |\vec{r}_\phi \times \vec{r}_\theta|\, dA$$

  $$= \int_0^{2\pi}\int_0^{\pi} \sin^2\phi\cos^2\theta \cdot \sin\phi\, d\phi\, d\theta$$

  $$= \int_0^{2\pi} \cos^2\theta\, d\theta \cdot \int_0^{\pi} \sin^3\phi\, d\phi$$

  $$= \frac{4\pi}{3}$$

use the identities

$\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$\sin^2\phi = 1 - \cos^2\phi$

Question. What is an application of surface integrals?

If a thin sheet of metal has the shape of a surface S and the density is given by $\rho(x,y,z)$, the total mass is $m = \iint_S \rho(x,y,z)\,dS$

Definition. If S is the graph of a function $g(x,y)$, how can we compute the surface integral of a function $f(x,y,z)$ over S?

A parametrization of S is given by $\vec{r}(x,y) = \langle x, y, g(x,y)\rangle$

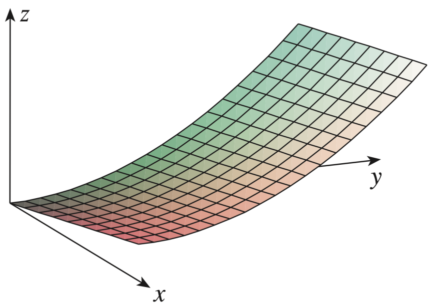
$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} i & j & k \\ 1 & 0 & g_x \\ 0 & 1 & g_y \end{vmatrix} = \langle -g_x, -g_y, 1\rangle$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \qquad \left(g_x = \frac{\partial z}{\partial x},\ g_y = \frac{\partial z}{\partial y}\right)$$

We then have

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\;dA$$

Example. Evaluate $\iint_S y\,dS$, where S is the surface $z = x + y^2, 0 \le x \le 1, 0 \le y \le 2$.



$\vec{r}(x,y) = \langle x, y, x+y^2 \rangle$

$\frac{\partial z}{\partial x} = 1 \qquad \frac{\partial z}{\partial y} = 2y$

$$\iint_S y\,dS = \iint_D y\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$= \int_0^1 \int_0^2 y\sqrt{1 + 1 + 4y^2}\,dy\,dx$$

$$= \int_0^1 dx \cdot \int_0^2 \sqrt{2}\,y\sqrt{1+2y^2}\,dy$$

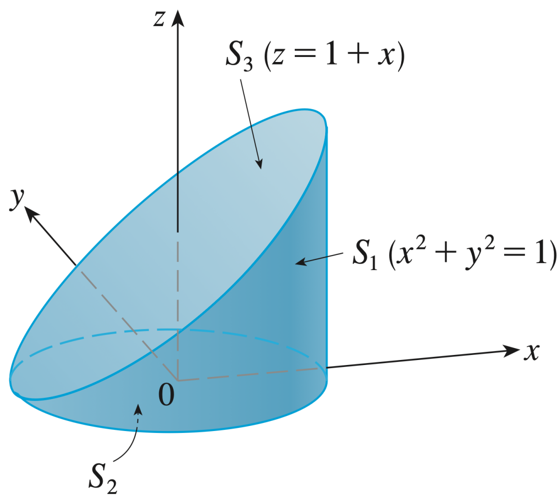
$$= \sqrt{2} \cdot \frac{1}{4} \cdot \frac{2}{3}(1+2y^2)^{3/2} \Big|_{y=0}^{y=2}$$

$$= \frac{13\sqrt{2}}{3}$$

Question. How can we evaluate a surface integral over S if S is a finite union of smooth surfaces that intersect only along their boundaries?

$$\iint_S f(x,y,z)\,dS = \iint_{S_1} f(x,y,z)\,dS + \cdots + \iint_{S_n} f(x,y,z)\,dS$$

Example. Evaluate $\iint_S z\,dS$, where S is the surface whose sides S_1 are given by the cylinder $x^2+y^2=1$, whose bottom S_2 is the disk $x^2+y^2 \le 1$ in the plane $z=0$, and whose top S_3 is the part of the plane $z=1+x$ that lies above S_2.



S_1:

$$\vec{r}(\theta, z) = \langle \cos\theta, \sin\theta, z \rangle$$

$$0 \le \theta \le 2\pi$$

$$0 \le z \le 1 + x = 1 + \cos\theta$$

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} i & j & k \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$$

$$|\vec{r}_\theta \times \vec{r}_z| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

Hence, $\displaystyle\iint_{S_1} z\,dS = \iint_D z\,|\vec{r}_\theta \times \vec{r}_z|\,dA = \int_0^{2\pi}\int_0^{1+\cos\theta} z\,dz\,d\theta$

$$= \frac{3\pi}{2}$$

S_2: Since S_2 lies in the plane $z=0$, we have

$$\iint_{S_2} z\, dS = \iint_{S_2} 0\, dS = 0$$

S_3: S_3 lies above the unit disk and is part of the plane $z = 1+x$

$$\iint_{S_3} z\, dS = \iint_D (1+x)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$= \iint_D (1+x)\sqrt{1+1+0}\, dA$$

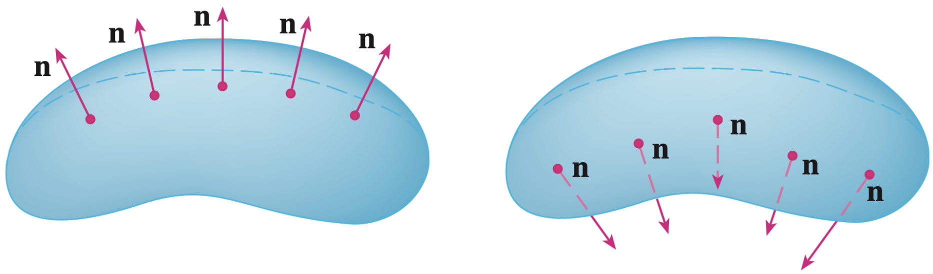
$$= \int_0^{2\pi}\int_0^1 (1+r\cos\theta)\sqrt{2}\, r\, dr\, d\theta$$

$$= \sqrt{2}\,\pi$$

Hence $$\iint_S z\, dS = \iint_{S_1} z\, dS + \iint_{S_2} z\, dS + \iint_{S_3} z\, dS$$

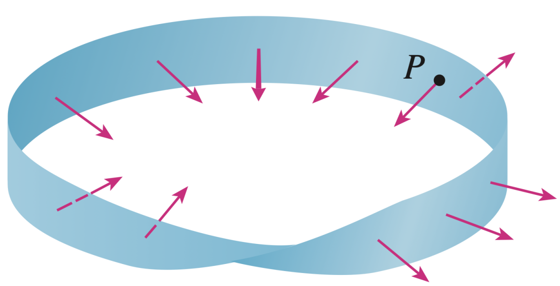
$$= \frac{3\pi}{2} + 0 + \sqrt{2}\,\pi$$

Definition. What is an orientable surface?



- Consider the tangent plane at a point (x,y,z) on S.
- There are two unit normal vectors $\vec{n}_1$ and $\vec{n}_2 = -\vec{n}_1$
- If we can choose a unit normal vector $\vec{n}$ at every point (x,y,z) so that $\vec{n}$ varies continuously over S, then S is called <u>orientable</u>.
- Giving a choice of $\vec{n}$ at every point provides S with an <u>orientation</u>.

Example. Give an example of a surface that is not orientable.



For the Mobius Strip, you cannot choose $\vec{n}$ at every point so that $\vec{n}$ varies continuously.

<u>Rmk</u>: From now on, we will only consider orientable (two-sided) surfaces.

Definition. If S is the graph of a function $g(x, y)$, how can we give S a natural orientation?

The choice of $\vec{n}$ is given by

$$\vec{n} = \frac{\langle -g_x, -g_y, 1 \rangle}{\sqrt{1 + (g_x)^2 + (g_y)^2}}$$

endows S with the upward orientation

(note: the $\vec{k}$-component is positive)

Definition. If S is a smooth orientable surface given in parametric form, how can we give S a natural orientation?

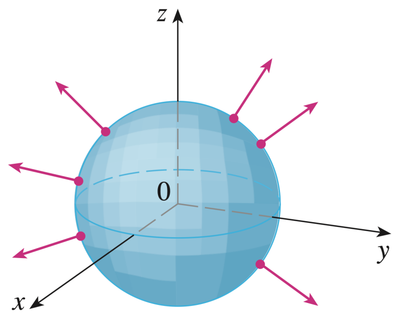
The normal vector at a point is given by

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

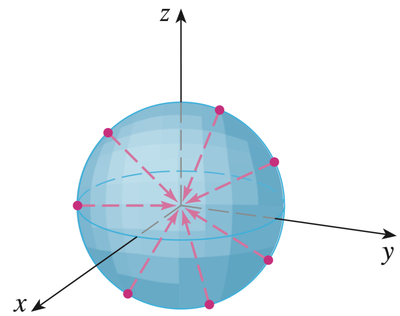
The opposite orientation is given by $-\vec{n}$.

the boundary of a solid region E

Definition. For a closed surface, what is the positive orientation?



positive orientation
(outward normal)



negative orientation
(inward normal)

Example. A parametric representation for the sphere $x^2 + y^2 + z^2 = a^2$ is given by

$$\mathbf{r}(\phi, \theta) = \langle a \sin\phi \cos\theta, a \sin\phi \sin\theta, a \cos\phi \rangle$$

What is the orientation induced by $\mathbf{r}(\phi, \theta)$?

In §12.6, we showed

$$\vec{r}_\phi \times \vec{r}_\theta = \langle a^2 \sin^2\phi \cos\theta, a^2 \sin^2\phi \sin\theta, a^2 \sin\phi \cos\phi \rangle$$

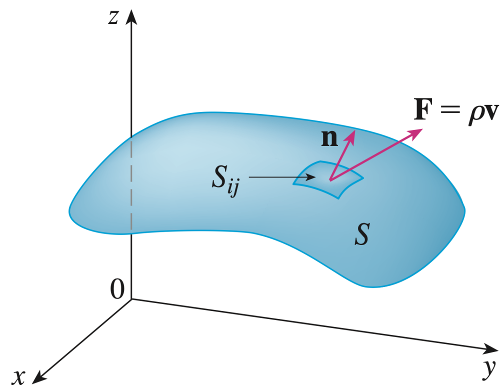
and $|\vec{r}_\phi \times \vec{r}_\theta| = a^2 \sin\phi$

$$\Rightarrow \vec{n} = \frac{\vec{r}_\phi \times \vec{r}_\theta}{|\vec{r}_\phi \times \vec{r}_\theta|} = \langle \sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi \rangle$$

$$= \frac{1}{a} \vec{r}(\phi, \theta)$$

Scale the position vector to have length 1. This is the positive orientation above.

Definition. If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, what is the surface integral of $\mathbf{F}$ over S?



$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} \, dS$$

- This integral is called the flux of $\vec{F}$ across S.
- In words, the surface integral of a vector field over S is equal to the surface integral of its normal component over S.

e.g. imagine a fluid with density $\rho(x,y,z)$ and velocity field $\vec{v}(x,y,z)$ flowing through S (like a fishing net)

We can approximate the mass of fluid per unit time crossing a small patch S_{ij} in the direction of the normal vector $\vec{n}$ by

$$(\rho \vec{v} \cdot \vec{n}) \, A(S_{ij})$$

The sum of all of these gives the surface integral $\iint_S \rho \vec{v} \cdot \vec{n} \, dS$

Note: $\vec{F} = \rho \vec{v}$ is a vector field on $\mathbb{R}^3$ and this surface integral measures the amount of fluid flowing through S.

Definition. If S is given by a vector function $\mathbf{r}(u, v)$, what does the surface integral of $\mathbf{F}$ over S look like?

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \, dS$$

$$= \iint_D \left[\vec{F}(\vec{r}(u,v)) \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \right] |\vec{r}_u \times \vec{r}_v| \, dA$$

$$= \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) \, dA$$

Definition. If S is the graph of a function $g(x, y)$, what does the surface integral of $\mathbf{F}$ over S look like?

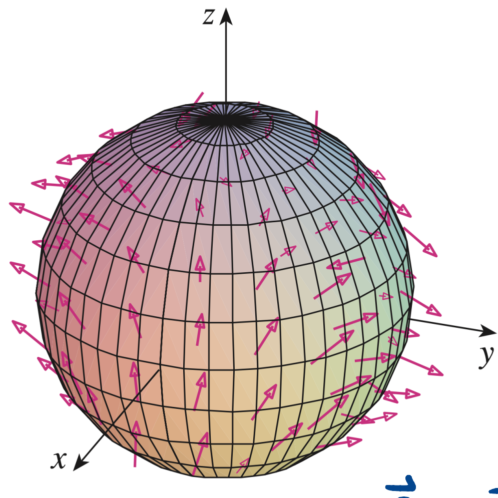
Using the parametrization $\vec{r}(x,y) = \langle x, y, g(x,y) \rangle$, we have

$$\vec{F} \cdot (\vec{r}_x \times \vec{r}_y) = \langle P, Q, R \rangle \cdot \langle -g_x, -g_y, 1 \rangle$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA$$

Note: This assumes the upward orientation of S. For the downward orientation, multiply by -1.

Example. Find the flux of the vector field $\mathbf{F}(x,y,z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ across the unit sphere $x^2+y^2+z^2=1$.



$\vec{r}(\phi,\theta) = \langle \sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi \rangle$

for $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$

$\vec{F}(\vec{r}(\phi,\theta)) = \langle \cos\phi,\ \sin\phi\sin\theta,\ \sin\phi\cos\theta \rangle$

$\vec{r}_\phi \times \vec{r}_\theta = \langle \sin^2\phi\cos\theta,\ \sin^2\phi\sin\theta,\ \sin\phi\cos\phi \rangle$

see §12.6 notes

$\vec{F}(\vec{r}(\phi,\theta)) \cdot (\vec{r}_\phi \times \vec{r}_\theta) = \cos\phi\sin^2\phi\cos\theta + \sin^3\phi\sin^2\theta + \sin^2\phi\cos\phi\cos\theta$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{r}_\phi \times \vec{r}_\theta)\,dA$$

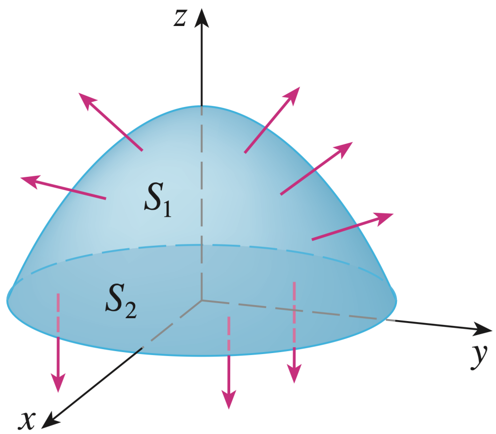
$$= \int_0^{2\pi}\int_0^{\pi} 2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta\,d\phi\,d\theta$$

$$= 2\int_0^{\pi}\sin^2\phi\cos\phi\,d\phi\int_0^{2\pi}\cos\theta\,d\theta + \int_0^{\pi}\sin^3\phi\,d\phi\int_0^{2\pi}\sin^2\theta\,d\theta$$

$\int_0^{2\pi}\cos\theta\,d\theta = 0$

$$= 0 + \frac{4\pi}{3}$$

Example. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.



We will use the positive (outward) orientation.

S_1: S_1 is the graph of $g(x,y) = 1 - x^2 - y^2$, so

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA$$

where $P(x,y,z) = y$, $Q(x,y,z) = x$, $R(x,y,z) = z = 1 - x^2 - y^2$

$$= \iint_D \left[-y(-2x) - x(-2y) + 1 - x^2 - y^2 \right] dA$$

$$= \iint_D 1 + 4xy - x^2 - y^2 \, dA$$

$$= \int_0^{2\pi} \int_0^1 (1 + 4r^2 \cos\theta \sin\theta - r^2)\, r\, dr\, d\theta$$

$$= \frac{\pi}{2}$$

S_2 : The disk S_2 is oriented downward, so its unit normal vector is $\vec{n} = -\vec{k}$

$$\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{S_2} \vec{F} \cdot (-\vec{k})\, dS$$

$$= \iint_{D} (-z)\, dA$$

$$= \iint_{D} 0 \, dA$$

Since $z=0$ on S_2.

Hence, $\displaystyle \iint_{S} \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S} = \frac{\pi}{2} + 0$